UMYU Scientifica, Vol. 1 NO. 2, December 2022, Pp 30 – 38
https://doi.org/10.56919/usci.1222.004
ISSN: 2955 – 1145 (print); 2955 – 1153 (online)
A periodical of the Faculty of Natural and Applied Sciences, UMYU, Katsina

ORIGINAL RESEARCH ARTICLE

Convergence and Order of the 2-Point Diagonally Implicit Block Backward Differentiation Formula with Two Off-Step Points

Alhassan Buhari[1]*, Musa Hamisu[2], Naghmeh Abasi[3]

[1]Department of Mathematics and Statistics, Al-Qalam University, Katsina, Nigeria
[2]Department of Mathematics and Statistics, Umaru Musa Yar'adua University, Katsina, Nigeria
[3]Institute for Mathematical Research, Universiti Putra, Serdang, Selangor, Malaysia

ABSTRACT

The development and formulation of a most reliable and efficient numerical schemes for the integration of stiff systems of ordinary differential equations in terms of order, convergence, stability requirements, accuracy, and computational expense has been a major challenged in the study of modern numerical analysis. In this paper, the order and convergence properties of the 2-point diagonally implicit block backward differentiation formula with two off-step points for solving first order stiff initial value problems have been studied, the method was derived and found to be of order five. The necessary and sufficient conditions for the convergence of the method have also been established. It has shown that the 2-point diagonally implicit block backward differentiation formula with two off-step points is both consistent and zero stable, having satisfied these two conditions of consistency and that of zero stability, it is therefore concluded that the method converges and suitable for the numerical integration of stiff systems.

ARTICLE HISTORY

Received November 23, 2022
Accepted December 27, 2022
Published December 30, 2022

KEYWORDS

Convergence, Order, Consistency, Zero-stability.

© The authors. This is an Open Access article distributed under the terms of the Creative Commons Attribution 4.0 License (http://creativecommons.org/licenses/by/4.0)

INTRODUCTION

In this paper, we shall be concerned with order and convergence properties of the method developed in (Musa, *et al.,* 2022) which has been derived to be of the form:

$$\left.\begin{aligned} y_{n+\frac{1}{2}} &= -\frac{1}{8}y_{n-1} + \frac{9}{8}y_n + \frac{3}{8}hf_{n+\frac{1}{2}} \\ y_{n+1} &= \frac{1}{21}y_{n-1} - \frac{4}{7}y_n + \frac{32}{21}y_{n+\frac{1}{2}} + \frac{2}{7}hf_{n+1} \\ y_{n+\frac{3}{2}} &= -\frac{3}{122}y_{n-1} + \frac{25}{61}y_n - \frac{75}{61}y_{n+\frac{1}{2}} + \frac{225}{122}y_{n+1} + \frac{15}{61}hf_{n+\frac{3}{2}} \\ y_{n+2} &= \frac{2}{135}y_{n-1} - \frac{1}{3}y_n + \frac{32}{27}y_{n+\frac{1}{2}} - 2y_{n+1} + \frac{32}{15}y_{n+\frac{3}{2}} + \frac{2}{9}hf_{n+2} \end{aligned}\right\} \quad (1)$$

The method (1) has been developed for approximate numerical solution of stiff systems in ordinary differential equations of the form

$$y' = f(x, y), \qquad y(x_0) = y_0, \qquad a \le x \le b \qquad (2)$$

where the function $f(x, y)$ is assumed to satisfy the Lipschitz conditions for the existence and uniqueness of solutions which guarantees that the ordinary differential equation (2) has a uniquely continuous differentiable solutions (Lambert, 1991). The solution of such equation is characterized by the presence of transient and steady state terms which restricts the step length of many numerical integration schemes except those numerical methods with A-stability properties (Suleiman *et al.,* 2014).

The system of ordinary differential equation (2) is said to be "stiff" when an extremely small step size is required to obtain correct numerical approximation. In other word, stiff problems are equations where certain implicit

Correspondence: Alhassan B. Department of Mathematics and Statistics, Al-Qalam University, Katsina, Nigeria.
buharialhassan@auk.edu.ng Phone +2347030308198
How to cite: Alhassan, B., Musa, H. and Naghmeh, A. (2022). Convergence and Order of the 2-Point Diagonally Implicit Block Backward Differentiation Formula with Two Off-Step Points. UMYU Scientifica, 1(1), 30 – 38. https://doi.org/10.56919/usci.1222.004

numerical methods, in particular backward differentiation formula (BDF) perform better than explicit numerical schemes (Curtiss and Hirschfelder, 1952). Most interesting and physically relevant real world stiff problems are difficult to solve analytically rather an alternative numerical methods are used in determining approximate solutions to the problems. However, in dealing with stiff ODEs, the stiffness property restricts the conventional explicit numerical integration methods from handling the problems efficiently. The stiff IVPs occur in many fields of science, engineering and technology, they are particularly found in chemical kinetics, thermodynamics and heat flow, control systems, vibration of the strings, electrical circuits, nuclear radioactive decay, weather prediction and forecasting (Bala and Musa, 2022).

Implicit linear multistep methods are known to be best and suitable for the treatment of stiff ODEs, the backward differentiation formula was developed by Curtiss and Hirschfelder, (1952), since then most of the improvements in the class of linear multistep methods are based on BDF, this is due to its special properties and better stability characteristics. The development of a most reliable and efficient numerical schemes for the integration of stiff systems of ordinary differential equations in terms of accuracy, stability requirements, convergence and computational expense has been a major challenged in the study of modern numerical analysis (Ibrahim *et al*, 2003). Although a few block numerical methods for the numerical integration of (2) have been proposed, there has remained a strong interest in developing standard fully and diagonally implicit block methods for solving stiff ODEs such as those found in (Suleiman, *et al.*, 2014; Musa and Bala, 2019; Ibrahim, *et al.*, 2007; Musa *et al.*, 2022; Abasi, *et al.*, 2014) and so on. However, this research will contribute to the existing literature as this paper focuses on the convergence properties such as the order, consistency and zero-stability of the 2-point diagonally implicit block backward differentiation formula with two off-step points presented in Musa *et al.*, (2022). The method approximates two solution values with two off-step points which are chosen when the step length is halved. Details on the derivations, stability analysis, implementation and performance of the method can be found in (Musa *et al.*, (2022). In the remaining sections of this paper, we shall derive the order and error constant of the method followed by consistency and zero-stability of the method.

ORDER AND ERROR OF THE METHOD

To derive the order of the method, we rearrange and rewrite the formula (1) in the following form:

$$\left.\begin{aligned}
y_{n+\frac{1}{2}}+\tfrac{1}{8}y_{n-1}-\tfrac{9}{8}y_n&=\tfrac{3}{8}hf_{n+\frac{1}{2}}\\
y_{n+1}-\tfrac{1}{21}y_{n-1}+\tfrac{4}{7}y_n-\tfrac{32}{21}y_{n+\frac{1}{2}}&=\tfrac{2}{7}hf_{n+1}\\
y_{n+\frac{3}{2}}+\tfrac{3}{122}y_{n-1}-\tfrac{25}{61}y_n+\tfrac{75}{61}y_{n+\frac{1}{2}}-\tfrac{225}{122}y_{n+1}&=\tfrac{15}{61}hf_{n+\frac{3}{2}}\\
y_{n+2}-\tfrac{2}{135}y_{n-1}+\tfrac{1}{3}y_n-\tfrac{32}{27}y_{n+\frac{1}{2}}+2y_{n+1}-\tfrac{32}{15}y_{n+\frac{3}{2}}&=\tfrac{2}{9}hf_{n+2}
\end{aligned}\right\} \tag{3}$$

The matrix associated with equation (3) is

$$\begin{bmatrix}0&\frac{1}{8}&0&-\frac{9}{8}\\0&-\frac{1}{21}&0&\frac{4}{7}\\0&\frac{3}{122}&0&-\frac{25}{61}\\0&-\frac{2}{135}&0&\frac{1}{3}\end{bmatrix}\begin{bmatrix}y_{n-\frac{3}{2}}\\y_{n-1}\\y_{n-\frac{1}{2}}\\y_n\end{bmatrix}+\begin{bmatrix}1&0&0&0\\-\frac{32}{21}&1&0&0\\\frac{75}{61}&-\frac{225}{122}&1&0\\-\frac{32}{27}&2&-\frac{32}{15}&1\end{bmatrix}\begin{bmatrix}y_{n+\frac{1}{2}}\\y_{n+1}\\y_{n+\frac{3}{2}}\\y_{n+2}\end{bmatrix}=h\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}f_{n-\frac{3}{2}}\\f_{n-1}\\f_{n-\frac{1}{2}}\\f_n\end{bmatrix}+$$

$$h\begin{bmatrix}\frac{3}{8}&0&0&0\\0&\frac{2}{7}&0&0\\0&0&\frac{15}{61}&0\\0&0&0&\frac{2}{9}\end{bmatrix}\begin{bmatrix}f_{n+\frac{1}{2}}\\f_{n+1}\\f_{n+\frac{3}{2}}\\f_{n+2}\end{bmatrix} \tag{4}$$

Let.

$$D_0=\begin{bmatrix}0\\0\\0\\0\end{bmatrix},D_1=\begin{bmatrix}\frac{1}{8}\\-\frac{1}{21}\\\frac{3}{122}\\-\frac{2}{135}\end{bmatrix},\quad D_2=\begin{bmatrix}0\\0\\0\\0\end{bmatrix},D_3=\begin{bmatrix}-\frac{9}{8}\\\frac{4}{7}\\-\frac{25}{61}\\\frac{1}{3}\end{bmatrix},\quad D_4=\begin{bmatrix}1\\-\frac{32}{21}\\\frac{75}{61}\\-\frac{32}{27}\end{bmatrix},\quad D_5=\begin{bmatrix}0\\1\\-\frac{225}{122}\\2\end{bmatrix},D_6=\begin{bmatrix}0\\0\\1\\-\frac{32}{15}\end{bmatrix},$$

$$D_7=\begin{bmatrix}0\\0\\0\\1\end{bmatrix},G_0=\begin{bmatrix}0\\0\\0\\0\end{bmatrix},G_1=\begin{bmatrix}0\\0\\0\\0\end{bmatrix},G_2=\begin{bmatrix}0\\0\\0\\0\end{bmatrix},G_3=\begin{bmatrix}0\\0\\0\\0\end{bmatrix},G_4=\begin{bmatrix}\frac{3}{8}\\0\\0\\0\end{bmatrix},G_5=\begin{bmatrix}0\\\frac{2}{7}\\0\\0\end{bmatrix},G_6=\begin{bmatrix}0\\0\\\frac{15}{61}\\0\end{bmatrix},\ G_7=\begin{bmatrix}0\\0\\0\\\frac{2}{9}\end{bmatrix}$$

Definition 1: The order of the block method (1) and its associated linear operator given by

$$L[y(x),h]=\sum_{j=0}^{7}\left[D_j y\left(x+j\frac{h}{2}\right)-hG_j y'\left(x+j\frac{h}{2}\right)\right] \qquad (5)$$

is a unique integer p such that $E_q=0$, $q=0(1)p$ and $E_{p+1}\neq 0$; where the E_q are constant (column) matrices defined by

$$\left.\begin{aligned}E_0&=D_0+D_1+D_2+\cdots+D_k\\E_1&=D_1+2D_2+\cdots+kD_k-2(G_0+G_1+G_2+\cdots+G_k)\\&\vdots\\E_q&=\frac{1}{q!}(D_1+2^qD_2+\cdots+k^qD_k)-\frac{2}{(q-1)!}(G_1+2^{q-1}G_2+\cdots+k^{q-1}G_k)\end{aligned}\right\}\qquad (6)$$

$$q=2,3,\ldots\ldots\ldots$$

For $q=0(1)6$, we have

$$E_0=\sum_{j=0}^{7}D_j=D_0+D_1+D_2+D_3+D_4+D_5+D_6+D_7$$

$$=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+\begin{bmatrix}\frac{1}{8}\\-\frac{1}{21}\\\frac{3}{122}\\-\frac{2}{135}\end{bmatrix}+\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+\begin{bmatrix}-\frac{9}{8}\\\frac{4}{7}\\-\frac{25}{61}\\\frac{1}{3}\end{bmatrix}+\begin{bmatrix}1\\-\frac{32}{21}\\\frac{75}{61}\\-\frac{32}{27}\end{bmatrix}+\begin{bmatrix}0\\1\\-\frac{225}{122}\\2\end{bmatrix}+\begin{bmatrix}0\\0\\1\\-\frac{32}{15}\end{bmatrix}+\begin{bmatrix}0\\0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$E_1=\sum_{j=0}^{7}\left(jD_j\right)-2\sum_{j=0}^{7}G_j=\left((0)D_0+(1)D_1+(2)D_2+(3)D_3+(4)D_4+(5)D_5+(6)D_6+(7)D_7\right)$$
$$-2\left(G_0+G_1+G_2+G_3+G_4+G_5+G_6+G_7\right)$$

$$=\left[(0)\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(1)\begin{bmatrix}\frac{1}{8}\\-\frac{1}{21}\\\frac{3}{122}\\-\frac{2}{135}\end{bmatrix}+(2)\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(3)\begin{bmatrix}-\frac{9}{8}\\\frac{4}{7}\\-\frac{25}{61}\\\frac{1}{3}\end{bmatrix}+(4)\begin{bmatrix}1\\-\frac{32}{21}\\\frac{75}{61}\\-\frac{32}{27}\end{bmatrix}+(5)\begin{bmatrix}0\\1\\-\frac{225}{122}\\2\end{bmatrix}+(6)\begin{bmatrix}0\\0\\1\\-\frac{32}{15}\end{bmatrix}+(7)\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right]$$

$$-2\left[\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+\begin{bmatrix}\frac{3}{8}\\0\\0\\0\end{bmatrix}+\begin{bmatrix}0\\\frac{2}{7}\\0\\0\end{bmatrix}+\begin{bmatrix}0\\0\\\frac{15}{61}\\0\end{bmatrix}+\begin{bmatrix}0\\0\\0\\\frac{2}{9}\end{bmatrix}\right]=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$E_2=\sum_{j=0}^{7}\frac{\left(j^2D_j\right)}{2!}-2\sum_{j=0}^{7}\left(jG_j\right)=\frac{1}{2!}\left((0)^2D_0+(1)^2D_1+(2)^2D_2+(3)^2D_3+(4)^2D_4+(5)^2D_5+(6)^2D_6+(7)^2D_7\right)$$
$$-2\left((0)G_0+(1)G_1+(2)G_2+(3)G_3+(4)G_4+(5)G_5+(6)G_6+(7)G_7\right)$$

$$=\frac{1}{2!}\left[(0)^2\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(1)^2\begin{bmatrix}\frac{1}{8}\\-\frac{1}{21}\\\frac{3}{122}\\-\frac{2}{135}\end{bmatrix}+(2)^2\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(3)^2\begin{bmatrix}-\frac{9}{8}\\\frac{4}{7}\\-\frac{25}{61}\\\frac{1}{3}\end{bmatrix}+(4)^2\begin{bmatrix}1\\-\frac{32}{21}\\\frac{75}{61}\\-\frac{32}{27}\end{bmatrix}+(5)^2\begin{bmatrix}0\\1\\-\frac{225}{122}\\2\end{bmatrix}+(6)^2\begin{bmatrix}0\\0\\1\\-\frac{32}{15}\end{bmatrix}+(7)^2\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right]$$

$$-2\left[(0)\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(1)\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(2)\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(3)\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(4)\begin{bmatrix}\frac{3}{8}\\0\\0\\0\end{bmatrix}+(5)\begin{bmatrix}0\\\frac{2}{7}\\0\\0\end{bmatrix}+(6)\begin{bmatrix}0\\0\\\frac{15}{61}\\0\end{bmatrix}+(7)\begin{bmatrix}0\\0\\0\\\frac{2}{9}\end{bmatrix}\right]=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$E_3=\sum_{j=0}^{7}\frac{\left(j^3D_j\right)}{3!}-2\sum_{j=0}^{7}\frac{\left(j^2G_j\right)}{2!}=\frac{1}{3!}\left((0)^3D_0+(1)^3D_1+(2)^3D_2+(3)^3D_3+(4)^3D_4+(5)^3D_5+(6)^3D_6+(7)^3D_7\right)$$
$$-\frac{2}{2!}\left((0)^2G_0+(1)^2G_1+(2)^2G_2+(3)^2G_3+(4)^2G_4+(5)^2G_5+(6)^2G_6+(7)^2G_7\right)$$

$$
=\frac{1}{3!}\left[(0)^3\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(1)^3\begin{bmatrix}\frac{1}{8}\\-\frac{1}{21}\\\frac{3}{122}\\-\frac{2}{135}\end{bmatrix}+(2)^3\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(3)^3\begin{bmatrix}-\frac{9}{8}\\\frac{4}{7}\\-\frac{25}{61}\\\frac{1}{3}\end{bmatrix}+(4)^3\begin{bmatrix}1\\-\frac{32}{21}\\\frac{75}{61}\\-\frac{32}{27}\end{bmatrix}+(5)^3\begin{bmatrix}0\\1\\-\frac{225}{122}\\2\end{bmatrix}+(6)^3\begin{bmatrix}0\\0\\1\\-\frac{32}{15}\end{bmatrix}+(7)^3\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right]
$$

$$
-\frac{2}{2!}\left[(0)^2\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(1)^2\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(2)^2\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(3)^2\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(4)^2\begin{bmatrix}\frac{3}{8}\\0\\0\\0\end{bmatrix}+(5)^2\begin{bmatrix}0\\\frac{2}{7}\\0\\0\end{bmatrix}+(6)^2\begin{bmatrix}0\\0\\\frac{15}{61}\\0\end{bmatrix}+(7)^2\begin{bmatrix}0\\0\\0\\\frac{2}{9}\end{bmatrix}\right]=\begin{bmatrix}-\frac{3}{8}\\0\\0\\0\end{bmatrix}
$$

$$
E_4=\sum_{j=0}^{7}\frac{\left(j^4D_j\right)}{4!}-2\sum_{j=0}^{7}\frac{\left(j^3G_j\right)}{3!}=\frac{1}{4!}\left((0)^4D_0+(1)^4D_1+(2)^4D_2+(3)^4D_3+(4)^4D_4+(5)^4D_5+(6)^4D_6+(7)^4D_7\right)
$$
$$
-\frac{2}{3!}\left((0)^3G_0+(1)^3G_1+(2)^3G_2+(3)^3G_3+(4)^3G_4+(5)^3G_5+(6)^3G_6+(7)^3G_7\right)
$$

$$
=\frac{1}{4!}\left[(0)^4\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(1)^4\begin{bmatrix}\frac{1}{8}\\-\frac{1}{21}\\\frac{3}{122}\\-\frac{2}{135}\end{bmatrix}+(2)^4\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(3)^4\begin{bmatrix}-\frac{9}{8}\\\frac{4}{7}\\-\frac{25}{61}\\\frac{1}{3}\end{bmatrix}+(4)^4\begin{bmatrix}1\\-\frac{32}{21}\\\frac{75}{61}\\-\frac{32}{27}\end{bmatrix}+(5)^4\begin{bmatrix}0\\1\\-\frac{225}{122}\\2\end{bmatrix}+(6)^4\begin{bmatrix}0\\0\\1\\-\frac{32}{15}\end{bmatrix}+(7)^4\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right]
$$

$$
-\frac{2}{3!}\left[(0)^3\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(1)^3\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(2)^3\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(3)^3\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(4)^3\begin{bmatrix}\frac{3}{8}\\0\\0\\0\end{bmatrix}+(5)^3\begin{bmatrix}0\\\frac{2}{7}\\0\\0\end{bmatrix}+(6)^3\begin{bmatrix}0\\0\\\frac{15}{61}\\0\end{bmatrix}+(7)^3\begin{bmatrix}0\\0\\0\\\frac{2}{9}\end{bmatrix}\right]=\begin{bmatrix}-\frac{9}{8}\\-\frac{4}{21}\\0\\0\end{bmatrix}
$$

$$
E_5=\sum_{j=0}^{7}\frac{\left(j^5D_j\right)}{5!}-2\sum_{j=0}^{7}\frac{\left(j^4G_j\right)}{4!}=\frac{1}{5!}\left((0)^5D_0+(1)^5D_1+(2)^5D_2+(3)^5D_3+(4)^5D_4+(5)^5D_5+(6)^5D_6+(7)^5D_7\right)
$$
$$
-\frac{2}{4!}\left((0)^4G_0+(1)^4G_1+(2)^4G_2+(3)^4G_3+(4)^4G_4+(5)^4G_5+(6)^4G_6+(7)^4G_7\right)
$$

$$
=\frac{1}{5!}\left[(0)^5\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(1)^5\begin{bmatrix}\frac{1}{8}\\-\frac{1}{21}\\\frac{3}{122}\\-\frac{2}{135}\end{bmatrix}+(2)^5\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(3)^5\begin{bmatrix}-\frac{9}{8}\\\frac{4}{7}\\-\frac{25}{61}\\\frac{1}{3}\end{bmatrix}+(4)^5\begin{bmatrix}1\\-\frac{32}{21}\\\frac{75}{61}\\-\frac{32}{27}\end{bmatrix}+(5)^5\begin{bmatrix}0\\1\\-\frac{225}{122}\\2\end{bmatrix}+(6)^5\begin{bmatrix}0\\0\\1\\-\frac{32}{15}\end{bmatrix}+(7)^5\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right]
$$

$$
-\frac{2}{4!}\left[(0)^4\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(1)^4\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(2)^4\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(3)^4\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(4)^4\begin{bmatrix}\frac{3}{8}\\0\\0\\0\end{bmatrix}+(5)^4\begin{bmatrix}0\\\frac{2}{7}\\0\\0\end{bmatrix}+(6)^4\begin{bmatrix}0\\0\\\frac{15}{61}\\0\end{bmatrix}+(7)^4\begin{bmatrix}0\\0\\0\\\frac{2}{9}\end{bmatrix}\right]=\begin{bmatrix}-\frac{279}{160}\\-\frac{24}{35}\\-\frac{15}{122}\\0\end{bmatrix}
$$

$$
E_6=\sum_{j=0}^{7}\frac{\left(j^6D_j\right)}{6!}-2\sum_{j=0}^{7}\frac{\left(j^5G_j\right)}{5!}=\frac{1}{6!}\left((0)^6D_0+(1)^6D_1+(2)^6D_2+(3)^6D_3+(4)^6D_4+(5)^6D_5+(6)^6D_6+(7)^6D_7\right)
$$

$$
-\frac{2}{5!}\left((0)^5G_0+(1)^5G_1+(2)^5G_2+(3)^5G_3+(4)^5G_4+(5)^5G_5+(6)^5G_6+(7)^5G_7\right)
$$

$$
=\frac{1}{6!}\left[(0)^6\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(1)^6\begin{bmatrix}\frac{1}{8}\\-\frac{1}{21}\\\frac{3}{122}\\-\frac{2}{135}\end{bmatrix}+(2)^6\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(3)^6\begin{bmatrix}-\frac{9}{8}\\\frac{4}{7}\\-\frac{25}{61}\\\frac{1}{3}\end{bmatrix}+(4)^6\begin{bmatrix}1\\-\frac{32}{21}\\\frac{75}{61}\\-\frac{32}{27}\end{bmatrix}+(5)^6\begin{bmatrix}0\\1\\-\frac{225}{122}\\2\end{bmatrix}+(6)^6\begin{bmatrix}0\\0\\1\\-\frac{32}{15}\end{bmatrix}+(7)^6\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right]
$$

$$
-\frac{2}{5!}\left[(0)^5\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(1)^5\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(2)^5\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(3)^5\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+(4)^5\begin{bmatrix}\frac{3}{8}\\0\\0\\0\end{bmatrix}+(5)^5\begin{bmatrix}0\\\frac{2}{7}\\0\\0\end{bmatrix}+(6)^5\begin{bmatrix}0\\0\\\frac{15}{61}\\0\end{bmatrix}+(7)^5\begin{bmatrix}0\\0\\0\\\frac{2}{9}\end{bmatrix}\right]=\begin{bmatrix}-\frac{37}{20}\\-\frac{80}{63}\\-\frac{125}{244}\\-\frac{4}{45}\end{bmatrix}
$$

Therefore, by the definition 1, we conclude that the order of the 2-point diagonally implicit block backward differentiation formula with two off-step points is 5 with error constant given by

$$E_6 = \begin{bmatrix} -\frac{37}{20} \\ -\frac{80}{63} \\ -\frac{125}{244} \\ -\frac{4}{45} \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

CONVERGENCE OF THE METHOD

A linear multistep method (LMM) must be convergent before it can be accepted and used for the numerical approximation of any stiff system due to the fact that any numerical method which does not converge has no practical importance. This section focuses on establishing the necessary and sufficient conditions for the method (1) to converge. According to Lambert, (1991), consistency and zero stability are the necessary and sufficient conditions for any numerical scheme to converge. We shall begin to show that the method (1) is consistent and zero stable by first presenting the following definitions and theorem related to the convergence of a linear multistep method.

Definition 2: A general k-step linear multistep method (LMM) has the form:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \qquad (7)$$

where α_j and β_j are constants and $\alpha_k \neq 0$, since α_0 and β_0 cannot both be zero at the same time. For any linear k-step method, α_k is normalized to 1.

Theorem 1: The necessary and sufficient conditions for the linear multistep method (7) to be convergent are that it be consistent and zero-stable.

To show that the 2-point diagonally implicit block BDF with two off-step points (DI2OBBDF) converge, we begin by showing that the method is consistent.

CONSISTENCY OF THE 2-POINT DIOBBDF METHOD

Definition 3: A linear multistep method (7) is said be consistent if it has order $p \geq 1$. It follows from (6) that the method (7) is said to be consistent if and only if the following conditions are satisfied:

$$\left.\begin{aligned} \sum_{j=o}^{k} D_j &= 0 \\ \sum_{j=o}^{k} jD_j &= 2\sum_{j=o}^{k} G_j \end{aligned}\right\} \qquad (8)$$

Based on these definitions, from section 2, we deduced that the order of the DI2OBBDF method is 5 which is greater than 1. Hence, by definition, the method is consistent.

Let $D_0, D_1, D_2, D_3, D_4, D_5, D_6, D_7$ and $G_0, G_1, G_2, G_3, G_4, G_5, G_6, G_7$ be as previously defined. Then

$$\sum_{j=0}^{7} D_j = D_0 + D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{8} \\ -\frac{1}{21} \\ \frac{3}{122} \\ -\frac{2}{135} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{9}{8} \\ \frac{4}{7} \\ -\frac{25}{61} \\ \frac{1}{3} \end{bmatrix} + \begin{bmatrix} \frac{1}{32} \\ -\frac{21}{75} \\ \frac{61}{32} \\ -\frac{32}{27} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{225} \\ -\frac{122}{2} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{32} \\ -\frac{15}{} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (9)$$

Hence, the first condition in (8) is satisfied.

$$\sum_{j=0}^{7} (jD_j) = ((0)D_0 + (1)D_1 + (2)D_2 + (3)D_3 + (4)D_4 + (5)D_5 + (6)D_6 + (7)D_7)$$

$$= \left[(0)\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (1)\begin{bmatrix} \frac{1}{8} \\ -\frac{1}{21} \\ \frac{3}{122} \\ -\frac{2}{135} \end{bmatrix} + (2)\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (3)\begin{bmatrix} -\frac{9}{8} \\ \frac{4}{7} \\ -\frac{25}{61} \\ \frac{1}{3} \end{bmatrix} + (4)\begin{bmatrix} 1 \\ -\frac{32}{21} \\ \frac{75}{61} \\ -\frac{32}{27} \end{bmatrix} + (5)\begin{bmatrix} 0 \\ 1 \\ -\frac{225}{122} \\ 2 \end{bmatrix} + (6)\begin{bmatrix} 0 \\ 0 \\ 1 \\ -\frac{32}{15} \end{bmatrix} + (7)\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right] = \begin{bmatrix} \frac{3}{4} \\ \frac{4}{7} \\ \frac{30}{61} \\ \frac{4}{9} \end{bmatrix} \quad (10)$$

$$2\sum_{j=0}^{7} G_j = 2\left(G_0 + G_1 + G_2 + G_3 + G_4 + G_5 + G_6 + G_7\right)$$

$$= 2\left[\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{3}{8} \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{2}{7} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{15}{61} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{2}{9} \end{bmatrix} \right] = \begin{bmatrix} \frac{3}{4} \\ \frac{4}{7} \\ \frac{30}{61} \\ \frac{4}{9} \end{bmatrix} \quad (11)$$

Hence, $\sum_{j=0}^{7} jD_j = 2\sum_{j=0}^{7} G_j$

Thus, the second condition in (8) is also satisfied. The consistency conditions are therefore met. Hence, the method is consistent. Next, we investigate the zero stability associated with method (1).

ZERO STABILITY OF THE 2-POINT DIOBBDF METHOD

Definition 4: A linear multi-step method (7) is said to be zero stable if all the roots of first characteristics polynomial have modulus less than or equal to unity and those roots with modulus unity are simple (Abasi *et al*, 2014).

To show that the method (1) is zero-stable, we present the stability polynomial of the method (1) as in Musa *et al.*, (2022).

$$R(t,\bar{h}) = -\frac{11}{1281}t^2 - \frac{121}{30744}t^2\bar{h} - \frac{1}{1464}t^2\bar{h}^2 - \frac{13595}{15372}t^3\bar{h} - \frac{3127}{15372}t^3\bar{h}^2 - \frac{10}{1281}t^3\bar{h}^3$$
$$-\frac{34705}{30744}t^4\bar{h} + \frac{2069}{4393}t^4\bar{h}^2 - \frac{221}{2562}t^4\bar{h}^3 + \frac{5}{854}t^4\bar{h}^4 + t^4 - \frac{1270}{1281}t^3 = 0 \quad (12)$$

we substitute $\bar{h} = 0$ in equation (12) to obtain the first characteristics polynomial as:

$$R(t,0) = -\frac{11}{1281}t^2 - \frac{1270}{1281}t^3 + t^4 = 0. \quad (13)$$

Solving equation (13) for t, we obtain the following roots as:

$$t = 0, t = 0, t = 1, t = -\frac{11}{1281}$$

Therefore, by the definition 5, the values of t above indicate that the method is zero-stable since no magnitude of the root is greater than one and the root $t = 1$ is simple.

The method satisfies the requirements of consistency and zero stability as stated in the above theorem, therefore the method converges.

CONCLUSION

In this paper, we established the convergence of the 2-point diagonally implicit block backward differentiation formula with two off-step points proposed by Musa *et al* (2022). It was also established that the method is of order 5. It has thus been shown that the method is zero stable and satisfied the consistency conditions. Having these two conditions, consistency and zero stability, the diagonally implicit 2-point block BDF with two off-step points is convergent.

ACKNOWLEDGEMENT

The authors are grateful to Al-Qalam University Katsina, Umaru Musa Yar'adua University, and Universiti Putra Malaysia for the supports they received from them during the compilation of this work.

REFERENCES

Abasi, N., Suleiman, M., B., Abassi, N., and Musa, H. (2014). 2-point block BDF with off-step points for solving stiff ODEs. International scientific publication and computing services, Pp 1-15. **[Crossref]**

Bala, N., Alhassan, B. and Musa, H. (2022). A new 3-point implicit block numerical method for solving first order stiff initial value problems, Timbou-Africa Academic Publications, International Journal of Science Research and Technology, 9(9), 219-231.

Bala N. and Musa H. (2022). Convergence of the Fourth Order Variable Step Size Super Class of Block Backward Differentiation Formula for Solving Stiff Initial Value Problems. UMYU Scientifica, 1(1), 178-183. **[Crossref]**

Curtiss, C. and Hirschfelder, J. (1952). Integration of stiff Equations, In: 38th National Academy of Sciences Conference, United States of America. p. 235-243. **[Crossref]**

Ibrahim, Z., B, Suleiman, M., B., Johari, R. and Ismail, F. (2003): On the stability of fully implicit block backward differentiation formulae. Jobatan Matematik, UTM. Pp 83-89.

Ibrahim, Z., B., Othman, K., I. and Suleiman, M., B. (2007). Implicit r-point Block Backward Differentiation Formula for First Order Stiff ODEs, Applied Mathematics and Computation. Pp 558-565. **[Crossref]**

Lambert, J., D. (1991). Numerical Methods for Ordinary Differential Systems: The Initial Value Problem. First Edition, John Wiley & Sons Ltd., New York, pp.124-245.

Musa, H., Alhassan, B. and Abasi, N. (2022). Diagonally implicit 2-point block backward differentiation formula with two off-step points for solving stiff initial value problems. Nigerian Journal of Mathematics and Application. 32 (II), pp 89-100 (Accepted)

Musa, H. and Bala, N. (2019). A 3-point Diagonally Implicit Super Class of Block Backward Differentiation Formula for Solving Stiff Initial Value Problems. Dutse Journal of Pure and Applied Science (DUJOPASS),5(1b),1-10.

Suleiman, M., B., Musa, H., Ismail, F., Senu, N. and Ibrahim, Z., B. (2014). A new super class of block backward differentiation formulas for stiff ODEs, Asian -European journal of mathematics.7(I), pp 1-17. **[Crossref]**