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ORIGINAL RESEARCH ARTICLE

The Type I Half Logistics-Topp-Leone-G Distribution Family: Model, its Properties and Applications.

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ABSTRACT

A number of new, upgraded, generalized, and extended distribution families have recently been developed to improve the distribution's applicability in a wider domain. The Type I Half Logistics-Topp Leone G family of distribution, otherwise known as (the TIHLTL-G) distribution family, was developed as a new generalized distribution family. Explicit expression, moment generating function, moments, probability weighted moment, hazard function, survival function, quantile function, and order statistics were also derived for the novel family. The exponential distribution was employed as a sub-model, and the novel distribution family provided great flexibility towards some sets of data. The methods of parameter estimation adopted are maximum likelihood (MLE) and maximum products of spacing (MPS) methods. Two data sets were examined, and simulation studies were conducted to exemplify the potential application and adaptability of the novel model compared with some of its existing counterparts. The MPS tends to perform better than the MLE in estimating the model parameters when the sample size is very small, but both did perform excellently when the sample sizes are moderate and large, as obtained in the simulation study. However, both methods of estimation of parameters support the novel model (TIHLTL-G) family of distribution through Akaike information and Bayesian information criterion as the best model.

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INTRODUCTION

Description and prediction of real-world phenomena are done with great accuracy by studying distributions and their statistical features. Statistical distributions are so valuable that their theory is researched extensively due to its validity and importance, and new distributions are developed. In real-life phenomena, the most well-known classical distributions rarely reveal the characteristics nor predict accurate data features. there is still a lot of interest in creating more adaptable statistical distributions. Numerous generalised classes of distributions arise and apply to real-life scenarios to characterize various phenomena, including those in medicine, demography, industrial statistics, engineering, actuarial science, environmental research, biological sciences, and economics. To track straightforward classical, continuous distributions are employed in data modelling. However, with the emergence of more sophisticated methods for lifespan analysis, it became evident that extended versions of these simple classical continuous univariate distributions were necessary. The logistic distribution has important uses in describing growth and as an alternative to the normal distribution. It has numerous significant applications in the area of population modeling, biology, geology, psychology, economics, medicine, finance, and engineering. It is evident that the classical distributions are assumed to approximately fit real-life data. Recently, many researchers have shown curiosity towards the new conceptual approach in developing new families of continuous distributions, adding reference one, two or more additional shape parameter(s) to the parent distribution. This parameter(s) induction has invariably been effective in inspecting the tail properties and enhancing the generator family's fitness. Although the existing distribution (parent) may have a monotonic failure rate, the advantage of these approaches for building a new probability model resides in their capacity to describe both all the required features of failure rate functions (monotonic, non-monotonic, etc.) Alzaatreh *et al.* (2013). Modeling in environmental sciences such as environmental pollution, actuarial science, modeling

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period of time without claims, modeling in engineering processes such as machine life cycles, and modeling the survival/recovery times of patients after surgery, are just a few examples of the research fields where statistical distributions are useful. There is a definite demand for extended forms of these distributions in many applicable fields, including lifetime analysis. Several statisticians have created new models as a result of the new families of continuous distributions that have emerged in recent years. These families are created by adding one, two, or more additional parameters (shape) to the parent distribution. The introduction of a new generalized distribution family is a significant and substantial advancement in probability distribution theory. This study focuses on developing a new compounding distribution family by introducing the Type I Half Logistic-G distribution family (TIHL-G distribution family) and Topp Leone-G distribution family (TL-G distribution family) together to form novel family known as Type I Half Logistic Topp Leone -G distribution family (TIHLTL-G distribution family). The newly developed families will showcase their potential to generate new models with more application flexibility. Over the preceding two decades, a wide variety of generalized family of distributions distribution family has been examined and investigated for use in activities involving data modeling in numerous applicable fields. Some well-known families include the Marshall-Olkin-G distribution was discussed by Marshall and Olkin (1997), the gamma G distributions discussed by Zografos and Balakrishnan (2009), and Ristic and Balakrishnan (2011) proposed The Gamma Topp-leone Exponential distribution. However, a number of techniques for creating fresh distribution families have been studied, resulting in the development of a number of compound distributions that are more adaptable than other types of distributions now in use. The newly generated families include but are not limited to the Marshall-Olkin-G distribution family, with two shape parameters, which were proposed by Cordeiro and de Castro (2011), The Kummer Beta Generalized (KB-G) distribution family proposed by Cordeiro *et al.* (2012) which contained the families such as KB-G -Normal, KB-G Weibull, and their sub-models. A novel technique for producing families of continuous distributions well recognized as "T-X" family and was proposed by Alzaatreh *et al.* (2013). Families such as the Gamma-X, Weibull-X, and Beta Exponential-X families were developed. A novel distribution family recognized as the Topp-leone "T-X" family with distinct parameter was introduced by Alzaghal *et al.* (2013). Silva *et al.* (2014) discussed the Nobel Weibull G distribution family. Cordeiro *et al.* (2014b) proposed the Lomax generator, new distribution family with two distinct parameters. In order to generalize any continuous parent distribution, Torabi and Montazari (2014) introduced the logistic-generated distributions, a new continuous distribution generator with location and scale parameters. The Type I Half Logistic distribution family with a parameter was proposed by Cordeiro *et al.* (2015). Alizadeh *et al.* (2015b) presented and investigated the Kumaraswamy Odd Logistic-G family, a novel distribution family with three distinct shape parameters. The logistic-X family of continuous distributions was first introduced by Tahir *et al.* (2016) and was derived from a logistic random variable. The topp leone distribution family is a new distribution family that was discussed by Al-Shomrani *et al.* (2016). The Noble Weibull-G distribution family is a new category of continuous distributions established considering the Weibull random variable that was researched and proposed by Tahir *et al.* (2016). An extension of the Weibull-G distribution family, the Kumaraswamy Weibull-generated distribution family with four parameters, was introduced by Hassan and Elgarhy (2016). A novel class of family of continuous distribution recognized as the Odd Topp-leone Half-Logistic-G Distribution Family was introduced by Afify *et al.* (2017). The " Generalized Odd Log-Logistic distribution family is a novel category of a typical continuous distributions was researched and proposed by Cordeiro *et al.* (2017). Makubate *et al.* (2018) proposed and discussed a novel family of generalized distributions recognized as the Beta Weibull-G with five distinct parameters. Yousof *et al.* (2018) obtained a generalized version of the Marshall-Olkin-G distribution family tagged as the Marshall-Olkin-G distribution Family. Hamedani *et al.* (2019) introduced a novel class of continuous distributions recognized as the Type II General Exponential distribution with two distinct positive parameters. Silva *et al.* (2019) studied and introduced a novel distribution family recognized as the Topp-leone Kumaraswamy class with three distinct positive parameters, which generalized the Kumaraswamy-G family. Ibrahim *et al.* (2020a) introduced a novel distribution family recognized as the Topp-leone Exponential-G distribution family with two distinct positive shape parameters, which generalized and also extended the Topp-Leone-G distribution family. Ibrahim *et al.* (2020b) proposed a new family of continuous distributions, recognized as the Topp Leone Kumaraswamy-G distribution family with three distinct positive shape parameters. Odd Chen-G distribution family was developed by Anzagra (2020). Nwezza *et al.* (2020). Proposed A New Gumbel Generated distribution family with its properties and application. Usman *et al.* (2020) introduced a new generalized distribution family called Weibull Odd Frechet distribution family with three distinct parameters. Half logistic odd Weibull Topp-leone-G distribution family was proposed by chipepa *et al.* (2020). Bello *et al.* (2020) proposed a Type I Half Logistic Exponentiated-G distribution family. Oluyede *et al.* (2021) developed a new generalization of the Weibull distribution family recognized as the Odd Weibull Topp-leone-G power series distribution family. Chipepa and Oluyede (2021) studied and developed a new generalized family of the Gompertz -G distribution recognized as the Marshall-Olkin-Gompertz-G distribution. Sengweni *et al.* (2021) introduced and studied a new distribution family recognized as the Topp-leone Half Logistic odd Lindley-G distribution. Makubate *et al.* (2021) proposed a new distribution family, namely the Marshal Olkin Half Logistic-G based on the generator pioneered by Marshall and Olkin (1997). Bello *et al.* (2021) proposed A Type Ii Half Logistic Exponentiated-G distribution family with application to survival analysis. Peter *et al.* (2022) studied The Half-Logistic Odd Power Generalized Weibull-G distribution family. Eghwerido *et al.* (2022) studied the Teissie-G distribution family, its properties and applications. Nanga *et al.* (2022) proposed the Tangent Topp-leone distribution family. Watthanawisut *et al.* (2022) discussed the Beta Topp-leone Generated distribution family with its Applications. Kadic *et al.* (2023) studied two families of continuous probability distributions generated from by the discrete lindley distribution. Alanzi *et al.* (2023) proposed a novel Mute generalised distribution family. One or more data sets were applied to the aforementioned distributions for illustrations and their flexibilities.

METHODS AND MATERIAL

This paper proposes the generalization method popularly known as Transform-Transformer (T-X) proposed by Alzaatreh *et al.* (2013), which was adopted to generate the new family of distributions. The procedure is expressed below;

Assuming there exists a random variable $K \in [y, z]$ for $-\infty \leq y < z < \infty$ with the probability density function (pdf) defined as r(k), then if D[A(x)] is a function of the cumulative distribution function of a random variable X such that D[A(x)] satisfies the conditions given below

a. D[A(x)] ∈ [y, z],
b. D[A(x)] is differentiable and monotonically non-decreasing, and
c. $D[A(x)] \to y$ as $x \to -\infty$ and $D[A(x)] \to z$ as $x \to \infty$.

The Alzaatreh *et al.* (2013) in their discussions, defined the T-X distribution family by

$$A(x) = \int_0^{D[A(x)]} r(k)dk \quad (1)$$

where D[A(x)] satisfies the three conditions a., b. and c. above. The corresponding probability density function to the equation above is given by differentiating the cdf. Thus,

$$a(x) = \left[\frac{d}{dx} D[B(x)]\right] r[DB(x)] \quad (2)$$

The Type I half logistic distribution Family was proposed by Cordeiro et.al. (2015), with cdf

and pdf given as:

$$F_{TIHL-G}(x;\ \zeta,\chi) = \frac{1-[1-G(x;\chi)]^{\zeta}}{1+[1-G(x;\chi)]^{\zeta}} \quad (3)$$

$$f_{TIHL-G}(x;\ \zeta,\chi) = \frac{2\zeta g(x;\chi)[1-g(x;\chi)]^{\zeta-1}}{\left[1+[1-g(x;\chi)]^{\zeta}\right]^2} \quad (4)$$

where x, ζ > 0, and g(x; χ) and G(x; χ) are the pdf and cdf of the parent distribution with parameter vector χ.

The Topp-Leone-G distribution family was proposed by Al-Shomrani *et al.* (2016) with cdf and pdf given as

$$F_{TL-G}(x;\ \theta) = \left[1-\left(1-H(x;\ \tau)\right)^2\right]^{\theta} \quad (5)$$

$$f_{TL-G}(x;\ \theta) = 2\theta h(x;\ \tau)[1-h(x;\ \tau)]\left[\left(1-H(x;\ \tau)\right)^2\right]^{\theta-1} \quad (6)$$

where θ > 0 is the parameter (shape) and H(x; τ) and h(x; τ) are the cdf and pdf of the parent distribution having parameter vector τ.

THE PROPOSED TYPE I HALF-LOGISTIC TOPP LEONE- G FAMILY OF DISTRIBUTION (TIHLTL-G) DISTRIBUTION FAMILY

Let the Topp Leone-G distribution family be the parent distribution family with cdf and pdf expressed in (5) and (6), respectively. From equation (1), the Type I Half-Logistic Topp Leone -G Family of Distribution (TIHLTL-G) distribution family has the cdf derived by substituting the cdf of the Topp Leone distribution family to the upper limit of the (1). Thus, the cdf of the proposed TIHLTL-G distribution family is defined as

$$F_{TIHLTL-G}(x;\ \zeta,\theta,\chi) = \int_0^{\left[1-(1-H(x;\tau))^2\right]^{\theta}} \frac{2\zeta g(x;\chi)[1-G(x;\chi)]^{\zeta-1}}{\left[1+[1-G(x;\chi)]^{\zeta}\right]^2} dx$$

and obtained as

$$F_{TIHLTL-G}(x;\ \zeta,\theta,\chi) = \frac{1-\left[1-\left\{1-\left(1-G(x;\chi)\right)^2\right\}^{\theta}\right]^{\zeta}}{1+\left[1-\left\{1-\left(1-G(x;,\chi)\right)^2\right\}^{\theta}\right]^{\zeta}} \quad (7)$$

The probability density function pdf of the proposed TIHLTL-G distribution family is obtained by differentiating the cdf with respect to x. Thus, the pdf of the proposed TIHLTL-G family is obtained as

$$f_{TIHLTL-G}(x;\ \zeta,\theta,\chi) == \frac{4\zeta\theta g(x;\chi)[1-G(x;\chi)]\left[1-\left(1-G(x;\chi)\right)^2\right]^{\theta-1}[1-[1-(1-G(x;\chi))^2]^{\theta}]^{\zeta-1}}{[1+[1-[1-(1-G(x;\chi))^2]^{\theta}]^{\zeta}]^2} \quad (8)$$

where $\zeta,\theta > 0$ are the shape parameters, and χ is a vector of parent distribution parameters

Fundamental Representation

This section contains a basic expansion of the TIHLTL-G. We will take into account the expansion of the cdf and pdf.

Expansion of Cdf

Consider then the cdf of the distribution family from (7). That is

$$F(x;\ \zeta,\theta,\chi) = \frac{1-\left[1-\left\{1-\left(1-G(x;\chi)\right)^2\right\}^\theta\right]^\zeta}{1+\left[1-\left\{1-\left(1-G(x;\chi)\right)^2\right\}^\theta\right]^\zeta}$$

Consider raising the cdf to the power of h, then

Assuming

$$[F(x;\ \zeta,\theta,\chi)]^h = \left[\frac{1-\left[1-\left\{1-\left(1-G(x;\chi)\right)^2\right\}^\theta\right]^\zeta}{1+\left[1-\left\{1-\left(1-G(x;\chi)\right)^2\right\}^\theta\right]^\zeta}\right]^h$$

$$[F(x;\ \zeta,\theta,\chi)]^h = \left[1-\left[1-\left\{1-\left(1-G(x;\chi)\right)^2\right\}^\theta\right]^\lambda\right]^h \left[1+\left[1-\left\{1-\left(1-G(x;\chi)\right)^2\right\}^\theta\right]^\lambda\right]^{-h}$$

From binomial expansion, we know that. $(1+x)^{-b} = \sum_{u=0}^{\infty}(-1)^u \binom{b+u-1}{u} x^u$

And

$$(1-x)^b = \sum_{v=0}^{\infty}(-1)^v \binom{b}{v} x^v$$

Therefore, the numerator now becomes

$$\left[1-\left[1-\left\{1-\left(1-G(x;\chi)\right)^2\right\}^\theta\right]^\zeta\right]^h = \sum_{u=0}^{h}(-1)^u \binom{h}{u}\left[1-\left\{1-\left(1-G(x;\chi)\right)^2\right\}^\theta\right]^{\zeta u}$$

Similarly

$$\left[1+\left[1-\left\{1-\left(1-G(x;\chi)\right)^2\right\}^\theta\right]^\zeta\right]^{-h} = \sum_{v=0}^{h}(-1)^v \binom{h+v-1}{v}\left[1-\left\{1-\left(1-G(x;\chi)\right)^2\right\}^\theta\right]^{\zeta v}$$

$$[F(x;\zeta,\theta,\chi)]^h = \sum_{u=0}^{h}(-1)^u \sum_{v=0}^{h}(-1)^v \binom{h}{u}\binom{h+v-1}{v}\left[1-\left\{1-\left(1-G(x;\chi)\right)^2\right\}^\theta\right]^{\zeta v}\left[1-\left\{1-\left(1-G(x;\chi)\right)^2\right\}^\theta\right]^{\zeta u}$$

$$[F(x;\zeta,\theta,\chi)]^h = \sum_{u=0}^{\infty}\sum_{v=0}^{h}(-1)^{u+v}\binom{h}{v}\binom{h+u-1}{u}\left[1-\left\{1-\left(1-G(x;\chi)\right)^2\right\}^\theta\right]^{\zeta(v+u)}$$

By expanding various brackets using binomial series, we will have

$$\left(F(x;\zeta,\theta,\chi)\right)^h = \sum_{u=0}^{h}(-1)^u \gamma_f G(x;\chi)^n \qquad (9)$$

Where $\gamma_f = \sum_{v=0}^{h}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\binom{h}{v}\binom{h+u-1}{u}\binom{\zeta(u+v)}{l}\binom{\theta k}{m}\binom{2m}{n}$

Expansion of Pdf

From (8). The pdf of the family can be rewritten as

$$f(x;\zeta,\theta,\chi) = 4\zeta\theta g(x;\chi)[1-G(x;\chi)]\left[1-\left(1-G(x;\chi)\right)^2\right]^{\theta-1}\left[1-\left[1-\left(1-G(x;\chi)\right)^2\right]^\theta\right]^{\zeta-1}\left[1+\left[1-\left[1-\left(1-G(x;\chi)\right)^2\right]^\theta\right]^\zeta\right]^{-2}$$

Now reduced to

$$f(x;\zeta,\theta,\chi) = 4\zeta\theta g(x;\chi)\sum_{y=0}^{\infty}\varphi_q\, G(x;\chi)^r \qquad (10)$$

Where

$\varphi_q = \sum_{z=0}^{\infty}\sum_{k=0}^{\infty}\sum_{p}^{\infty}\sum_{r}^{\infty}\binom{1+y}{y}\binom{\zeta z}{z}\binom{\zeta-1}{k}\binom{\theta(k+z+1)-1}{p}\binom{2p+1}{r}$

STATISTICAL PROPERTIES

This section deals with the derivation of some of the statistical properties of the novel distribution family.

Probability Weighted Moments (PWMs)

This category is used to construct inverse form estimators for a distribution's parameters and quantiles. Using the following relationship given below, the PWMs, denoted by $\zeta_{r,s}$, may be obtained for any random variable X.

$$\zeta_{r,s} = E[x^r F(x)^s] = \int_{-\infty}^{\infty} x^r f(x)(F(x))^s\, dx \qquad (11)$$

The TIHLTL-G PWMs are obtained by inserting equation (9) and equation (10) into equation (11) and interchange h and s. then, we will have

$$\zeta_{r,s} = \int_0^\infty x^r\, 4\zeta\theta g(x;\chi)\sum_{y=0}^{\infty}\varphi_q\, G(x;\chi)^r \sum_{u=0}^{s}\gamma_f G(x;\chi)^n dx$$

$$= \int_0^\infty x^r\, 4\zeta\theta g(x;\chi)\sum_{y=0}^{\infty}\varphi_q \sum_{u=0}^{s}\gamma_f G(x;\chi)^{r+n} dx$$

$$\zeta_{r,s} = \sum_{y=0}^{\infty}\sum_{u=0}^{s} 4\zeta\theta \int_0^\infty x^r g(x;\chi)\,\varphi_q\gamma_f G(x;\chi)^{r+n} dx$$

$$\zeta_{r,s} = \sum_{y=0}^{\infty}\sum_{u=0}^{s} 4\zeta\theta\varphi_q\gamma_f \int_0^\infty x^r g(x;\chi)\, G(x;\chi)^{r+n} dx$$

$$\zeta_{r,s} = \sum_{y=0}^{\infty}\sum_{u=0}^{s} 4\zeta\theta\varphi_q\gamma_f\, \mu_\partial \qquad (12)$$

Where $\mu_\partial = \int_0^\infty x^r g(x;\chi) G(x;\chi)^{r+n} dx$

Moments (Ms)

The moments are required and vital in every statistical study, particularly in applications. As a result, we will derive the r^{th} moment for the new family.

$$\upsilon_r' = E(x^r) = \int_0^\infty x^r f(x)dx$$

The moment of TIHLTL -G distribution family can be achieved by adopting the expanded pdf in equation (10)

Therefore, the moment of the continuous distribution Family is

$$\upsilon_r' = \int_0^\infty x^r\,\zeta\theta g(x;\chi)\sum_{y=0}^\infty \varphi_q\,G(x;\chi)^r dx$$

$$\upsilon_r' = \sum_{y=0}^\infty 4\zeta\theta\varphi_q\,\vartheta_r' \quad (13)$$

Where

$$\vartheta_r' = \int_0^\infty x^r g(x;\chi)G(x;\chi)^r\,dx$$

Moment Generating Function (MGF)

The MGF of any random variable x is expressed as

$$M_x^t = \int_0^\infty e^{tx} f(x)dx$$

By employing the expanded pdf in (10), then the moment generating function of the TIHLTL-G distribution family is obtained as

$$M_x^t = \int_0^\infty e^{tx}\,4\zeta\theta g(x;\chi)\sum_{i=0}^\infty \varphi_q\,G(x;\chi)^r$$

$$M_x^t = \sum_{i=0}^\infty 4\zeta\theta\varphi_q\eta_r' \quad (14)$$

Where

$$\eta_r' = \int_0^\infty e^{tx}\,g(x;\chi)G(x;\chi)^r$$

Reliability/Survival Function (SF)

The reliability function, sometimes called the survivor function, is usually employed to determine the likelihood that a patient will survive for a longer time above a specific period of time.

$$R(x) = 1 - F(x)$$

$$R(x;,\zeta,\theta,\chi) = \frac{2\left[1-\left[1-\left(1-G(x;\chi)\right)^2\right]^\theta\right]^\zeta}{1+\left[1-\left[1-\left(1-G(x;\chi)\right)^2\right]^\theta\right]^\zeta} \quad (15)$$

Determining the survival probabilities for various time x gives an important overview from survival data.

Hazard Function (HF)

The popular function known as HF, also regarded as conditional failure rate, is defined as the probability of occurrence of an event of main interest in a relatively concise time frame. This is defined as:

$$Z(x;\zeta,\theta,\chi) = \frac{f(x;\zeta,\theta,\chi)}{R(x;\zeta,\theta,\chi)}$$

$$= \frac{\dfrac{4\zeta\theta g(x;\chi)[1-G(x;\chi)]\left[1-\left(1-G(x;\chi)\right)^2\right]^{\theta-1}[1-[1-\left(1-G(x;\chi)\right)^2]^\theta]^{\zeta-1}}{[1+[1-[1-(1-G(x;\chi))^2]^\theta]^\zeta]^2}}{\dfrac{2\left[1-\left[1-\left(1-G(x;\chi)\right)^2\right]^\theta\right]^\zeta}{1+\left[1-\left[1-\left(1-G(x;\chi)\right)^2\right]^\theta\right]^\zeta}}$$

$$Z(x;\zeta,\theta,\chi) = \frac{4\zeta\theta g(x;\chi)[1-G(x;\chi)]\left[1-\left(1-G(x;\chi)\right)^2\right]^{\theta-1}[1-[1-\left(1-G(x;\chi)\right)^2]^\theta]^{\zeta-1}}{2\left[1-\left[1-\left(1-G(x;\chi)\right)^2\right]^\theta\right]^\zeta[1+[1-[1-(1-G(x;\chi))^2]^\theta]^\zeta]} \quad (16)$$

Order Statistics (OS)

Order statistics has been an important aspect in which numerous areas of statistics, including Survival and life testing, have employed order statistics substantially. Assuming that $X_1, X_2, X_3, \dots, X_n$ is a random variable which is independent and identically distributed, corresponding to cdf $F(x)$. Assuming that $X_1, X_2, X_3, \dots, X_n$ is a random variable which is independent and identically distributed (IID) generated from the TIHLTL-G distribution family. Also assuming that $F_{rn}(x)$ and $f_{rn}(x)$, such that $r = 1, 2, \dots, n$ represent the cdf and pdf of the r^{th} OS X_{rn}. The X_{rn} pdf is expressed by

$$f_{r,n}(x) = \frac{1}{B(r,n-r+1)} F_x^{r-1}[1-F(x)]^{n-r} f(x) \quad (17)$$

replacing the quantity in equation (7) and equation (8) into equation (17), we have,

$$f_{r,n}(x;\zeta,\theta,\chi) = \frac{1}{B(r,n-r+1)}\left[\frac{1-\left[1-\left[1-(1-G(x;\chi))^2\right]^{\theta}\right]^{\zeta}}{1+\left[1-\left[1-(1-G(x;,\chi))^2\right]^{\theta}\right]^{\zeta}}\right]^{r-1} \times \left[\frac{2\left[1-\left[1-(1-G(x;\chi))^2\right]^{\theta}\right]^{\zeta}}{1+\left[1-\left[1-(1-G(x;\chi))^2\right]^{\theta}\right]^{\zeta}}\right]^{n-r}$$
$$\times \frac{4\lambda\theta g(x;\chi)[1-G(x;\chi)]\left[1-(1-G(x;\chi))^2\right]^{\theta-1}[1-[1-(1-G(x;\chi))^2]^{\theta}]^{\zeta-1}}{[1+[1-[1-(1-G(x;,\chi))^2]^{\theta}]^{\zeta}]^2} \quad (18)$$

The equation (18) is called the TIHLTL-G distribution family's r^{th} order statistics. Now, assuming we wish to obtain the pdf of the maximum and minimum, respectively. Then, by setting $r = n$ and $r = 1$, the maximum and minimum order statistics will be respectively obtained as given below

$$f_{n,n}(x;\zeta,\theta,\chi) = \frac{4n\zeta\theta g(x;\chi)[1-G(x;\chi)]\left[1-(1-G(x;\chi))^2\right]^{\theta-1}[1-[1-(1-G(x;\chi))^2]^{\theta}]^{\zeta-1}}{[1+[1-[1-(1-G(x;\chi))^2]^{\theta}]^{\zeta}]^2}$$
$$\times \left[\frac{1-\left[1-\left[1-(1-G(x;\chi))^2\right]^{\theta}\right]^{\zeta}}{1+\left[1-\left[1-(1-G(x;\chi))^2\right]^{\theta}\right]^{\zeta}}\right]^{n-1} \quad (19)$$

$$f_{1,n}(x;\zeta,\theta,\chi) = \frac{4n\zeta\theta g(x;\chi)[1-G(x;\chi)]\left[1-(1-G(x;\chi))^2\right]^{\theta-1}[1-[1-(1-G(x;\chi))^2]^{\theta}]^{\zeta-1}}{[1+[1-[1-(1-G(x;\chi))^2]^{\theta}]^{\zeta}]^2}$$
$$\times \left[\frac{2\left[1-\left[1-(1-G(x;\chi))^2\right]^{\theta}\right]^{\zeta}}{1+\left[1-\left[1-(1-G(x;\chi))^2\right]^{\theta}\right]^{\zeta}}\right]^{n-1} \quad (20)$$

Quantile Function (QF)

To obtain the quantile function by inverting the TIHLTL-G distribution Family cdf.

$$F(x) = v$$

$$F(x;\zeta,\theta,\chi) = \frac{1-\left[1-\left\{1-(1-G(x;\chi))^2\right\}^{\theta}\right]^{\zeta}}{1+\left[1-\left\{1-(1-G(x;\chi))^2\right\}^{\theta}\right]^{\zeta}} = v$$

Can be rewritten as

$$1-\left[1-\left\{1-(1-G(x;\chi))^2\right\}^{\theta}\right]^{\zeta} = v + v\left[1-\left\{1-(1-G(x;\chi))^2\right\}^{\theta}\right]^{\zeta}$$

now reduced to

$$\left[1-\left\{1-(1-G(x;\chi))^2\right\}^{\theta}\right]^{\zeta} = \frac{1-v}{v+1}$$

$$Q(x) = G^{-1}\left[1-\left(1-\left(1-\left(\frac{1-v}{v+1}\right)^{1/\zeta}\right)^{1/\theta}\right)^{1/2}\right] \quad (21)$$

METHOD OF PARAMETER ESTIMATION

Two methods will be employed to estimate the parameters of the TIHLTL-G distribution family; thus, Maximum Likelihood and Maximum product of space are considered.

Maximum Likelihood Estimation (MLE)

Assuming $x_1, x_2, x_3 \dots, x_m$ randomly sampled from the tihltl-g distribution family with pdf $f(x;\zeta,\theta,\chi) = \frac{4\zeta\theta g(x;\zeta,\theta,\chi)[1-G(x;\chi)]\left[1-(1-G(x;\chi))^2\right]^{\theta-1}[1-[1-(1-G(x;\chi))^2]^{\theta}]^{\zeta-1}}{[1+[1-[1-(1-G(x;\chi))^2]^{\theta}]^{\zeta}]^2}$

Then, the loglikelihood becomes

$$log(L) = nlog4 + nlog\zeta + +nlog\,\theta + \sum_{i=0}^{n} logg(x;\chi) + \sum_{i=0}^{n} log(1-G(x;\chi)) + (\theta-1)\sum_{i=0}^{n} log\left(1-(1-G(x;\chi))^2\right) + (\zeta-1)\sum_{i=0}^{n} log\left[1-\left(1-(1-G(x;\chi))^2\right)^{\theta}\right] - 2\sum_{i=0}^{n} log\left[1+\left[1-\left(1-(1-G(x;\chi))^2\right)^{\theta}\right]^{\zeta}\right] \quad (22)$$

$$\frac{\delta logL}{\delta\zeta} = \frac{n}{\zeta} + \sum_{i=0}^{n} log\left[1-\left(1-(1-G(x;\chi))^2\right)^{\theta}\right] - 2\sum_{i=1}^{n} log\left[1+\left[1-\left[1-(1-G(x;\chi))^2\right]^{\theta}\right]^{\zeta}\right] \times \frac{log\left[1+\left[1-\left[1-(1-G(x;\chi))^2\right]^{\theta}\right]^{\zeta}\right]}{1+\left[1-\left[1-(1-G(x;\chi))^2\right]^{\theta}\right]^{\zeta}}$$

$$\frac{\delta logL}{\delta\theta} = \frac{n}{\theta} + \sum_{i=1}^{n} log\left[1-(1-G(x;\chi))^2\right](\zeta+1)\sum_{i=1}^{n}\left[1-(1-G(x;\chi))^2\right]^{\theta} \times \frac{log\left[1-\left[1-(1-G(x;\chi))^2\right]^{\theta}\right]}{1-\left[1-(1-G(x;\chi))^2\right]^{\theta}}$$

$$\frac{\delta logL}{\delta\chi} = \sum_{i=1}^{n}\frac{g(x;\chi)^{\chi}}{g(x;\chi)} + \sum_{i=1}^{n}\frac{G(x;\chi)^{\chi}}{(1-(x;\chi))} + 2(\theta-1)\sum_{i=1}^{n}\frac{(1-G(x;\chi))G(x;\chi)^{\chi}}{1-(1-G(x;\chi))^2}$$
$$+\frac{2\theta(\zeta-1)\sum_{i=1}^{n}[1-[1-G(x;\chi)]^2]^{\theta-1}[1-G(x;\chi)]G(x;\chi)^{\chi}}{1-\left[1-(1-G(x;\chi))^2\right]^{\theta}}$$
$$-2\sum_{i=0}^{n}\frac{2\zeta\theta\left[1-\left[1-(G(x;\chi))^2\right]^{\theta}\right]^{\zeta-1}\left[(1-G(x;\chi))^2\right]^{\theta-1}(1-G(x;\chi))G(x;\chi)^{\chi}}{1+\left[1-\left[1-(1-G(x;\chi))^2\right]^{\theta}\right]^{\zeta}} \quad (23)$$

Maximum Product of Spacing (MPS)

Assuming $x_1, x_2, x_3 \dots, x_m$ is randomly sampled from the TIHLTL-G distribution family with cdf $R(x; \zeta, \theta, \chi)$, and also $x_{(1)}, x_{(2)}, x_{(3)} \dots, x_{(m)}$ correspond to the order sample. The spacing

$$A_p = R(x_{(p)}) - R(x_{(p-1)}) \quad \text{for } p = 1,2,3,\dots,m+1.$$

Where $R(x_{(0)}) = 0$ and $R(x_{(m+1)}) = 1$

Therefore $R(x_{(p)}; \zeta, \theta, \chi) = \dfrac{1-\left[1-\left\{1-\left(1-R(x_{(p)}); \chi\right)^2\right\}^\theta\right]^\zeta}{1+\left[1-\left\{1-\left(1-R(x_{(p)}); \chi\right)^2\right\}^\theta\right]^\zeta}$

and $R(x_{(p-1)}; \zeta, \theta, \chi) = \dfrac{1-\left[1-\left\{1-\left(1-R(x_{(p-1)}); \chi\right)^2\right\}^\theta\right]^\zeta}{1+\left[1-\left\{1-\left(1-R(x_{(p-1)}); \chi\right)^2\right\}^\theta\right]^\zeta}$

then $A_p = R(x_{(p)}) - R(x_{(p-1)}) = \left(\dfrac{1-\left[1-\left\{1-\left(1-R(x_{(p)}); \chi\right)^2\right\}^\theta\right]^\zeta}{1+\left[1-\left\{1-\left(1-R(x_{(p)}); \chi\right)^2\right\}^\theta\right]^\zeta}\right) - \left(\dfrac{1-\left[1-\left\{1-\left(1-R(x_{(p-1)}); \chi\right)^2\right\}^\theta\right]^\zeta}{1+\left[1-\left\{1-\left(1-R(x_{(p-1)}); \chi\right)^2\right\}^\theta\right]^\zeta}\right)$

and by maximizing

$$D(x; \zeta, \theta, \chi) = \frac{1}{m+1}\sum_{p=1}^{m+1} \log A_p \quad (24)$$

the parameter estimates can be obtained by differentiating D with regard to those parameters involved then solve the non-linear equations, which will produce parameter estimates of $\hat{\lambda}_{mps}$, $\hat{\theta}_{mps}$ and $\hat{\chi}_{mps}$

$$\frac{\partial D(x; \zeta, \theta, \chi)}{\partial \zeta} = \frac{1}{m+1}\sum_{p=1}^{m+1}\frac{1}{A_p}\left[L_1(x_{(p)}; \chi) - L_2(x_{(p-1)}; \chi)\right] \quad (25)$$

$$\frac{\partial D(x; \zeta, \theta, \chi)}{\partial \theta} = \frac{1}{m+1}\sum_{p=1}^{m+1}\frac{1}{A_p}\left[M_1(x_{(p)}; \chi) - M_2(x_{(p-1)}; \chi)\right] \quad (26)$$

$$\frac{\partial D(x; \zeta, \theta, \chi)}{\partial \chi} = \frac{1}{m+1}\sum_{p=1}^{m+1}\frac{1}{A_p}\left[N_1(x_{(p)}; \chi) - N_2(x_{(p-1)}; \chi)\right] \quad (27)$$

Where $L_1(x_{(p)}; \chi), L_2(x_{(p)-1}; \chi), M_1(x_{(i)}; \chi), M_2(x_{(i)-1}; \chi), N_1(x_{(i)}; \chi)$ *and* $N_2(x_{(i-1)}; \chi)$ are the outcome of the differentiatial expression. Setting equations (26), (26), and (27) to zero and solve the equation numerically will produce the MPS of the parameters ζ, θ and χ

Sub-model

we provide exponential distribution as a sub-model to the TIHLTL- G distribution family.

THE TYPE I HALF LOGISTIC TOPP LEONE EXPONENTIAL DISTRIBUTION (TIHLTLExp)

Considering the exponential distribution as the parent distribution with cdf and pdf with parameter β is defined as

$F(x, \beta) = 1 - e^{-\beta x}$ and $f(x, \beta) = \beta e^{-\beta x}$ respectively. Then the cdf and pdf of the new Type I Half Logistic Topp Leone exponential distribution with parameters ζ, θ and β are obtained and expressed as

$$F(x; \zeta, \theta, \beta) = \frac{1-\left[1-\left[1-\left(1-\left(1-e^{-\beta x}\right)\right)^2\right]^\theta\right]^\zeta}{1+\left[1-\left[1-\left(1-\left(1-e^{-\beta x}\right)\right)^2\right]^\theta\right]^\zeta} \quad (28)$$

and

$$f(x; \zeta, \theta, \beta) = \frac{4\zeta\theta\beta e^{-\beta x}\left(1-\left(1-e^{-\beta x}\right)\right)\left[1-\left(1-\left(1-e^{-\beta x}\right)\right)^2\right]^{\theta-1}\left[1-\left[1-\left(1-\left(1-e^{-\beta x}\right)\right)^2\right]^\theta\right]^{\zeta-1}}{\left[1+\left[1-\left[1-\left(1-\left(1-e^{-\beta x}\right)\right)^2\right]^\theta\right]^\zeta\right]^2} \quad (29)$$

$$x > 0, \qquad \zeta, \theta, \beta > 0$$

respectively, where ζ, θ are the shape parameters and the β being the scale parameter

APPLICATIONS

This section reveals the result obtained from pdf plots, Hazard function plots, simulation studies, Estimation of Parameters and analysis of the models using two distinct data sets.

Data set I

These data consist of 100 annual maximum precipitation (inches) for one rain gauge in Fort Collins, Colorado, from 1900 through 1999: Katz, (2002).

239, 232, 434, 85, 302, 174, 170, 121, 193, 168, 148, 116, 132, 132, 144, 183, 223, 96, 298, 97, 116, 146, 84, 230, 138, 170, 117, 115, 132, 125, 156, 124, 189, 193, 71, 176, 105, 93, 354, 60, 151, 160, 219, 142, 117, 87, 223, 215, 108, 354, 213, 306, 169, 184, 71, 98, 96, 218, 176, 121, 161, 321, 102, 269, 98, 271, 95, 212, 151, 136, 240, 162, 71, 110, 285, 215, 103, 443, 185, 199, 115, 134, 297, 187, 203, 146, 94, 129, 162, 112, 348, 95, 249, 103, 181, 152, 135, 463, 183, 241

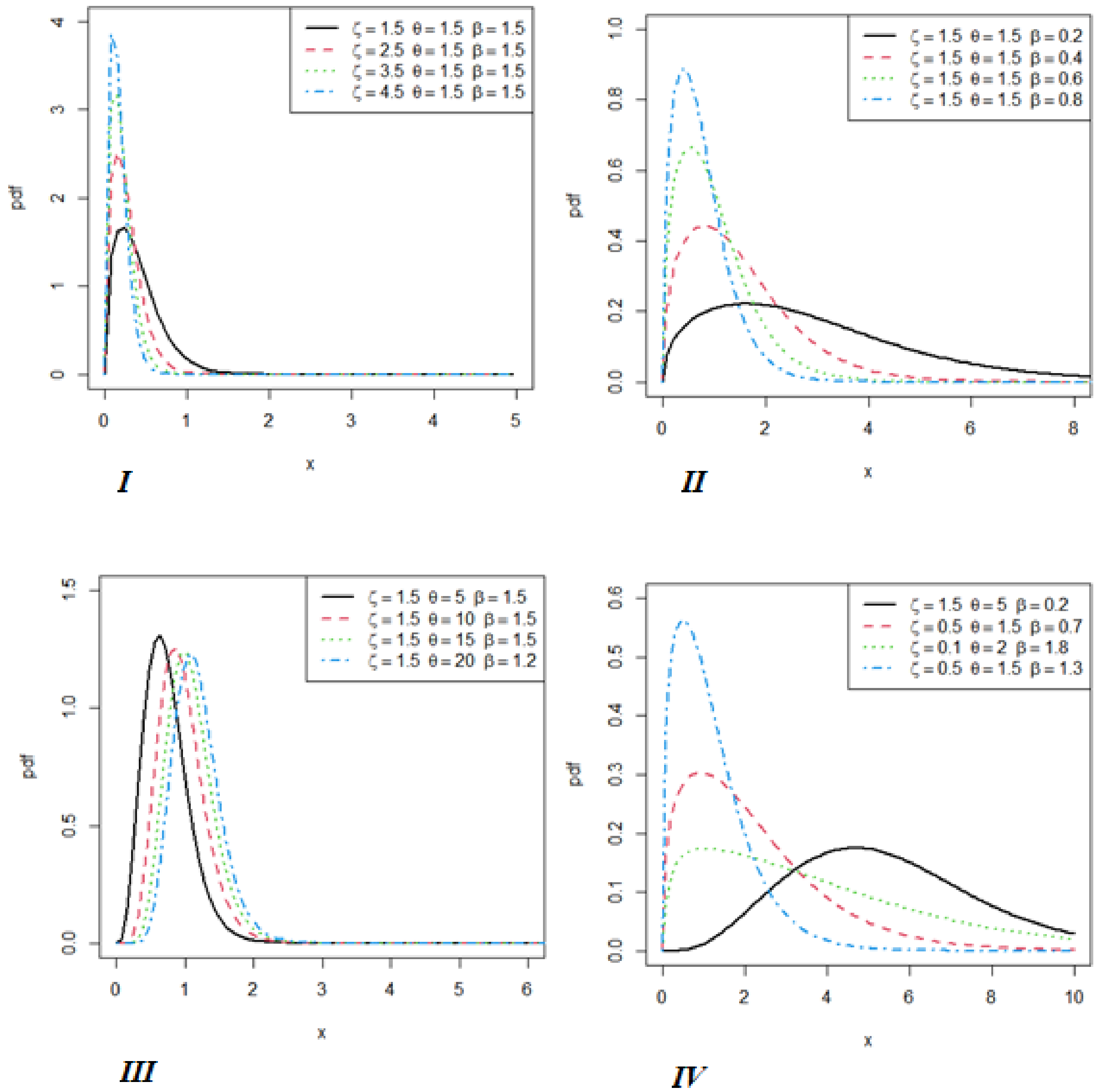
Data set II

The maximum stress per cycle

The data set II consists of 101 observations with a maximum stress per cycle 31,000 psi (Birnbaum and Saunders, 1969), studied by Sangsanit and Bodhisuwan (2016)

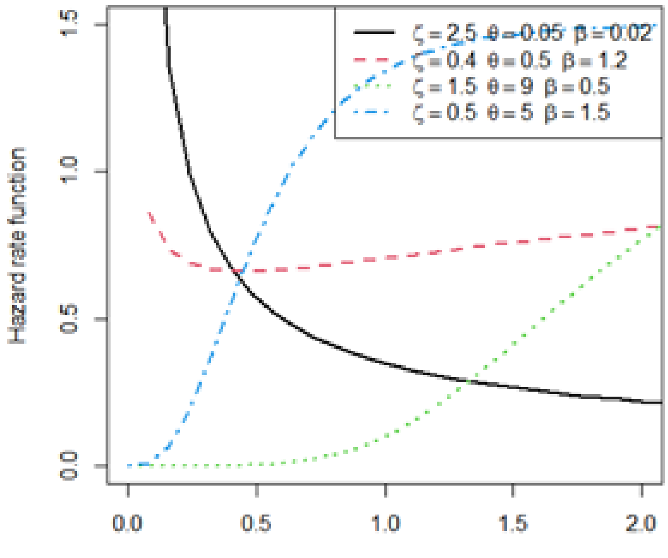
70 90 96 97 99 100 103 104 104 105 107 108 108 108 109 109 112 112 113 114 114 114 116 119 120 120 120 121 121 123 124 124 124 124 124 128 128 129 129 130 130 130 131 131 131 131 131 132 132 132 133 134 134 134 134 136 136 137 138 138 138 139 139 141 141 142 142 142 142 142 142 144 144 145 146 148 148 149 151 151 152 155 156 157 157 157 157 158 159 162 163 163 164 166 166 168 170 174 201 212

The pdf plot of the TIHLTLExp distribution



The Figure: I, II, III, IV above depict the pdf shape of the TIHLTLExp distribution with distinct values of parameters.

Hazard rate function of the TIHLTLExp distribution



V

The Figure: V, above depict different hazard shapes of the TIHLTLExp distribution.

Table I: Simulation Study

(Two Method of Parameter estimation was used; Maximum Likelihood (ML) and Maximum Products of Spacing (MPS) methods**)**

Sample size (n)	measures	MLE			MPS		
		0.9	1.5	1	0.9	1.5	1
10	Estimates	0.7587	1.5636	1.5811	0.9257	1.6460	1.0387
	Biases	-0.1413	0.0636	0.5811	0.0257	0.1460	0.0387
	RMSE	0.3945	0.8464	1.3138	0.3233	0.4557	0.7865
50	Estimates	0.8276	1.6766	1.0024	0.9254	1.6552	0.9619
	Biases	-0.0724	0.1766	0.0024	0.0254	0.1552	-0.0381
	RMSE	0.1763	0.2676	0.3978	0.1599	0.2364	0.3504
150	Estimates	0.8357	1.7015	0.9212	0.9113	1.6442	0.9692
	Biases	-0.0643	0.2015	-0.0788	0.0113	0.1442	-0.0308
	RMSE	0.1066	0.2171	0.1575	0.0895	0.1579	0.1435
250	Estimates	0.8373	1.7059	0.9104	0.9071	1.6403	0.9780
	Biases	-0.0627	0.2059	-0.0896	0.0071	0.1403	-0.0220
	RMSE	0.0906	0.2120	0.1333	0.0697	0.1487	0.1073
500	Estimates	0.8406	1.7085	0.9000	0.9056	1.6331	0.9838
	Biases	-0.0594	0.2085	-0.1000	0.0056	0.1331	-0.0162
	RMSE	0.0739	0.2120	0.1187	0.0475	0.1387	0.0719
1000	Estimates	0.8429	1.7066	0.8958	0.9050	1.6304	0.9891
	Biases	-0.057	0.2066	-0.1042	0.0050	0.1304	-0.0109
	RMSE	0.0656	0.2088	0.1138	0.0352	0.1348	0.0520

Table II: Analysis of Data I (MLE and MPS Estimations, loglikehood, AIC and BIC of TIHLTLExp and competing Distributions)

Distribution	MPS					MLE				
	Estimate		Loglikehood	AIC	BIC	Estimate		Loglikehood	AIC	BIC
TIHLTLExp	ζ θ β	0.2839 11.8016 0.0251	-572.4783	1150.957	1158.772	ζ θ β	0.5539 11.6347 0.0137	-567.1815	1140.36	1148.17
TIHLEtExp	ζ θ β	0.2844 61.2768 0.0502	620.041	1246.082	1253.898	ζ θ β	0.1035 60.3289 0.0762	-610.2366	1226.473	1234.289
KExp	ζ θ β	0.0364 34.890 0.3510	-597.5134	1204.027	1208.842	ζ θ β	0.0354 34.994 0.3490	-574.7132	1158.426	1163.242
TIHLExp	ζ β	1.3395 0.0062	-603.5467	1211.093	1216.304	ζ β	1.3226 0.0064	-603.5355	1211.071	1216.281
LOExp	ζ β	3.8378 0.0063	601.5431	1207.086	1212.297	ζ β	3.9654 0.0063	-567.5988	1159.198	1164.408
TOLExp	θ β	0.0077 7.9725	-597.5134	1201.027	1208.842	θ β	0.0081 8.9523	-576.4513	1158.903	1166.718
EXExp	ζ β	8.0009 0.0155	-591.5859	1187.172	1192.382	ζ β	8.9521 0.0162	-572.4513	1148.903	1154.113

Table III: Analysis for Data II (MLE and MPS Estimations, loglikehood, AIC and BIC of TIHLTLExp and competing distributions)

Distribution	MPS					MLE				
	Estimate		Loglikehood	AIC	BIC	Estimate		Loglikehood	AIC	BIC
TIHLTLExp	ζ	0.2628	-452.4487	910.8974	918.7428	ζ	0.0654	-452.2216	910.4432	918.2886
	θ	0.3298				θ	0.7331			
	β	0.0122				β	0.0848			
TIHLEtExp	ζ	0.0624	-585.9302	1177.86	1185.706	ζ	0.0615	-580.4983	1166.997	1174.842
	θ	0.0331				θ	0.1049			
	β	0.1583				β	0.1608			
KExp	ζ	0.1661	-593.3931	1180.786	1172.941	ζ	0.1675	-592.0975	1190.195	1198.04
	θ	0.3749				θ	0.4999			
	β	0.0427				β	0.0431			
TIHLExp	ζ	0.0572	-570.9346	1145.862	1151.092	ζ	0.0564	-570.9309	1145.862	1151.092
	β	0.1990				β	0.2030			
LOExp	ζ	0.1516	-782.109	1568.218	1573.448	ζ	0.2150	-780.539	1565.078	1570.308
	β	0.0075				β	0.0075			
TOLExp	θ	0.0218	-458.8607	921.7214	926.9516	θ	0.0228	-458.6398	921.2796	926.5098
	β	213.9767				β	275.0160			
EXExp	ζ	214.5643	-458.8594	921.7188	926.949	ζ	276.9756	-458.6402	921.2804	926.5106
	β	0.0438				β	0.0458			

DISCUSSION OF RESULT

The Figure. I to IV reveals the flexibility and adaptability of the TIHLTLExp distribution, considering different parameter values, by investigating the shape of the new model, it reveals the fitness of the model ability to handle different data with different shape. The hazard rate function in Figure V also shows different shapes such as monotonic increasing, monotonic increasing and bathub shape for different parameter value. The two methods of parameter estimation (MLE and MPS) employed shows the consistency of the two methods even though the MPS reveals minimum error compared to MLE. The novel model TIHLTLExp distribution performs better while comparing with its counterparts using the two data sets.

CONCLUSION

The novel family of distribution tagged as the Type I Half Logistic Topp Leone-G distribution family (TIHLTL-G distribution family was proposed and discussed. Explicit expressions, quantile function, Moments, weighted moment, Moment generating function, Survival function, Hazard function, as well as the order statistics are among the statistical properties discussed. The novel family's sub-model (exponential) was studied. Maximum likelihood and maximum product of spacing approach were used to estimate the novel model parameters. Pdf , hazard function were graphed and their behaviors were studied via different parameters values. Some well known existing flexible models were examined and compared with the proposed model using two distinct data sets to investigate the competency, flexibility and adaptability of the novel model. The results of the studies indicate that the novel model is superb to the competing models. Hence, it appears to be more flexible in modelling different data.

FUTURE RESEARCH

This new distribution family is open for modification/extension such as adding more parameter(s) to form a new family(s) or distribution(s). Also, a regression model or control chart can be designed and applied to the new distribution such as the case of survival analysis and process monitoring.

DATA AVAILABILITY

All data used are included in the paper.

CONFLICTS OF INTEREST

The authors declared no conflicts of interest.

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