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ORIGINAL RESEARCH ARTICLE

An Order Quantity Model for Delayed Deteriorating Items with Time-Varying Demand Rate, Linear Holding Cost, Complete Backlogging Rate and Two-Level Pricing under Trade Credit Policy.

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ABSTRACT
In some classical inventory models for non-instantaneous deteriorating items, it is tacitly assumed that the selling price before and after deterioration sets in is the same. However, in real practice, when deterioration sets in, the retailer may decide to reduce the selling price to encourage more sales, reduce the cost of holding stock, attract new customers and reduce losses due to deterioration. This research developed an economic order quantity model for non-instantaneous deteriorating items with two-phase demand rates, linear holding cost, complete backlogging rate and two-level pricing strategies under trade credit policy. It is assumed that the holding cost is linear time-dependent, the unit selling price before deterioration sets in is greater than that after deterioration sets and the demand rate before deterioration sets in is considered as continuous time-dependent quadratic, after which it is considered as constant up to when the inventory is completely exhausted. Shortages are allowed and completely backlogged. The proposed model determines the optimal time with positive inventory, cycle length and order quantity such that the total profit of the inventory system has a maximum value. The necessary and sufficient conditions for the existence and uniqueness of optimal solutions have been established. Numerical experiments have been conducted to illustrate the theoretical result of the model. Sensitivity analysis of some model parameters on the decision variables has been carried out, and suggestions towards maximising the total profit were also given.

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Two-phase demand rates, Linear Holding, two-level pricing, complete backlogging rate, Trade Credit Policy.

INTRODUCTION

"Inventory" encompasses the materials used in production and the finished products available for sale. Efficiently managing the procurement, storage, and accessibility of goods is known as inventory control. The primary goal is maintaining an appropriate stock level that prevents shortages and excesses. Poor inventory control can lead to either inadequate supplies or surplus products. Inadequate inventory management can result in missed deliveries, lost sales, dissatisfied customers, and production bottlenecks.

On the other hand, overstocking ties up funds that could be used more productively and occupy valuable storage space. Harris introduced the Economic Order Quantity (EOQ) model in 1913 to address the balance between understocking and overstocking. This model aimed to determine the optimal order quantity that minimises variable costs, assuming a constant demand rate. However, many subsequent researchers, such as Giri *et al.* (2000) and Kar *et al.* (2001), adapted the EOQ model for scenarios with time-dependent linear demand rates. These adaptations considered changing demand rates over time, a more realistic scenario for various products compared to the steady rise or fall in demand rate assumed by the original model. Similar to this, some researchers such as Ahmed and Musa (2016) and Malumfashi *et al.* (2022) and others modify the assumption of the traditional EOQ model in the case of a time-dependent exponential demand rate, which is also hardly observed to occur for any product because the demand rate of a significant number of products may not change at the higher rate of change as exponential. An accelerated/retarded rise or decrease in demand rates, which is best represented by a quadratic function of time, would be the best substitute for linear or exponential demand rates. An accelerated or retarded rise or decrease in demand rates, which is best represented by a quadratic function of time, would be a preferable option to linear or exponential demand rates. New things like technology, trendy items, and so forth typically have an accelerated surge in demand. While

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accelerated rise and fall in demand rates typically happen with seasonal products, whose demand rises quickly to a peak in the middle of the season and then falls quickly as the season ends, accelerated fall in demand rates typically happen with outmoded automobiles, clothes, computers, and other items. Khanra and Chaudhuri (2003) were the first to study an inventory model with a time-dependent quadratic demand rate. They assumed that only a fixed percentage of the available inventory degrades over both infinite and finite time horizons, and they provided a detailed justification for choosing a time-dependent quadratic demand rate over linear or exponential demand rates. Later, researchers like Uthayakumar and Karuppasamy (2017), Priya and Senbagam (2018), Rahman and Uddin (2021) and so on developed inventory models with time-dependent quadratic demand rates.

The traditional EOQ model (created by Harris, 1913) assumed that goods had an indefinite lifespan and that inventory was depleted only due to constant demand rates. However, occasionally, inventory products are depleted due to deterioration. Therefore, the impact of deterioration on inventory items cannot be disregarded. The first inventory model for items that deteriorate at the end of the allotted storage term was studied by Whitin in 1957. For products that deteriorate exponentially at constant rates, Ghare and Schrader (1963) presented an improved version of the EOQ model, in which the consumption rate of deteriorating items is based on a negative exponential function of time. After that, Covert and Philip (1974) developed an EOQ model for instantaneously deteriorating items using a modified version of Ghare and Schrader's (1963) model, where the rate of deterioration is distributed according to a two-parameter Weibull distribution. Philip (1974) modified Covert and Philip's (1974) model by developing an inventory model for instantaneously deteriorating items, where the rate of deterioration is distributed according to a three-parameter Weibull distribution and shortages are prohibited. Additionally, Baraya and Sani (2016), Mandal and Venkataraman (2019), Jaggi et al. (2019), and other studies on inventory models assume that deterioration begins as soon as items are received.

A widely accepted belief in academic circles is that once inventory items are received, they immediately start to deteriorate. However, this notion is not universally applicable, particularly to products like perishables such as vegetables, fruits, fish, and meat. These items often have a period during which their quality remains intact, and deterioration is not yet evident. Assuming that deterioration begins immediately upon stocking can lead retailers to adopt unsuitable restocking strategies, exaggerating the estimation of overall inventory costs. Addressing this issue, Wu *et al.* (2006) developed an effective method for replenishing items that deteriorate gradually over time. Their approach factors in the influence of stock levels on-demand rates, managing shortages through variable backlogging rates based on time until the next replenishment. Geetha and Udayakumar (2016) similarly tackled this challenge by designing an optimal strategy for determining batch sizes of deteriorating items. They considered factors like pricing and advertising that affect demand rates and accommodated shortages with partial backlogging. In another study, Babangida and Baraya (2018) introduced an inventory model for items undergoing gradual deterioration, characterised by quadratic demand rates. Their model incorporated trade credit policies and accounted for time-dependent demand rates before deterioration. Following deterioration, a constant demand rate was assumed until inventory depletion. Furthermore, researchers including Bello and Baraya (2018), Bello and Baraya (2019), Babangida and Baraya (2021b), Mustapha and Majid (2023), and others have extensively explored inventory models for items undergoing non-instantaneous deterioration. These studies encompass diverse assumptions and scenarios, collectively enhancing the comprehension of efficient management strategies for items with gradual deterioration under various conditions.

It is typically believed in some traditional EOQ models that the retailer must pay for goods as soon as they are received from the supplier or manufacturer. However, such an assumption could not be valid in today's highly competitive market environment. Harley and Higgins initially presented the concept of trade credit in the inventory literature in 1973. Tripathy et al. (2022) designed an inventory model for non-instantaneous deteriorating items with steady demand under progressive financial trade credit facilities. In addition, Jaggi et al. (2015), Shaikh et al. (2018), Babangida and Baraya (2020), and others have published relevant studies on inventory models under trade credit policy with a variety of assumptions.

Many inventory models assume constant holding costs, but in reality, these costs often change due to shifts in the time value of money and price indices. Most items held in stock have holding costs that increase linearly with the length of time they are stored. Researchers have addressed this issue in various ways. Selvaraju and Ghuru (2018) developed Economic Order Quantity (EOQ) models for items that deteriorate instantly, considering constant, linear, and quadratic holding costs. Babangida and Baraya (2019a) extended the EOQ model to non-instantaneously deteriorating items, incorporating two demand components and linear time-dependent holding costs under a trade credit policy. They found that models with time-varying holding costs tend to have higher total variable costs compared to those with constant holding costs. Malumfashi *et al.* (2021) devised an EOQ-like model, known as the Economic Production Quantity (EPQ), for delayed deteriorating items. This model considers a two-phase production period, a variable demand rate, and linearly increasing time-dependent holding costs. It has four stages: two production phases with different rates but the same demand rate, a period after inventory build-up with a quadratic time demand rate, and a deterioration period with demand dependent on stock levels, with no backorders allowed. Deo *et al.* (2022) developed an inventory model that accounts for selling prices, time-dependent demand, variable holding costs, and two storage facilities. Babangida and Baraya (2022) expanded the EOQ model for non-instantaneously deteriorating items, considering two-phase demand rates, time-dependent linear holding costs, and addressing shortages with partial backlogging under a trade credit policy. Their model aims to simultaneously determine the optimal time for positive inventory, cycle length, and economic order quantity to minimise total variable costs.

In the classical inventory model, shortages are not allowed. However, sometimes, customers' demands cannot be fulfilled by the supplier from the current stocks; this situation is known as stock out or shortage condition. Roy (2008) developed an EOQ model for instantaneous deteriorating items with a price-dependent demand rate, where deterioration rate and holding cost are considered linearly increasing time functions. Shortages are allowed and completely backlogged. Choudhury et al. (2013) developed an inventory model for non-instantaneous deteriorating items with stock-dependent demand rates, time-varying holding costs, and completely backlogged shortages. Babangida and Baraya (2019b) developed an inventory model for non-instantaneous deteriorating items with two-phase demand and shortages under the trade credit policy. Shortages are allowed and completely backlogged. The optimal time with positive inventory, cycle length and order quantity are determined such that the total variable cost has a minimum value.

Most inventory models for non-instantaneous deteriorating items assume that the unit selling price before and after deterioration sets in is the same. However, in real practice, the unit selling price before and after deterioration sets in differs and this assumption needs to be considered in developing inventory policies for non-instantaneous deteriorating items, where the objective function is to maximise the total profit of the inventory system. Pang *et al.* (2022) developed an inventory model for perishable items with two-stage pricing. Babangida and Baraya (2021a) developed an EOQ model for non-instantaneous deteriorating items with two-phase demand rates and two-level pricing strategies under trade credit policy.

This study focuses on an Economic Order Quantity (EOQ) model designed for items with non-instantaneous deterioration. The model considers two-phase demand rates, a linear holding cost, a complete backlogging rate, and two-level pricing strategies, all operating under a trade credit policy. The research establishes the necessary and sufficient conditions for optimal solutions. The aim is to determine the optimal time for maintaining positive inventory, the cycle length, and the order quantity, collectively maximising the total profit per unit of time. The paper presents several numerical experiments to demonstrate how the theoretical model works in practice. Additionally, sensitivity analysis is conducted on specific parameters within the proposed models to understand the impact of parameter changes on decision variables. Recommendations for maximising total profit in light of these effects are also provided in this work.

NOTATIONS AND ASSUMPTIONS

Notation:

The inventory system is developed using the following notations.

A The fixed ordering cost per order

C The purchasing cost per unit time

S_1 Unit selling price during the interval $[0,\ t_d]$

S_2 Unit selling price during the interval $[t_d, T]$, where $S_1 > S_2 > C$

C_b Shortage cost per unit time

I_C The interest charged in stock by the supplier.

I_e The interest earned

M The trade credit period (in year for settling account)

θ The constant deterioration rate function

t_d The length of time in which the product exhibit more deterioration

t_1 Length of time in which the inventory has no shortage

T The length of replenishment cycle time

Q_m The maximum inventory level

B_m The backorder level during the shortage period

Q The order quantity during the cycle length i.e.$Q = Q_m + B_m$

Assumptions

In addition to assumptions 8 and 9, which are not taken into consideration in Babangida and Baraya (2021), this model develops under the following assumptions, which have been adapted from the aforementioned research.

1. The replenishment rate is infinite, i.e., the replenishment rate is instantaneous, and the lead time is zero.
2. During the fixed period,t_d, there is no deterioration and at the end of this period, the inventory item deteriorates at the rate θ.
3. There is no replacement or repair for deteriorating items.
4. The demand rate before deterioration begins is assumed to be continuous time-dependent quadratic and is given by $a + bt + t^2$, where $a \geq 0, b \neq 0, c \neq 0 c \neq 0$. Here a is the initial demand rate, b

is the rate at which the demand rate changes and c is the accelerated change in the demand rate.

5. The demand rate after deterioration sets in is assumed to be constant and is given by d,$d > 0$.
6. During the trade credit period $M(0 < M < 1)$, the e account is not settled; generated sales revenue is deposited in an interest-bearing account. At the end of the period, the retailer pays off all units bought and starts to pay the capital opportunity cost for the items in stock. No interest is earned after the trade credit period.
7. The unit selling price is not the same as the unit purchasing cost. It is assumed that the unit selling price before deterioration sets in is greater than that after deterioration sets n $(S_1 > S_2 > C)$.
8. Shortages are allowed and completely backlogged.
9. Holding cost $C_1(t)$per unit time is linear time-pendent and is assumed to be $C_1(t) = h_1 + h_2 t$; where $h_1 > 0$ and $h_2 > 0$.

FORMULATION OF THE MODEL

Q_m Units of items are ordered at the beginning of the cycle (i.e., at time $t = 0$). During the interval $[0, t_d]$, the inventory level is depleting gradually due to market demand only and the demand rate is assumed to be time-dependent quadratic. At time interval $[t_d, t_1]$, the inventory level is depleting due to the combined effects of customer demand and deterioration, and the demand rate reduces to d. At time $t = t_1$, the inventory level depletes to zero. Shortages occur at the time interval $[t_1, T]$ and are completely backlogged. The behaviour of the inventory system is described in Figure 1 below.

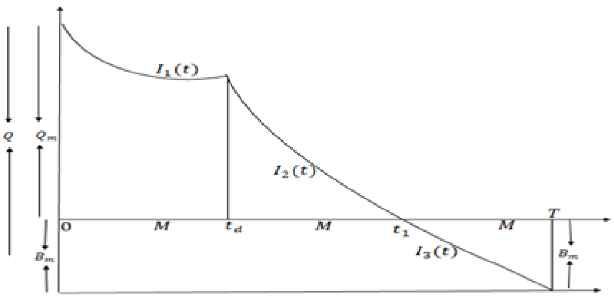

Fig. 1: Graphical representation of the inventory system with a complete backlogging rate

During the time interval $[0, T]$, the change of inventory at any time t is represented by the following differential equations,

$$\frac{dI_1(t)}{dt} = -(a + bt + ct^2), \qquad 0 \le t \le t_d \qquad (1)$$

with boundary conditions $I_1(0) = Q_m$ and $I_1(t_d) = Q_d$.

$$\frac{dI_2(t)}{dt} + \theta I_2(t) = -d, \qquad t_d \le t \le t_1 \qquad (2)$$

with boundary conditions $I_2(t_1) = 0$ and $I_2(t_d) = Q_d$.

$$\frac{dI_3(t)}{dt} = -d, \qquad t_1 \le t \le T \qquad (3)$$

with condition $I_3(t_1) = 0$ at $t = t_1$.

The solution of equations (1), (2) and (3) are respectively given by

$$I_1(t) = \frac{d}{\theta}\left(e^{\theta(t_1 - t_d)} - 1\right) + a(t_d - t) + \frac{b}{2}(t_d^2 - t^2) + \frac{c}{3}(t_d^3 - t^3) \quad 0 \le t \le t_d \qquad (4)$$

$$I_2(t) = \frac{d}{\theta}\left(e^{\theta(t_1 - t)} - 1\right), \qquad t_d \le t \le t_1 \qquad (5)$$

$$I_3(t) = d(t_1 - t) \qquad (6)$$

From Fig.1, using the condition $I_1(0) = Q_m$ In equation (4), the maximum stock level is given by

$$Q_m = \frac{d}{\theta}\left(e^{\theta(t_1 - t_d)} - 1\right) + \left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3}\right) \qquad (7)$$

Similarly, the value of Q_d can be derived at $t = t_d$, then it follows from equation (5) that

$$Q_d = \frac{d}{\theta}\left(e^{\theta(t_1 - t_d)} - 1\right) \qquad (8)$$

The maximum back-ordered inventory B_m It is obtained at $t = T$, and then from equation (6), it follows that.

$$B_m = d(T - t_1) \qquad (9)$$

Therefore, the order size Q during the period $[0, T]$ is obtained as the sum of the maximum inventory level Q_m And maximum back-ordered inventory B_m

$$Q = \frac{d}{\theta}\left(e^{\theta(t_1 - t_d)} - 1\right) + \left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3}\right) + d(T - t_1) \qquad (10)$$

(i) *The total demand during the period* $[t_d, t_1]$ *is given by*

$$D_M = \int_{t_d}^{t_1} d\,dt = d(t_1 - t_d) \qquad (11)$$

(ii) *The total number of deteriorated items per cycle is obtained as the difference between* Q_d and D_M

$$D_P = \frac{d}{\theta}\left[e^{\theta(t_1 - t_d)} - 1 - \theta(t_1 - t_d)\right] \qquad (12)$$

(iii) *Total number of items sold*

$$SN = Q - D_P = \left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3}\right) + d(t_1 - t_d) + d(T - t_1) \quad (13)$$

(iv) Sale revenue (SR)

$$SR = S_1\left[\int_0^{t_d}(a + bt + ct^2)dt\right] + S_2\left[\int_{t_d}^{t_1} d\,dt + \int_{t_1}^{T} d\,dt\right]$$
$$= S_1\left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3}\right) + S_2 d(t_1 - t_d) + S_2 d(T - t_1) \quad (14)$$

(v) Purchasing cost *(PC)*

$$PC = CQ = C\left[\frac{d}{\theta}\left(e^{\theta(t_1 - t_d)} - 1\right) + \left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3}\right) + d(T - t_1)\right] \quad (15)$$

(iv) *The fixed ordering cost per order is given by* A

(v) *The inventory holding cost for the entire cycle is given by*

$$C_H = \int_0^{t_d}(h_1 + h_2 t)I_1(t)dt + \int_{t_d}^{t_1}(h_1 + h_2 t)I_2(t)dt \quad (16)$$

Substituting (5)and(4) into (16) to obtain

$$C_H = \int_0^{t_d}(h_1 + h_2 t)\left[\frac{d}{\theta}\left(e^{\theta(t_1 - t_d)} - 1\right) + a(t_d - t) + \frac{b}{2}(t_d^2 - t^2) + \frac{c}{3}(t_d^3 - t^3)\right]dt + \int_{t_d}^{t_1}(h_1 + h_2 t)\left[\frac{d}{\theta}\left(e^{\theta(t_1 - t)} - 1\right)\right]dt$$
$$= h_1\left[\frac{dt_d}{\theta}e^{\theta(t_1 - t_d)} + \frac{a}{2}t_d^2 + \frac{b}{3}t_d^3 + \frac{c}{4}t_d^4 + \frac{d}{\theta^2}e^{\theta(t_1 - t_d)} - \frac{d}{\theta^2} - \frac{dt_1}{\theta}\right] + h_2\left[\frac{dt_d^2}{2\theta}e^{\theta(t_1 - t_d)} + \frac{a}{6}t_d^3 + \frac{b}{8}t_d^4 + \frac{c}{10}t_d^5 + \frac{dt_d}{\theta^2}e^{\theta(t_1 - t_d)} - \frac{dt_1}{\theta^2} - \frac{d}{\theta^3} + \frac{d}{\theta^3}e^{\theta(t_1 - t_d)} - \frac{dt_1^2}{2\theta}\right] \quad (17)$$

(vi) *The back-ordered cost per cycle is given by*

$$SC = C_b\int_{t_1}^{T} -I_3(t)dt = \frac{C_b d}{2}(T - t_1)^2 \quad (18)$$

(vii) *The total profit per unit time for a replenishment cycle (denoted by* $TP(t_1, T)$ *is given by*

$$TP(t_1, T) = \begin{cases} TP_1(t_1, T) & 0 < M \le t_d \\ TP_2(t_1, T)t_d & < M \le t_1 \\ TP_3(t_1, T) & M > t_1 \end{cases} \quad (19)$$

where $TP_1(t_1, T)$, $TP_2(t_1, T)$, and $TP_3(t_1, T)$ are discussed for three different cases follows.

Case 1: ($0 < M \le t_d$)

The interest payable

This is the stage before deterioration begins, and goods are settled with the rate I_c for the items in stock. Therefore, the interest payable is given below.

$$I_{P1} = CI_c\left[\int_M^{t_d} I_1(t)dt + \int_{t_d}^{t_1} I_2(t)dt\right]$$
$$= CI_c\left[\frac{d(t_d - M)}{\theta}\left(e^{\theta(t_1 - t_d)} - 1\right) + \frac{a}{2}(t_d - M)^2 + \frac{b}{6}(2t_d + M)(t_d - M)^2 + \frac{c}{12}(3t_d^2 + 2t_d M + M^2)(t_d - M)^2 + \frac{d}{\theta^2}\left(e^{\theta(t_1 - t_d)} - 1 - \theta(t_1 - t_d)\right)\right] \quad (20)$$

The Interest Earned

In this case, the retailer can earn interest up to the time M. The interest earned is

$$I_{E1} = S_1 I_e\left[\int_0^M (a + bt + ct^2)t\,dt\right] = S_1 I_e\left(a\frac{M^2}{2} + b\frac{M^3}{3} + c\frac{M^4}{4}\right) \quad (21)$$

The total profit per unit time for case 1($0 < M \le t_d$) is

$TP_1(t_1, T) = \frac{1}{T}${Sales Revenue-Purchasing cost-Ordering cost - inventory holding cost- back ordered cost - interest payable during the permissible delay period + interest earned during the cycle}

$$= \frac{1}{T}\left\{(S_1 - C)\left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3}\right) + S_2 d(t_1 - t_d) + (S_2 - C)d(T - t_1) - C\left[\frac{d}{\theta}\left(e^{\theta(t_1-t_d)} - 1\right)\right] - A - h_1\left[\frac{dt_d}{\theta}e^{\theta(t_1-t_d)} + \frac{a}{2}t_d^2 + \frac{b}{3}t_d^3 + \frac{c}{4}t_d^4 + \frac{d}{\theta^2}e^{\theta(t_1-t_d)} - \frac{d}{\theta^2} - \frac{dt_1}{\theta}\right] - h_2\left[\frac{dt_d^2}{2\theta}e^{\theta(t_1-t_d)} + \frac{a}{6}t_d^3 + \frac{b}{8}t_d^4 + \frac{c}{10}t_d^5 + \frac{dt_d}{\theta^2}e^{\theta(t_1-t_d)} - \frac{dt_1}{\theta^2} - \frac{d}{\theta^3} + \frac{d}{\theta^3}e^{\theta(t_1-t_d)} - \frac{dt_1^2}{2\theta}\right] - \frac{C_b d}{2}(T - t_1)^2 - cI_c\left[\frac{d(t_d - M)}{\theta}\left(e^{\theta(t_1-t_d)} - 1\right) + \frac{a}{2}(t_d - M)^2 + \frac{b}{6}(2t_d + M)(t_d - M)^2 + \frac{c}{12}(3t_d^2 + 2t_d M + M^2)(t_d - M)^2 + \frac{d}{\theta^2}\left(e^{\theta(t_1-t_d)} - 1 - \theta(t_1 - t_d)\right)\right] + S_1 I_e\left(a\frac{M^2}{2} + b\frac{M^3}{3} + c\frac{M^4}{4}\right)\right\} \quad (22)$$

Case 2: ($t_d < M \leq t_1$)

The interest payable

This is when the credit period is greater than the period with no deterioration but shorter than or equal to the period with positive inventory. The interest payable is

$$I_{P2} = cI_c\left[\int_M^{t_1} I_2(t)dt\right] = cI_c\left[\frac{d}{\theta^2}\left(e^{\theta(t_1-M)} - 1 - \theta(t_1 - M)\right)\right] \quad (23)$$

The interest earned

In this case, the retailer can earn interest up to the trade credit period M. The interest earned is

$$I_{E2} = S_1 I_e\left[\int_0^{t_d}(a + bt + ct^2)tdt\right] + S_2 I_e\left[\int_{t_d}^{M} dtdt\right] = S_1 I_e\left(a\frac{t_d^2}{2} + b\frac{t_d^3}{3} + c\frac{t_d^4}{4}\right) + S_2 I_e\left(\frac{dM^2}{2} - \frac{dt_d^2}{2}\right) \quad (24)$$

The total profit per unit time for case 2($t_d < M \leq t_1$) is

$TP_2(t_1, T) = \frac{1}{T}${Sales Revenue - Purchasing cost - Ordering cost - inventory holding cost - back ordered cost - interest payable during the permissible delay period + interest earned during the cycle}

$$= \frac{1}{T}\left\{(S_1 - C)\left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3}\right) + S_2 d(t_1 - t_d) + (S_2 - C)d(T - t_1) - C\left[\frac{d}{\theta}\left(e^{\theta(t_1-t_d)} - 1\right)\right] - A - h_1\left[\frac{dt_d}{\theta}e^{\theta(t_1-t_d)} + \frac{a}{2}t_d^2 + \frac{b}{3}t_d^3 + \frac{c}{4}t_d^4 + \frac{d}{\theta^2}e^{\theta(t_1-t_d)} - \frac{d}{\theta^2} - \frac{dt_1}{\theta}\right] - h_2\left[\frac{dt_d^2}{2\theta}e^{\theta(t_1-t_d)} + \frac{a}{6}t_d^3 + \frac{b}{8}t_d^4 + \frac{c}{10}t_d^5 + \frac{dt_d}{\theta^2}e^{\theta(t_1-t_d)} - \frac{dt_1}{\theta^2} - \frac{d}{\theta^3} + \frac{d}{\theta^3}e^{\theta(t_1-t_d)} - \frac{dt_1^2}{2\theta}\right] - \frac{C_b d}{2}(T - t_1)^2 - cI_c\left[\frac{d}{\theta^2}\left(e^{\theta(t_1-M)} - 1 - \theta(t_1 - M)\right)\right] + S_1 I_e\left(a\frac{t_d^2}{2} + b\frac{t_d^3}{3} + c\frac{t_d^4}{4}\right) + S_2 I_e\left(\frac{dM^2}{2} - \frac{dt_d^2}{2}\right)\right\} \quad (25)$$

Case 3: ($M > t_1$)

The interest payable

In this case, the delay in payment is greater than the period with positive inventory. In this case, the retailer pays no interest. Therefore,$I_{P3} = 0$.

The interest earned

In this case, the period of delay in payment (M) is greater than the period with positive inventory(t_1). Interest earned for the time period $[0, T]$

$$I_{E3} = S_1 I_e \left[\int_0^{t_d} (a + bt + ct^2) t dt + (M - t_1) \int_0^{t_d} (a + bt + ct^2) dt \right] + S_2 I_e \left[\int_{t_d}^{t_1} d t dt + (M - t_1) \int_{t_d}^{t_1} d dt \right]$$

$$= S_1 I_e \left[\left(a\frac{t_d^2}{2} + b\frac{t_d^3}{3} + c\frac{t_d^4}{4} \right) + (M - t_1)\left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3} \right) \right] + S_2 I_e \left[-\frac{d}{2}(t_1 - t_d)^2 + Md(t_1 - t_d) \right] \quad (26)$$

The total profit per unit time for case 3 $(M > t_1)$ is

$TP_3(t_1, T) = \frac{1}{T}${Sales Revenue - Purchasing cost - Ordering cost - inventory holding cost - back ordered cost + interest earned during the cycle}

$$= \frac{1}{T}\left\{ (S_1 - C)\left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3} \right) + S_2 d(t_1 - t_d) + (S_2 - C)d(T - t_1) - C\left[\frac{d}{\theta}\left(e^{\theta(t_1 - t_d)} - 1 \right) \right] - A - h_1 \left[\frac{dt_d}{\theta} e^{\theta(t_1 - t_d)} + \frac{a}{2}t_d^2 + \frac{b}{3}t_d^3 + \frac{c}{4}t_d^4 + \frac{d}{\theta^2} e^{\theta(t_1 - t_d)} - \frac{d}{\theta^2} - \frac{dt_1}{\theta} \right] - h_2 \left[\frac{dt_d^2}{2\theta} e^{\theta(t_1 - t_d)} + \frac{a}{6}t_d^3 + \frac{b}{8}t_d^4 + \frac{c}{10}t_d^5 + \frac{dt_d}{\theta^2} e^{\theta(t_1 - t_d)} - \frac{dt_1}{\theta^2} - \frac{d}{\theta^3} + \frac{d}{\theta^3} e^{\theta(t_1 - t_d)} - \frac{dt_1^2}{2\theta} \right] - \frac{C_b d}{2}(T - t_1)^2 + S_1 I_e \left[\left(a\frac{t_d^2}{2} + b\frac{t_d^3}{3} + c\frac{t_d^4}{4} \right) + (M - t_1)\left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3} \right) \right] + S_2 I_e \left[-\frac{d}{2}(t_1 - t_d)^2 + Md(t_1 - t_d) \right] \right\} \quad (27)$$

Since $0 < \theta < 1$, by utilising a quadratic approximation for the exponential terms in equations (22), (25) and (27) to obtain

$$TP_1(t_1, T) = \frac{1}{T}\left\{ \left[(S_1 - C)\left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3} \right) - (S_2 - C)dt_d - \frac{Cd\theta t_d^2}{2} - A - h_1\left(\frac{a}{2}t_d^2 + \frac{b}{3}t_d^3 + \frac{c}{4}t_d^4 - \frac{dt_d^2}{2} + \frac{dt_d^3\theta}{2} \right) - h_2\left(\frac{a}{6}t_d^3 + \frac{b}{8}t_d^4 + \frac{c}{10}t_d^5 + \frac{dt_d^4\theta}{4} \right) - CI_c\left(\frac{a}{2}(t_d - M)^2 + \frac{b}{6}(2t_d + M)(t_d - M)^2 + \frac{c}{12}(3t_d^2 + 2t_d M + M^2)(t_d - M)^2 + dMt_d - \frac{dt_d^2}{2} + \frac{d}{2}(t_d - M)\theta t_d^2 \right) + S_1 I_e\left(a\frac{M^2}{2} + b\frac{M^3}{3} + c\frac{M^4}{4} \right) \right] + d\left[h_1 t_d^2\theta + \frac{h_2}{2}(1 + t_d\theta)t_d^2 + Ct_d\theta + cI_c(M + (t_d - M)\theta t_d) \right] t_1 - \frac{d}{2}\left[h_1(t_d\theta + 1) + h_2\left(\frac{t_d\theta}{2} + 1 \right)t_d + C\theta + C_b + cI_c(\theta(t_d - M) + 1) \right] t_1^2 - \frac{C_b dT^2}{2} + C_b dt_1 T + (S_2 - C)dT \right\}$$

$$= \frac{d}{T}\left\{ -\frac{1}{2}A_1 t_1^2 + \mathrm{B}_1 t_1 - \mathrm{C}_1 - \frac{C_b T^2}{2} + C_b t_1 T + (S_2 - C)T \right\} \quad (28)$$

Where

$A_1 = \left[h_1(t_d\theta + 1) + h_2\left(\frac{t_d\theta}{2} + 1 \right)t_d + C\theta + C_b + cI_c(\theta(t_d - M) + 1) \right]$,

$$\mathrm{B}_1 = \left[h_1 t_d^2\theta + \frac{h_2}{2}(1 + t_d\theta)t_d^2 + Ct_d\theta + cI_c(M + (t_d - M)\theta t_d) \right]$$

and

$$C_1 = -\frac{1}{d}\Bigg[(S_1 - C)\left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3}\right) - (S_2 - C)dt_d - \frac{Cd\theta t_d^2}{2} - A - h_1\left(\frac{a}{2}t_d^2 + \frac{b}{3}t_d^3 + \frac{c}{4}t_d^4 - \frac{dt_d^2}{2} + \frac{dt_d^3\theta}{2}\right) - h_2\left(\frac{a}{6}t_d^3 + \frac{b}{8}t_d^4 + \frac{c}{10}t_d^5 + \frac{dt_d^4\theta}{4}\right) - CI_c\left(\frac{a}{2}(t_d - M)^2 + \frac{b}{6}(2t_d + M)(t_d - M)^2 + \frac{c}{12}(3t_d^2 + 2t_dM + M^2)(t_d - M)^2 + dMt_d - \frac{dt_d^2}{2} + \frac{d}{2}(t_d - M)\theta t_d^2\right) + S_1I_e\left(a\frac{M^2}{2} + b\frac{M^3}{3} + c\frac{M^4}{4}\right)\Bigg]$$

Similarly,

$$TP_2(t_1, T) = \frac{d}{T}\left\{-\frac{1}{2}A_2t_1^2 + B_2t_1 - C_2 - \frac{C_bT^2}{2} + C_bt_1T + (S_2 - C)T\right\} \quad (29)$$

Where

$$A_2 = \left[h_1(t_d\theta + 1) + h_2\left(\frac{t_d\theta}{2} + 1\right)t_d + C\theta + C_b + CI_c\right],$$

$$\left[h_1(t_d\theta + 1) + h_2\left(\frac{t_d\theta}{2} + 1\right)t_d + C\theta + (C_b + C_\pi\delta) + cI_c\right]$$

$$B_2 = \left[h_1t_d^2\theta + \frac{h_2}{2}(1 + t_d\theta)t_d^2 + Ct_d\theta + cI_cM\right]$$

and

$$C_2 = -\frac{1}{d}\Bigg[(S_1 - C)\left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3}\right) - (S_2 - C)dt_d - \frac{Cd\theta t_d^2}{2} - A - h_1\left(\frac{a}{2}t_d^2 + \frac{b}{3}t_d^3 + \frac{c}{4}t_d^4 - \frac{dt_d^2}{2} + \frac{dt_d^3\theta}{2}\right) - h_2\left(\frac{a}{6}t_d^3 + \frac{b}{8}t_d^4 + \frac{c}{10}t_d^5 + \frac{dt_d^4\theta}{4}\right) - CI_c\frac{d}{2}M^2 + S_1I_e\left(a\frac{t_d^2}{2} + b\frac{t_d^3}{3} + c\frac{t_d^4}{4}\right) + S_2I_e\left(\frac{dM^2}{2} - \frac{dt_d^2}{2}\right)\Bigg]$$

and

$$TP_3(t_1, T) = \frac{d}{T}\left\{-\frac{1}{2}A_3t_1^2 + B_3t_1 - C_3 - \frac{C_bT^2}{2} + C_bt_1T + (S_2 - C)T\right\} \quad (30)$$

Where

$$A_3 = \left[h_1(t_d\theta + 1) + h_2\left(\frac{t_d\theta}{2} + 1\right)t_d + C\theta + C_b + S_2I_e\right],$$

$$B_3 = \left[h_1t_d^2\theta + \frac{h_2}{2}(1 + t_d\theta)t_d^2 + Ct_d\theta - \frac{1}{d}\left\{S_1I_e\left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3}\right)\right\} + S_2I_et_d + S_2I_eM\right]$$

and

$$C_3 = -\frac{1}{d}\Bigg[(S_1 - C)\left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3}\right) - (S_2 - C)dt_d - \frac{Cd\theta t_d^2}{2} - A - h_1\left(\frac{a}{2}t_d^2 + \frac{b}{3}t_d^3 + \frac{c}{4}t_d^4 - \frac{dt_d^2}{2} + \frac{dt_d^3\theta}{2}\right) - h_2\left(\frac{a}{6}t_d^3 + \frac{b}{8}t_d^4 + \frac{c}{10}t_d^5 + \frac{dt_d^4\theta}{4}\right) + S_1I_e\left[\left(a\frac{t_d^2}{2} + b\frac{t_d^3}{3} + c\frac{t_d^4}{4}\right) + \left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3}\right)M\right] - S_2I_e\frac{d}{2}t_d^2 - S_2I_eMdt_d\Bigg]$$

OPTIMAL DECISION

This section determines the optimal ordering policies that maximise the total profit per unit time. The necessary and sufficient conditions for optimal solutions' existence and uniqueness will be established. The necessary conditions for the total profit per unit time $TP_i(t_1, T)$ to be maximum are $\frac{\partial TP_i(t_1,T)}{\partial t_1} = 0$ and $\frac{\partial TP_i(t_1,T)}{\partial T} = 0$ for $i = 1, 2, 3$. The value of (t_1, T) obtained from $\frac{\partial TP_i(t_1,T)}{\partial t_1} = 0$ and $\frac{\partial TP_i(t_1,T)}{\partial T} = 0$ and for which the sufficient condition $\left\{\left(\frac{\partial^2 TP_i(t_1,T)}{\partial t_1^2}\right)\left(\frac{\partial^2 TP_i(t_1,T)}{\partial T^2}\right) - \left(\frac{\partial^2 TP_i(t_1,T)}{\partial t_1 \partial T}\right)^2\right\} > 0$ is satisfied, gives a maximum value for the total profit per unit time $TP_i(t_1, T)$.

For case 1($0 < M \leq t_d$)

The necessary condition for the total profit $TP_1(t_1, T)$ in equation (28) to be the maximum are $\frac{\partial TP_1(t_1,T)}{\partial t_1} = 0$ and $\frac{\partial TP_1(t_1,T)}{\partial T} = 0$, which gives

$$\frac{\partial TP_1(t_1, T)}{\partial t_1} = \frac{d}{T}\{-A_1t_1 + B_1 + C_bT\}$$

Setting $\frac{\partial TP_1(t_1,T)}{\partial t_1} = 0$ gives

$$\{-A_1t_1 + B_1 + C_bT\} = 0 \qquad (31)$$

and

$$T = \frac{1}{C_b}(A_1t_1 - B_1) \qquad (32)$$

Since $(t_d - M) \geq 0, (t_1 - t_d) > 0, (t_1 - M) > 0$, it should be noted that

$$(A_1t_1 - B_1) = \Big[h_1(t_d\theta(t_1 - t_d) + t_1) + h_2\left(t_1 - \frac{t_d}{2}\right)t_d + \frac{h_2t_d\theta}{2}(t_1 - t_d)t_d + C\theta(t_1 - t_d) + C_bt_1 + cI_c\big((t_1 - M) + \theta(t_d - M)(t_1 - t_d)\big)\Big] > 0$$

Similarly,

$$\frac{\partial TP_1(t_1,T)}{\partial T} = -\frac{d}{T^2}\left\{-\frac{1}{2}A_1t_1^2 + B_1t_1 - C_1 + \frac{C_bT^2}{2}\right\} \qquad (33)$$

Setting $\frac{\partial TP_1(t_1,T)}{\partial T} = 0$ to obtain

$$-\frac{d}{T^2}\left\{-\frac{1}{2}A_1t_1^2 + B_1t_1 - C_1 + \frac{C_bT^2}{2}\right\} = 0 \qquad (34)$$

Substituting T from equation (32) into equation (34) yields

$$\{A_1(C_b - A_1)t_1^2 - 2B_1(C_b - A_1)t_1 - (B_1^2 - 2C_bC_1)\} = 0 \qquad (35)$$

Let $\Delta_1 = \{A_1(C_b - A_1)t_d^2 - 2B_1(C_b - A_1)t_d - (B_1^2 - 2C_bC_1)\}$, then the following result is obtained.

Lemma 1

(i) If $\Delta_1 \geq 0$, then the solution of $t_1 \in [t_d, \infty)$ (say t_{11}^*), which satisfies equation (35) not only exists but also is unique.
See the proof in Appendix 1a

(ii) If $\Delta_1 < 0$, then the solution of $t_1 \in [t_d, \infty)$ Which satisfies equation (35) does not exist.

See the proof in Appendix 1b

Therefore, the value of t_1 (denoted by t_{11}^*) can be found from equation (35) and is given by

$$t_{11}^* = \frac{B_1}{A_1} + \frac{1}{A_1}\sqrt{\frac{(2A_1C_1 - B_1^2)C_b}{(A_1 - C_b)}} \qquad (36)$$

Once the value of t_{11}^* is obtained, then the value of T (denoted by T_1^*) can be found from (32) and is given by

$$T_1^* = \frac{1}{C_b}(A_1t_{11}^* - B_1) \qquad (37)$$

Equations (36) and (37) give the optimal values of t_{11}^* and T_1^* for the profit function in equation (28) only if B_1 satisfies the inequality given in equation (38)

$$2A_1C_1 > B_1^2 \qquad (38)$$

Theorem 1

(i) If $\Delta_1 \geq 0$, then the total profit $TP_1(t_1, T)$ is concave and reaches its global maximum at the point (t_{11}^*, T_1^*), where (t_{11}^*, T_1^*) is the point which satisfies equations (35) and (31), if all principal minors are negative definite, i.e., if

$$\left(\left.\frac{\partial^2 TP_1(t_1,\ T)}{\partial t_1^2}\right|_{(t_{11}^*,\ T_1^*)}\right) < 0, \left(\left.\frac{\partial^2 TP_1(t_1,\ T)}{\partial T^2}\right|_{(t_{11}^*,\ T_1^*)}\right) < 0$$

and

$$\begin{vmatrix} \left.\frac{\partial^2 TP_1(t_1,\ T)}{\partial t_1^2}\right|_{(t_{11}^*,\ T_1^*)} & \left.\frac{\partial^2 TP_1(t_1,\ T)}{\partial t_1 \partial T}\right|_{(t_{11}^*,\ T_1^*)} \\ \left.\frac{\partial^2 TP_1(t_1,\ T)}{\partial t_1 \partial T}\right|_{(t_{11}^*,\ T_1^*)} & \left(\left.\frac{\partial^2 TP_1(t_1,\ T)}{\partial T^2}\right|_{(t_{11}^*,\ T_1^*)}\right) \end{vmatrix} > 0.$$

See the proof in Appendix 1c

(ii) If $\Delta_1 < 0$, then the total profit $TP_1(t_1, T)$ has a maximum value at the point (t_{11}^*, T_1^*) where $t_{11}^* = t_d$ and $T_1^* = \frac{1}{C_b}(A_1t_d - B_1)$

See the proof in Appendix 1d

For case 2 ($t_d < M \leq t_1$)

The necessary condition for the total profit $TP_2(t_1, T)$ in equation (29) to be the maximum are $\frac{\partial TP_2(t_1,T)}{\partial t_1} = 0$ and $\frac{\partial TP_2(t_1,T)}{\partial T} = 0$, which gives

$$\frac{\partial TP_2(t_1,T)}{\partial t_1} = \frac{d}{T}\{-A_2t_1 + B_2 + C_bT\}$$

Setting $\frac{\partial TP_2(t_1,T)}{\partial t_1} = 0$ gives

$$\{-A_2t_1 + B_2 + C_bT\} = 0 \qquad (39)$$

and

$$T = \frac{1}{C_b}A_2t_1 - B_2 \qquad (40)$$

Since $(t_1 - t_d) > 0, (t_1 - M) \geq 0$, it should be noted that

$$(A_2t_1 - B_2) = [h_1(t_d\theta(t_1 - t_d) + t_1) + h_2\left(t_1 - \frac{t_d}{2}\right)t_d + \frac{h_2t_d\theta}{2}(t_1 - t_d)t_d + C\theta(t_1 - t_d) + C_bt_1 + CI_C(t_1 - M) > 0$$

Similarly

$$\frac{\partial TP_{2(t_{1,T})}}{\partial T} = -\frac{d}{T^2}\left\{-\frac{1}{2}A_2t_1^2 + B_2t_1 - C_2 + \frac{C_bT^2}{2}\right\} \qquad (41)$$

Setting $\frac{\partial TP_2(t_{1,}T)}{\partial T} = 0$ to obtain

$$-\frac{d}{T^2}\left\{-\frac{1}{2}A_2t_1^2 + B_2t_1 - C_2 + \frac{C_bT^2}{2}\right\} = 0 \qquad (42)$$

Substituting T from equation (40) into equation (42) yields

$$A_2(C_b - A_2)t_1^2 - 2B_2(C_b - A_2)t_1 - (B_2^2 - 2C_bC_2)\} = 0 \qquad (43)$$

Let $\Delta_2 = \{A_2(C_b - A_2)M^2 - 2B_2(C_b - A_2)M - (B_2^2 - 2C_bC_2)\}$, then the following result is obtained.

Lemma 2

(i) If $\Delta_2 \geq 0$, then the solution of $t_1 \in [M, \infty)$ $(say\ t_{12}^*)$, which satisfies equation (43) not only exists but also is unique.

The proof is similar to Appendix 1a. hence is omitted

(ii) If $\Delta_2 < 0$, then the solution of $t_1 \in [M, \infty)$, which satisfies equation (43), does not exist.
The proof is similar to Appendix 1b; hence is omitted

Therefore, the value of t_1 (denoted by t_{12}^*) can be found from equation (43) and is given by

$$t_{12}^* = \frac{B_2}{A_2} + \frac{1}{A_2}\sqrt{\frac{(2A_2C_2 - B_2^2)C_b}{(A_2 - C_b)}} \qquad (44)$$

Once the value of t_{12}^* is obtained, then the value of T (denoted by T_2^*) can be found from

$$T_2^* = \frac{1}{C_b}(A_2t_{12}^* - B_2) \qquad (45)$$

Equations (44) and (45) give the optimal of t_{12}^* and T_2^* for the profit function in equation (29) only if B_2 satisfies the inequality given in equation (46)

$$2A_2C_2 > B_2^2 \qquad (46)$$

Theorem 2

(i) If $\Delta_2 \geq 0$, then the total profit $TP_2(t_1, T)$It is a concave and reaches its global maximum at the point (t_{12}^*, T_2^*), where (t_{12}^*, T_2^*) is the point which satisfies equations (45) and (41) if all principal minors are negative definite, i.e. if

$$\left(\left.\frac{\partial^2 TP_2(t_1,\ T)}{\partial t_1^2}\right|_{(t_{12}^*,\ T_2^*)}\right) < 0, \left(\left.\frac{\partial^2 TP_2(t_1,\ T)}{\partial T^2}\right|_{(t_{12}^*,\ T_2^*)}\right) < 0$$

The proof is similar to Appendix 1c; hence is omitted

(ii). If $\Delta_2 < 0$, then the total profit $TP_2(t_1, T)$ has a maximum value at the point (t_{12}^*, T_2^*) where $t_{12}^* = t_d$ and $T_2^* = \frac{1}{C_b}(A_2t_d - B_2)$

The proof is similar to Appendix 1d; hence is omitted.

For case 3 ($M > t_1$)

The necessary condition for the total profit $TP_3(t_{1,}T)$ in equation (30) to be the maximum is $\frac{\partial TP_3(t_{1,}T)}{\partial t_1}$ and

$\frac{\partial TP_3(t_1,T)}{\partial T} = 0$, which gives

$$\frac{\partial TP_3(t_1, T)}{\partial t_1} = \frac{d}{T}\{-A_3t_1 + B_3 + C_bT\}$$

Setting $\frac{\partial TP_3(t_{1,}T)}{\partial t_1} = 0$ gives

$$\{-A_3t_1 + B_3 + C_bT\} = 0 \qquad (47)$$

and

$$T = \frac{1}{C_b}(A_3t_1 - B_3) \qquad (48)$$

Since $(t_1 - t_d) > 0$, it should be noted that

$$(A_3t_1 - B_3) = h_1(t_d\theta(t_1 - t_d) + t_1) + h_2\left(t_1 - \frac{t_d}{2}\right)t_d + \frac{h_2t_d\theta}{2}(t_1 - t_d)t_d + C\theta(t_1 - t_d) + C_bt_1 + \frac{1}{d}\left\{S_1I_e\left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3}\right)\right\} - S_2I_e(t_d + M)] > 0$$

Similarly,

$$\frac{\partial TP_3(t_1,T)}{\partial T} = -\frac{d}{T^2}\left\{-\frac{1}{2}A_3t_1^2 + B_3t_1 - C_3 + \frac{C_bT^2}{2}\right\} \quad (49)$$

Setting $\frac{\partial TP_3(t_1,T)}{\partial T} = 0$ to obtain

$$-\frac{d}{T^2}\left\{-\frac{1}{2}A_3t_1^2 + B_3t_1 - C_3 + \frac{C_bT^2}{2}\right\} = 0 \quad (50)$$

Substituting T from equation (48) into equation (50) yields

$$\{A_3(C_b - A_3)t_1^2 - 2B_3(C_b - A_1)t_1 - (B_3^2 - 2C_bC_3)\} = 0 \quad (51)$$

Let $\Delta_{3a} = \{A_3(C_b - A_3)t_d^2 - 2B_3(C_b - A_3)t_d - (B_3^2 - 2C_bC_3)\} > 0$

and

$$\Delta_{3b} = A_3(C_b - A_3)M^2 - 2B_3(C_b - A_3)M - (B_3^2 - 2C_bC_3) < 0$$

Then, the following result is obtained.

Lemma 3

(i) If $\Delta_{3b} \leq 0 \leq \Delta_{3a}$, then the solution of $t_1 \in [t_d, M](say\ t_{13}^*)$, which satisfies equation (55) not only exists but also is unique.
The proof is similar to Appendix 1a; hence is omitted

(ii) If $\Delta_{3a} < 0$, then the solution of $t_1 \in [t_d, M]$ Which satisfies equation (51) does not exist.
The proof is similar to Appendix 1b; hence is omitted

Therefore, the value of t_1 (denoted by t_{13}^*) can be found from equation (51) and is given by

$$t_{13}^* = \frac{B_3}{A_3} + \frac{1}{A_3}\sqrt{\frac{(2C_3C_3 - B_3{}^2)C_b}{(A_3 - C_b)}} \quad (52)$$

Once the value of t_{13}^* is obtained, then the value of T (denoted by T_3^*) can be found from (48) and is given by

$$T_3^* = \frac{1}{C_b}(A_3t_{13}^* - B_3) \quad (53)$$

Equations (52) and (53) give the optimal values of t_{13}^* and T_3^* for the profit function in equation (30) only if B_3 satisfies the inequality given equation (54)

$$2A_3C_3 > B_3^2 \quad (54)$$

Theorem 3

(i) if $\Delta_{3a} \geq 0$, then the total profit $TP_3(t_1,T)$ is concave and reaches its global maximum at the point (t_{13}^*, T_3^*) where (t_{13}^*, T_3^*) is the point which satisfies equation (47) and equation (51) if all principal minors are negative definite, i.e. if

$$\left(\left.\frac{\partial^2 TP_3(t_1,\ T)}{\partial t_1^2}\right|_{(t_{13}^*,\ T_3^*)}\right) < 0, \left(\left.\frac{\partial^2 TP_3(t_1,\ T)}{\partial T^2}\right|_{(t_{13}^*,\ T_3^*)}\right) < 0$$

and

$$\begin{vmatrix} \left.\frac{\partial^2 TP_3(t_1,\ T)}{\partial t_1^2}\right|_{(t_{13}^*,\ T_3^*)} & \left.\frac{\partial^3 TP_3(t_1,\ T)}{\partial t_1 \partial T}\right|_{(t_{13}^*,\ T_3^*)} \\ \left.\frac{\partial^2 TP_1(t_1,\ T)}{\partial t_1 \partial T}\right|_{(t_{13}^*,\ T_3^*)} & \left(\left.\frac{\partial^2 TP_3(t_1,\ T)}{\partial T^2}\right|_{(t_{13}^*,\ T_3^*)}\right) \end{vmatrix}$$

The proof is similar to Appendix 1c; hence is omitted

(ii) if $\Delta_{3a} < 0$, then the total profit $TP_3(t_1,T)$ has a maximum value at the point (t_{13}^*, T_3^*) where $t_{13}^* = M$ and $T_3^* = \frac{1}{C_b}(A_3M - B_3)$.
The proof is similar to Appendix 1d; hence is omitted

(iii) If $\Delta_{3b} > 0$, then the total profit $TP_3(t_1,T)$ has a maximum value at the point (t_{13}^*, T_3^*) where $t_{13}^* = t_d$ and $T_3^* = \frac{1}{C_b}(A_3t_d - B_3)$

The proof is similar to Appendix 1d; hence is omitted.

Thus, the optimal Economic Order Quantity (denoted by EOQ^*) corresponding to t_1^* and T^* Can be computed as follows:

$$EOQ^* = Total\ demand\ before\ deterioration\ sets\ in + total\ demand\ after\ deterioration\ sets\ in + total\ number\ of\ deteriorated\ items + the\ total\ number\ of\ items\ back-ordered$$

$$= \int_0^{t_d}(a + bt + dt^2)dt + \int_{t_d}^{t_1^*} ddt + \left[\frac{d}{\theta}\left(e^{\theta(t_1^*-t_d)} - 1\right) - d(t_1^* - t_d)\right] + d(T^* - t_1^*)$$

$$= at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3} + \frac{d}{\theta}\left(e^{\theta(t_1^*-t_d)} - 1\right) + d(T^* - t_1^*) \quad (55)$$

Note: It is obvious when $t_d = t_1 = M$ that $TP_1(t_1,T) = TP_2(t_1,T) = TP_3(t_1,T)$. When $t_d = M$, $TP_1(t_1,T) = TP_2(t_1,T)$. When $t_1 = M, TP_2(M,T) =$

$TP_3(M,T)$. Hence, the profit function $TP(t_1,T)$ is continuous and well-defined.

NUMERICAL RESULTS

Example 5.1 ($M \leq t_d$)

The following parameters are adopted from Babangida and Baraya (2021) in addition to h_1 and C_b which are not considered in their work. The parameters and their values are as follows:

Table 5.1: Parameters and their values

Parameter(s)	Value(s)
A	$\$250$/order
h_1	$\$2$ unit/year
h_2	$\$15$ unit/year
θ	0.01 unit/year
a	180 unit
b	30 unit
c	15 unit
d	120 unit
t_d	0.1354 year
M	0.0888 year
I_c	0.1
I_e	0.08
C_b	$\$30$

It is seen that $M \leq t_d$, Δ_1 is solved numerically to test the conditions of lemma 1, which states that if $\Delta_1 \geq 0$ solution exists and is unique, and if $\Delta_1 < 0$ solution does not exist.

$\Delta_1 = 39.8867 > 0$, which implies that the solution exists and is unique.

Likewise $2A_1C_1 = 51.4381$ and $B_1^2 = 0.0851$ were solved numerically to see if equation (38) $2A_1C_1 > B_1^2$ is satisfied. Substituting the above values in equation (36), (37), (28) and (55). The result is obtained in the table 5.2 below.

Table 5.2: Optimal Solutions, for example, 5.1

Parameters	Values
t_{11}^*	0.4694 (171 days)
T_1^*	0.5487 (200 days)
$TP_1(t_{11}^*, T_1^*)$	$\$314.7020$
EOQ_1^*	74.3213 unit.

Example 5.2 ($M > t_d$)

The values of the parameters are the same as in example 5.1 [as in Babangida and Baraya (2021)] except that $M = 0.1523$. It is seen that $M > t_d$, Δ_2 is solved numerically to check the condition of lemma 2, which states that if $\Delta_2 \geq 0$ solution exists and is unique, and if $\Delta_2 < 0$ solution does not exist. $\Delta_2 = 38.50112 > 0$, which implies that the solution exists. Likewise

$2A_2C_2 = 50.7758$ and $B_2^2 = 0.1496$ were solved numerically to check if equation (46) is satisfied $2A_2C_2 > B_2^2$. Substituting the above values in equation (44), (45), (29) and (55). The result is obtained in the table 5.3 below

Table 5.3: Optimal Solutions for example 5.2

Parameters	Values
t_{12}^*	0.4689 (171 days)
T_2^*	0.5448 (199 days)
$TP_2(t_{12}^*, T_2^*)$	$\$326.5504$
EOQ_2^*	73.8575 unit.

Example 5.3 ($M > t_1$)

The values of the parameters are the same as in Example 5.1, except that $M = 0.36$. It is seen that $M > t_d$, Δ_{3a} and Δ_{3b} were solved numerically to check the condition of lemma 3, which states that if $\Delta_{3b} \leq 0 \leq \Delta_{3a}$ the solution not only exists but is also unique, and if $\Delta_{3a} < 0$ The solution does not exist. $\Delta_{3a} = 19.5432 > 0$, $\Delta_{3b} = -2.0879 < 0$, which implies that the solution exists and is unique. Likewise $2A_3C_3 = 27.1778$ and $B_3^2 = 0.2916$ were solved numerically to check if equation (54) $2A_3C_3 > B_3^2$ is satisfied. Substituting the above values in equation (52), (53), (30) and (55). The result is obtained in the table 5.4 below.

Table 5.4: Optimal Solutions

Parameters	Values
t_{13}^*	0.3451 (126 days)
T_3^*	0.3936 (144 days)
$TP_3(t_{13}^*, T_3^*)$	$\$425.2604$
EOQ_3^*	55.6691 unit.

Thus, the optimal total profit is given by

$$TP(t_1^*, T^*) = Max\left(TP_1(t_{11}^*, T_1^*), TP_2(t_{12}^*, T_2^*), TP_3(t_{13}^*, T_3^*)\right) = TP_3(t_{13}^*, T_3^*) = \$425.2604.$$

Table 5.5: Comparison of the proposed model and Babangida and Bature (2021)

Models	Average total profit per unit for case 1	Average total profit per unit for case 2	Average total profit per unit for case 3
Babangida Baraya (2021)	\$4.1341	\$4.3176	-
Proposed Model	\$4.2343	\$4.4214	\$7.6391

It is clearly seen from the table 5.5 above that the average total profit per unit for case 1 and case 2 of model 1 is greater than that of Babangida and Baraya (2021).

SENSITIVITY ANALYSIS

The sensitivity analysis of some model parameters has been carried out by changing each of these parameters from -5% to $+5\%$ taking one parameter at a time and keeping the remaining parameters unchanged. The effects of changes in these parameters on decision variables are summarised in Tables 6.1.

Table 6.1: Effect of changes of some parameters on decision variables.

Parameter	% change in Parameter	% change in t_{13}^*	% change in T_{13}^*	% change in EOQ_3^*	% change in $TP_3(t_{13}^*, T_3^*)$
θ	-5%	0.0594	0.0485	0.0388	0.0121
	5%	-0.0593	-0.0483	-0.0387	-0.0121
C	-5%	-6.8911	-7.2181	-6.1342	25.0845
	5%	6.4091	6.7169	5.7094	-24.8227
S_1	-5%	30.7285	31.9598	27.1758	-16.7289
	5%	-47.1617	-49.1444	-41.7406	25.9853
S_2	-5%	-19.4973	-20.2416	-17.1991	-17.7268
	5%	15.7682	16.4243	13.9627	19.5527
I_e	-5%	2.8609	2.9129	2.4760	-1.3487
	5%	-2.8636	-2.9291	-2.4895	1.3948
A	-5%	-14.3321	-14.9876	-12.7356	8.0730
	5%	12.4479	13.0172	11.0656	-7.0117
C_b	-5%	-0.2355	0.4007	0.3396	0.1327
	5%	0.2152	-0.3639	-0.3084	-0.1212

DISCUSSION ON SENSITIVITY ANALYSIS

The following managerial insights are discovered based on the results shown in Table 6.1.

(i) From Table 6.1, it is obviously seen that the higher the rate of deterioration (θ), the lower the optimal time with positive inventory (t_1^*), cycle length (T^*), order quantity (EOQ^*) and the total profit $TP(T^*)$ and vice versa. This implies that the retailer needs to take all the necessary measures to avoid or reduce deterioration in order to maximise higher profit.

(ii) From Table 6.1, it is visibly seen that as the unit purchasing cost (C) increases, the total profit $TP(T^*)$ decreases while the optimal time with positive inventory (t_1^*), cycle length (T^*) and order quantity (EOQ^*) increase and vice versa. This result reveals that when the unit purchasing cost increases, the retailer will order smaller quantities to enjoy the benefits of permissible delay in payments more frequently, which will consequently shorten the cycle length.

(iii) From Table 6.1, it is apparently seen that as the unit selling price before deterioration sets in (S_1) increases, the optimal time with positive inventory (t_1^*), cycle length (T^*) and order quantity (EOQ^*) decrease while the total profit $TP(T^*)$ Increases and vice versa. This implies that as the selling price increases, the retailer will order less quantity to enjoy the benefits of trade credit more frequently.

(iv) From Table 6.1, it is evidently seen that as the unit selling price after deterioration sets in (S_2) increases, the optimal time with positive inventory (t_1^*), cycle length (T^*), order quantity (EOQ^*) and the total profit $TP(T^*)$ Increase and vice versa. This implies that as the selling price increases, the retailer maximises higher profit.

(v) From Table 6.1, it is seen that as the interest earned. (I_e) is increasing, the total profit $TP(T^*)$ is also increasing while the optimal time with positive inventory (t_1^*), cycle length (T^*) and order quantity (EOQ^*) are decreasing and vice versa. This implies that when the interest earned is high, the retailer should order less quantity of inventory to enjoy the benefits of trade credit more frequently.

(vi) From Table 6.1, it is obviously seen that as the ordering cost (A) is increasing the total profit $TP(T^*)$ is decreasing while the optimal time with positive inventory (t_1^*), cycle length (T^*) and order quantity (EOQ^*) increase. This implies that the retailer should order large quantity when the ordering cost per order is high.

(vii) From table 6.1, it is clearly seen that as the shortage cost (C_b) increases the total profit $TP(T^*)$, the economic order quantity (EOQ^*), the optimal cycle length (T^*) decreases while the time with positive inventory increases.

CONCLUSION

This research developed an economic order quantity model for non-instantaneous deteriorating items with two phase demand rates, linear holding cost, complete backlogging rate and two-level pricing strategies under trade credit policy. The purpose of the model is to determine the optimal time with positive inventory, cycle length and order quantity such that the total profit of the inventory system has a maximum value. Some numerical examples have been given to illustrate the theoretical result of the model. Sensitivity analysis of some model parameters on the decision variables has been carried out, and suggestions towards maximising the total profit were also given. The retailer can maximise the total profit by ordering less quantity and shortening the cycle length if the rate of deterioration, unit purchasing cost, ordering cost and shortage cost increase and unit selling price before deterioration starts, unit selling price after deterioration starts, and interest earned decrease. The model can be used in inventory control and management of items such as food items (e.g., beans, maise, corns, millet), electronics (e.g., mobile phones, computers), automobiles, fashionable items, etc. The proposed model can be extended by considering factors such as variable deterioration, inflation and time value of money, quantity discount, order size dependent trade credit, etc.

APPENDIX 1a: Proof of Lemma 1(i)

From equation (35), a new function $F_1(t_1)$ is defined as follows

$$F_1(t_1) = \{A_1(C_b - A_1)t_1^2 - 2B_1(C_b - A_1)t_1 - (B_1^2 - 2C_bC_1)\}, \quad t_1 \in [t_d, \infty) \qquad (56)$$

Taking the first-order derivative of $F_1(t_1)$ with respect to $t_1 \in [t_d, \infty)$, it follows that

$$\frac{F_1(t_1)}{dt_1} = 2(A_1t_1 - B_1)(C_b - A_1) < 0$$

Because$(A_1t_1 - B_1) > 0$

and

$$\{C_b - A_1\} = -\left[h_1(t_d\theta + 1) + h_2\left(1 + \frac{t_d\theta}{2}\right)t_d + C\theta + cI_c(\theta(t_d - M) + 1)\right] < 0$$

Hence $F_1(t_1)$ is a strictly decreasing function of t_1 in the interval $[t_d, \infty)$.Moreover, $\lim_{t_1\to\infty} F_1(t_1) = -\infty$ and $F_1(t_d) = \Delta_1 \geq 0$. Therefore, by applying intermediate value theorem, there exists a unique t_1 say $t_{11}^* \in [t_d, \infty)$ such that $F_1(t_{11}^*) = 0$. Hence t_{11}^* is the unique solution of equation (35).

APPENDIX 1b: Proof of Lemma 1(ii)

If $\Delta_1 < 0$, then from equation (36), $F_1(t_1) < 0$. Since $F_1(t_1)$ is a strictly decreasing function of $t_1 \in [t_d, \infty)$ and $F_1(t_1) < 0$ for all $T \in [t_d, \infty)$.Therefore, a value of $T \in [t_d, \infty)$ such that $F_1(t_1) = 0$ cannot found. This completes the proof.

APPENDIX 1c: Proof of Theorem 1(i)

When $\Delta_1 \geq 0$, it is seen that t_{11}^* and T_1^* are the unique solutions of equations (35) and equation (31) respectively from Lemma l(i). Taking the second derivative of $TP_1(t_1, T)$ with respect to t_1 and T, and then finding the values of these functions at the point (t_{11}^*, T_1^*), it follows that

$$\left.\frac{\partial^2 TP_1(t_1, T)}{\partial t_1^2}\right|_{(t_{11}^*, T_1^*)} = -\frac{d}{T_1^*}A_1 < 0$$

$$\left.\frac{\partial^2 TP_1(t_1, T)}{\partial t_1 \partial T}\right|_{(t_{11}^*, T_1^*)} = \frac{d}{T_1^*}C_b$$

$$\left.\frac{\partial^2 TP_1(t_1, T)}{\partial T^2}\right|_{(t_{11}^*, T_1^*)} = -\frac{d}{T_1^*}C_b < 0$$

and

$$\left(\left.\frac{\partial^2 TP_1(t_1, T)}{\partial t_1^2}\right|_{(t_{11}^*, T_1^*)}\right)\left(\left.\frac{\partial^2 TP_1(t_1, T)}{\partial T^2}\right|_{(t_{11}^*, T_1^*)}\right) - \left(\left.\frac{\partial^2 TP_1(t_1, T)}{\partial t_1 \partial T}\right|_{(t_{11}^*, T_1^*)}\right)^2 =$$

$$\frac{d^2C_b}{T_1^{*2}}\left(\left[h_1(t_d\theta + 1) + h_2\left(1 + \frac{t_d\theta}{2}\right)t_d + C\theta + C_b + cI_c(\theta(t_d - M) + 1)\right]\right) > 0 \qquad (57)$$

It is therefore conclude from equation (57) and Lemma 1 that$TP_1(t_{11}^*, T_1^*)$is concave and $(t_{11}^*, T_1^*)$is the global maximum point of$TP_1(t_1, T)$. Hence the values of$t_1$and Tin equation (36) and equation (37) are optimal.

APPENDIX 1d: Proof of Theorem 1(ii)

When $\Delta_1 < 0$,then $F_1(t_1) < 0$for all $t_1 \in [t_d, \infty)$.Therefore,$\frac{\partial TP_1(t_1, T)}{\partial T} = \frac{F_1(t_1)}{T^2} < 0$ for all $t_1 \in [t_d, \infty)$ which implies $TP_1(t_1, T)$ is astrictly decreasing function of t_1. Therefore,$TP_1(t_1, T)$ has a maximum value when t_1 is minimum. Therefore, $TP_1(t_1, T)$ has a maximum value at the point(t_{11}^*, T_1^*) where $t_{11}^* = t_d$ and$T_1^* = \frac{1}{C_b}(A_1t_d - B_1)$. This completes the proof.

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