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ORIGINAL RESEARCH ARTICLE

Solution of System of First Order Nonlinear Non-Homogeneous Fuzzy Ordinary Differential Equations by Embedding Method.

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ABSTRACT
In this study, a system of first order nonlinear non-homogeneous fuzzy ordinary differential equations will be examine in fuzzy environment and solved using embedding method. The results of nonlinear non-homogeneous fuzzy ordinary differential equations are established which followed the form of $S \times S$ matrices and all the components of the matrices are real functions of time denoted by t. The accuracy of the results obtained is tested on some constructed example and recommended that further study should consider odd and even system of nonlinear non-homogeneous fuzzy ordinary differential equations by embedding method.

INTRODUCTION

As the result of the advent of modern computers, numerical solutions to FDEs has attracted researchers' interest because analytical solutions to such problems are available only for some two specific problems. The concept of the fuzzy derivative was first introduced by Zadeh (1965). However, the first formulations of FDEs were presented by Kaleva (1987) and Seikkala (1987). Embedding theorem was generalized by Puri and Ralescu (1983) to define the differential concept of a fuzzy function. Goetschel and Voxman (1986) introduced a new idea in which fuzzy numbers were considered in a topological vector space setting. Friedman et al. (1996) introduced a new approach in defining fuzzy derivatives and provided sufficient conditions for the existence and uniqueness of solutions to fuzzy initial value problems. Buckley (2004) used classical techniques, which are available for solving non-fuzzy differential equations, in solving FDEs. Dubois and Prade (1982) presented a concept of fuzzy derivatives based on the extension principle. Following the extension principle, Bede and Stefanini (2013) introduced new generalized differentiability concepts for fuzzy-valued functions. Mazandarani and Kamyad (2013) defined Caputo-type fuzzy fractional derivatives and adopted a modified fractional Euler method for solving fuzzy fractional initial value problems. There is often a need to interpret and solve the problems operating in the environment's inherent uncertainties and vagueness in many real-world problems, which may be addressed using stochastic, statistical, or fuzzy models. Stochastic and statistical uncertainty occurs due to the natural randomness in the process, which may be generally expressed by a probability density or frequency distribution function, which requires sufficient information about the variables and parameters for the estimation of the distribution. On the other hand, fuzzy set theory refers to the uncertainty when we may have a lack of knowledge or incomplete information about the variables and parameters. In general, science and engineering systems are governed by ordinary and partial differential equations Agarwal et al., (2010). Hence, this topic has been paid more and more attention from many scientists and mathematicians. Ahmadian et al, (2017) state that in the development flow, scientists have proposed many techniques to solve the analysis and numerical solutions of fractional fuzzy differential equations.

Solution of second order linear homogeneous ODEs in fuzzy environment based on the concept of gH derivatives was considered. The linear homogeneous second order ODEs

$\frac{d^2x(t)}{dt^2} = kx(t)$ with the fuzzy initial conditions $x(t_0) = \tilde{a} = (a_1, a_2, a_3, a_4, \omega)$ and

$\frac{d\,x(t_0)}{d\,t} = \tilde{b} = (b_1, b_2, b_3, b_4, \omega)$ was solved with fuzzy number as generalized trapezoidal. However,

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limiting the work to generalized trapezoidal fuzzy numbers without considering the case when $\omega = 1$ or $b_2 = b_4, a_1 = a_3$. This need to be addressed by applying gH derivative concept with generalized triangular fuzzy number (GTFN). In addition to that, also establishing the existence and uniqueness of solution to the given equations is deeming to be necessary. Sankar and Tapan (2015)

In fuzzy differential equations, much attentions have been given to different types of problems with nonlocal conditions. These conditions were used to state the motional phenomena with better effect than the classical conditions, see Ahmadian et al, (2015) and Ahmadian et al, (2018). Therefore, this paper utilizes the Embedding method to solve a system of first-order nonlinear non-homogeneous fuzzy ordinary differential equations. The results of this study will give a clear distinction between this study and the existing ones.

METHOD

Formulation of Third Order ODEs with Fuzzy Initial Conditions

Based on the works of Sankar and Tapan (2015), we therefore, formulated third order nonlinear ODE (1) with fuzzy initial conditions (2)

$$y''' + yy' = ky + y_c \tag{1}$$

where $y(t) = \left(\underline{y}(t,\alpha), \overline{y}(t,\alpha)\right)$, with $y \neq y_c$ and k is constant with the following fuzzy initial condition

$$\begin{cases} y(0) = x_0 = (\underline{x}_0, \overline{x}_0) \\ y'(0) = y_0 = (\underline{y}_0, \overline{y}_0). \\ y''(0) = z_0 = (\underline{z}_0, \overline{z}_0) \end{cases} \tag{2}$$

Equation (1) is transformed into a system of three first order ODEs

$$\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = -y_1 y_2 + k y_1 + y_c \end{cases} \tag{3}$$

where y_c is the inhomogeneous term and k is constant. Equation can therefore transform in the form of system of first order FODE below,

$$\sum_{i=1}^{3} A_i(t) Y^i(t) = F(t),\ Y^{(i)}(0) = G_i, i = 1,2,3 \tag{4}$$

where

$$Y^{(i)} = \begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} \tag{5}$$

$$F(t) = \begin{pmatrix} y_2 \\ y_3 \\ -y_1 y_2 + k y_1 + y_c \end{pmatrix} \tag{6}$$

$A_1(t), A_2(t), A_3(t)$ are $S \times S$ matrices and every component of them is real function of t and $A_n(t) = I$ denote the $S \times S$ identity matrix, Y, G_i and F are fuzzy S - dimensional vectors. The rth components will be represented by the following equations

$$Y = [Y_1(t,r), Y_2(t,r), Y_3(t,r)]^T \tag{7}$$

$$F = [F_1(t,r), F_2(t,r), F_3(t,r)]^T \tag{8}$$

$$G_i = [g_{i1}(r), g_{i2}(r), g_{i3}(r)]^T \tag{9}$$

Equations (7), (8) and (9) are in transpose form

Fuzzy Derivative

Let $f : (a,b) \to R_F$ and $t_0 \in (a,b)$, if there exists $f'(t_0) \in R_F$ such that for all $h > 0$ sufficiently small, $\exists\, f'(t_0)$ such that:

(i) $f'(t_0)$ is H -differentiable, then $f'(t_0)$ is called fuzzy derivative of $f(t)$; and

(ii) if $f'(t_0)$ is (ii)- gH -differentiable, then $f'(t_0)$ is called generalized fuzzy derivative of $f(t)$

RESULTS

Solving equation (3) by the Embedding method and transforming in parametric system form we have.

$$\sum_{i=1}^{3} \underline{a}_{d,k}^{i}(t) \underline{y}_k^{(i)}(t,r) = \underline{f}_d(t,r),\ d = 1,2,3,\ i = 1,2,3 \tag{10}$$

$$\sum_{i=1}^{3} \overline{a}_{d,k}^{i}(t) \overline{y}_k^{(i)}(t,r) = \overline{f}_d(t,r),\ d = 1,2,3,\ i = 1,2,3 \tag{11}$$

$$\underline{y}_j^{(i)}(0,r) = g_{i,j}(r), i = 1,2,3 \tag{12}$$

$$\overline{y}_j^{(i)}(0,r) = \overline{g}_{i,j}(r), i = 1,2,3 \tag{13}$$

For some $r \in [0,1]$. Define the functions $b^i_{d,k}(t)$ and $c^i_{d,k}(t)$ as

$$b^i_{d,k}(t) = \begin{cases} 0, & a^i_{j,k} < 0 \\ a^i_{j,k}(t) & a^i_{j,k}(t) \geq 0 \end{cases} \quad 14)$$

$$c^i_{j,k}(t) = \begin{cases} a^i_{j,k}(t), & a^i_{j,k} < 0 \\ 0, & a^i_{j,k}(t) \geq 0 \end{cases} \quad (15)$$

Now, equations (10), (11), (12) and (13) can be transformed into the following systems of equations

$$\sum_{i=1}^{3} \left[b^i_{d,k}(t) \left(Y_l\right)^{(i)}_k (t,r) + c^i_{d,k}(t) \left(Y_l\right)^{(i)}_k (t,r) \right] = \left(f_l \right)_d (t,r),\ d = 1,2,3,\ i = 1,2,3 \quad (16)$$

$$\sum_{i=1}^{3} \left[c^i_{d,k}(t) \left(Y_l\right)^{(i)}_k (t,r) + b^i_{d,k}(t) \left(Y_l\right)^{(i)}_k (t,r) \right] = \left(f_u \right)_d (t,r),\ d = 1,2,3,\ i = 1,2,3 \quad (17)$$

$$(Y_l)^{(i)}_j (0,r) = (g_l)_{i,j}(r), i = 1,2,3 \quad (18)$$

$$(Y_u)^{(i)}_j (0,r) = (g_u)_{i,j}(r), i = 1,2,3 \quad (19)$$

$$\sum_{i=1}^{3} A_i(t) Z^{(i)}(t,r) - F(t,r) = 0 \quad (20)$$

where

$$Z(t,r) = \left[Z_1(t,r), Z_2(t,r), Z_3(t,r) \right]^T \quad (21)$$

$$F(t,r) = \left[F_1(t,r), F_2(t,r), F_3(t,r) \right]^T \quad (22)$$

Such that $A_0(t), A_1(t), A_2(t), A_3(t)$ are $S \times S$ matrices and all components of the matrices are real function of time denoted by t and $A_p(t) = I_s$ where $p = 1,2,3$ and I_s denote the $S \times S$ identity matrix. The system of equation (20) can be written in the form

$$N_i \left[Z_1(t,r), Z_2(t,r), Z_3(t,r) \right] = 0,\ i = 1,2,3 \quad (23)$$

where N_i are either linear or nonlinear operators depending on N_i that represent the whole of equation (1).

Constructed Example

Consider the following equation, using the embedding method we have

$$(1-q) L_i \left[\phi_i(t,r;q) - Z_{i,0}(t,r) \right] = qh \quad (24)$$

where $q \in [0,1]$ is an embedding parameter, h is a non-zero auxiliary parameter, L_i, $i = 1,2,3$ are auxiliary linear or nonlinear operators, $Z_{i,0}(t,r)$ are initial guesses, $Z_i(t,r)$ and $\phi_i(t,r;q)$, putting $q = 0$ and $q = 1$ in equation (24) we have $\phi_i(t,r;0) = Z_{i,o}(t,r)$ and $\phi_i(t,r;1) = Z_i(t,r)$. By using Taylor series expansion of we have

$$\phi_1(t,r;q) = Z_{i,0}(t,r) + \sum_{m=1}^{3} Z_{i,m}(t,r) q^m \quad (25)$$

Substituting $q = 1$ in equation (24) we have

$$\phi_1(t,r;1) = Z_{i,0}(t,r) + \sum_{m=1}^{3} Z_{i,m}(t,r) \quad (26)$$

Differentiating equation (26) and divide by $m!$, we have

$$L_i \left[Z_{i,m}(t,r) - X_m Z_{i,m-1}(t,r) \right] = hR_{i,m} \left[Z_{1,m-1}, Z_{2,m-2},\ Z_{3,m-3} \right], i = 1,2,3 \quad (27)$$

Validation of the Results Based on the work of Sadeghi et al., (2011)

Consider the initial value problem (28) in Sadeghi et al., (2011)

$$\dot{X}(t)=\alpha(A-I_n)\left[t(A-I_n)+I_n\right]^{-1}X(t),\quad X(0)=X_0 \qquad (28)$$

where A is $n\times n$ matrix that has nonnegative eigenvalues in real axis, X_0 is fuzzified and embedding method is applied on the fuzzified initial value problem. Using some special functions of matrix, the necessary and sufficient condition for the existence of fuzzy solution $X(t)$ is presented.

Consider the homogeneous system of FODE (28) and the following equation

$$\dot{X}(t)=\alpha(A-I_n)\left[t(A-I_n)+I_n\right]^{-1}X(t)+f(t,X),\ X(0)=X_0 \qquad (29)$$

Transforming equation (29) into system using equations (16) and (17). By means of generalizing the embedding method using the relation obtained in equation (23) with the constructed example (24)

$(1-q)L_i\left[\phi_i(t,r;q)-Z_{i,0}(t,r)\right]=qh$ where $q\in[0,1]$ is an embedding parameter, h is a non-zero auxiliary parameter, L_i, $i=1,2,3$ are auxiliary nonlinear operators, $Z_{i,0}(t,r)$ are initial guesses, $Z_i(t,r)$ and $\phi_i(t,r;q)$, when $q=0$ and $q=1$, then by substituting values of q in equation (29) we have $\phi_i(t,r;0)=Z_{i,o}(t,r)$ and $\phi_i(t,r;1)=Z_i(t,r)$ hold. The solutions $\phi_i(t,r;q)$ varies from the initial guesses $Z_{i,0}(t,r)$ to the solutions $Z_i(t,r)$. By taylor series expansion $\phi_i(t,r;q)$ we have $X(t)=\left[t(A-I_n)+I_n\right]^{\alpha}X_0$. Which corresponds to the equations below

$$\phi_1(t,r;q)=Z_{i,0}(t,r)+\sum_{m=1}^{3}Z_{i,m}(t,r)q^m$$

DISCUSSION

In this study, we consider equation (1) and transformed into system of FODE (3). It is discovered that FLT cannot solve system of FODEs. In this regard, the use the FEM is considered as an option. The first order system is transformed in to a Matrix form as equation (6) using FEM and the results are what appeared in equations (20) and (23) respectively by using the fact that $A_0(t),A_1(t),A_2(t),A_3(t)$ are $S\times S$ matrices and all components of the matrices are real function of time denoted by t and $A_p(t)=I_s$ where $p=1,2,3$ and I_s denote the $S\times S$ identity matrix. Example constructed using the main results shows that FEM is a very essential method for solving system of first order fuzzy ordinary differential equation.

CONCLUSION AND RECOMMENDATIONS

In this study, first order nonlinear non homogeneous fuzzy ordinary differential equation was transformed into system and solved by Embedding method. Hence the results obtained allowed us to conclude that the embedding method is very efficient in finding the solution of system of FODEs. It is therefore recommended that further study should consider odd and even system of nonlinear non-homogeneous fuzzy ordinary differential equations by embedding method.

CONTRIBUTION TO KNOWLEDGE

i. Based on the works of Sankar and Tapan (2015), we therefore, formulated third order nonlinear ODE (1) with fuzzy initial conditions.

ii. First order nonlinear non homogeneous fuzzy ordinary differential equation was transformed into system and solved by Embedding method.

REFERENCES

Agarwal, R.P. Lakshmikantham, V. Nieto, J.J. (2010). On the concept of solution for fractional differential equations with uncertainty, Nonlinear Analysis., **72**, 2859-2862. **[Crossref]**

Ahmadian, A. Chang, C.S. Salahshour, S. (2017). Fuzzy approximate solutions to fractional differential equations under uncertainty: Operational matrices approach, IEEE Tran. Fuzzy Syst., **25**, 218-236. **[Crossref]**

Ahmadian, A. Salahshour, S. Baleanu, D. Amirkhani, H. Yunus, R. (2015). Tau method for the numerical solution of a fuzzy fractional kinetic model and its application to the Oil Palm Frond as a promising source of xylose, Journal of Computer Physics, **294**, 562-584. **[Crossref]**

Ahmadian, A. Salahshour, S. Chan, C. S., Baleanu, D. (2018). Numerical solutions of fuzzy differential equations by an effcient Runge-Kutta method with generalized differentiability, Fuzzy Sets System, **331**, 47-67. **[Crossref]**

Bede, B., Stefanini, L. (2013). Generalized differentiability of fuzzy-valued functions. Fuzzy Sets Syst. 230, 119–141 **[Crossref]**

Buckley, J.J. (2004) Fuzzy Probabilities and Fuzzy Sets for Web Planning. Springer BerlinnHeidelberg, Berlin, Heidelberg

Dubois, D., Prade, H. (1982). Towards fuzzy differential calculus part 3: Differentiation. Fuzzy Sets Syst. 8, 225–233 [**Crossref**]

Friedman, M., Ma, M., Kandel, A. (1996). Fuzzy Derivatives and Fuzzy Cauchy Problems Using LP Metric. 57–72. [**Crossref**]

Goetschel, R. and Voxman, W. (1986) Elementary Fuzzy Calculus. Fuzzy Sets and Systems, 18, 31-43. [**Crossref**]

Kaleva, O. (1987). Fuzzy differential equations. Fuzzy Sets Syst. 24, 301–317 [**Crossref**]

Mazandarani, M., Kamyad, A.V. (2013). Modified fractional Euler method for solving Fuzzy Fractional Initial Value Problem. Commun. Nonlinear Sci. Numer. Simul. 18, 12–21 [**Crossref**]

Puri, M.L., Ralescu, D.A. (1983). Differentials of fuzzy functions. J. Math. Anal. Appl. 91, 552–558 . [**Crossref**]

Sadeghi, A., Izani, A. & Jameel, A. F. (2011). Solving systems of fuzzy differential equation. International mathematical forum, 6(42), 2087 – 2100.

Sankar, P. M. & Tapan, K. R. (2015). Solution of second order linear differential equation in fuzzy environment. Journal of Fuzzy Mathematics and Informatics. ISSN: 2093–9310 (print version) ISSN: 2287–6235 (electronic version) http://www.afmi.or.kr.

Seikkala, S. (1987). On the fuzzy initial value problem. Fuzzy Sets Syst. 24, 319–330 [**Crossref**]

Zadeh, L.A. (1965). Fuzzy sets. Inf. Control. 8, 338–353 [**Crossref**]