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ORIGINAL RESEARCH ARTICLE

Numerical Solution of Eighth Order Two Point Boundary Value Problems by Taylor Series Method.

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ABSTRACT
A Taylor series method which is taught at undergraduate levels and which has been hitherto used for the solution of initial value problem is successfully used in this work for the solutions of eighth-order two point boundary-value problems. The method is based on successive differentiation of the governing equation to obtain high order derivatives and then evaluated at the boundary point x=a. The solution is expressed inform a Taylor series with the unknown coefficients at a point x=a. By applying boundary conditions at x=b in the Taylor series solution, the system of unknown coefficient is obtained. After solving the system, then unknown coefficient are determined. The procedure is applied on both linear and nonlinear boundary-value problems. A comparison of the results obtained by the present method with results obtained by other methods reveals that the present method is simple, effective and also is in good agreement with the previous result and exact solution as showing in the tables and figures.

INTRODUCTION

Many engineering problems are models by eight order two point boundary value problems. The eight order boundary value problems occur in the study of fluid dynamics, astrophysics, hydrodynamic, hydromagetic stability, astronomy, beam and long wave theory, induction motors engineering and applied physicis El-Gamel and Abdrabou, (2019). Also in (Chandrasekhar 1961) stated that when an infinite horizontal layer of fluid is heated from below and is under the action of rotation, instability sets, when the instability sets in as over stability, it is modeled by an eighth-order ordinary differential equation. Furthermore (Agarwal 1986) discussed the existence and uniqueness for the solution of higher order ordinary differential equation. Solving such boundary value problems analytically is likely only in very rare cases. As a result, numerous numerical methods have been developed over time to approximate the solution for these boundary value problems.

Due it is importance many authors have developed numerical methods for the solution of these problems. Here are only a limited authors who have provided numerical methods for solving the BVPs. Twizell et al., (1994) have develop second order finite-difference methods for solving eight - tenth-and twelfth order value problem arising in study of the effect of rotation on a horizontal layer of fluid heated from below. He, (2007) has applied Variational iteration method for the Numerical solution of eight order initial-boundary value problems. (Mestrovic 2007) used A modified Adomian decomposition method for the numerical solution of eight order boundary value problems. Siddiqi and Iftikhar, (2013) have used the homotopy analysis method (HAM) for the solution of seventh, eighth and tenth order boundary value problems and obtained the approximate solution in terms of convergence series. Xu and Zhou, (2015) have proposed A Collection method based on the second kind chebyshev wavelets for the numerical solution of eight order two point boundary value problems and initial value problems and also derived second kind chebyshev wavelets operation matrix of integration and used it to transform the problem to a system of algebraic equation.

In Elahi et al, (2016) the Authors have introduced Galerking method using legendary polynomial as a basis function to solve the eight order linear two point boundary value problems. Porshokouhi et al, (2011) have considered Variational iteration method (VIM) to solve eighth order boundary value problems. Al-mamun, el al, (2019) have introduced some basic idea of Variational iteration method for the numerical solution of eighth order boundary value by using suitable transformation to the equivalent system of integral equation. Khalid et al, (2019) the Authors studied the Cubic B-spline to find the numerical solution of linear and non-linear eight order boundary value problems. Reddy et al, (2017) proposed

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Haar wavelet collocation method (HWCM) which is robust and accurate for numerical solution of 8th order boundary value problems. El-Gamel and Abdrabou, (2019) have used sinc-Galerkin method for the numerical solution of eight order boundary value problems. In Amin et al, (2021) the Authors study Haar technique for the solution of both nonlinear and linear eight-order boundary value problems. *Haq. and* Sohaib, (2021) suggested Legendre wavelet collocation method for solving high-order boundary value problems numerically. Raji et al., (2023) proposed the use of First-kind Chebyshev polynomials as the basis functions for the approximate solution of eight order boundary value problems. . He (2020) applied Taylor series to solve third - order boundary value problems appearing in Architectural engineering. Motivated by He (2020), we investigated numerical solution of eight order boundary value problem by using Taylor series expansion.

This paper is organized as follows: Section 2 Presents the methodology. Section 3 presents the Numerical results while Section 4 highlights the conclusion of the paper.

METHODOLOGY OF TAYLOR SERIES EXPANSION

In this paper, linear and nonlinear eighth-order boundary value problems are consider in the form of

$$y^{(8)}(x) = f\left(x, y, y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}, y^{(5)}, y^{(6)}, y^{(7)}\right) \quad (2.1)$$

Subject to the following boundary conditions,

$$\left.\begin{array}{ll} y(a) = \alpha_0 & y(b) = \beta_0 \\ y^{(1)}(a) = \alpha_1 & y^{(1)}(b) = \beta_1 \\ y^{(2)}(a) = \alpha_2 & y^{(2)}(b) = \beta_2 \\ y^{(3)}(a) = \alpha_3 & y^{(3)}(b) = \beta_3 \end{array}\right\} \quad (2.2)$$

Where f *and* $y(x)$ assumed real and as many times differentiable as required for $x \in [a, b]$and $\alpha i, \beta i$, i =0, 1, 2, 3, are finite real constant.

Having differentiating (2.1) n times successively the following results were obtained:

$$y^{(8+1)}(x) = f'\left(x, y, y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}, y^{(5)}, y^{(6)}, y^{(7)}\right) = f_1(x, y, y', y'', y''', \dots, y^{(8)}) \quad (2.3)$$

$$y^{(8+2)}(x) = f''\left(x, y, y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}, y^{(5)}, y^{(6)}, y^{(7)}\right) = f_2(x, y, y', y'', y''', \dots, y^{(9)}) \quad (2.4)$$

$$y^{(8+3)}(x) = f'''\left(x, y, y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}, y^{(5)}, y^{(6)}, y^{(7)}\right) = f_3(x, y, y', y'', y''', \dots, y^{(10)}) \quad (2.5)$$

$$y^{(8+n)}(x) = f^n\left(x, y, y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}, \dots, y^{(7)}\right) = f_n(x, y, y', y'', y''', \dots, y^{(8+(n-1))}) \quad (2.6)$$

Assumed that:

$$\left.\begin{array}{l} y^{(4)}(a) = \gamma_1 \\ y^{(5)}(a) = \gamma_2 \\ y^{(6)}(a) = \gamma_3 \\ y^{(7)}(a) = \gamma_4 \end{array}\right\} \quad (2.7)$$

Where $\gamma_1, \gamma_2, \gamma_3$ *and* γ_4 are unknown's parameters to be determined?

Evaluating (2.1) and (2.3-2.6) at x=a, using equation (2.7), and considering (2.2) at x=a, the following differentials equation were obtained:

$$y^{(8)}(a) = f_0\left(x, y, y^{(1)}, y^{(2)}, y^{(3)}, \dots, y^{(7)}\right)|_{x=a} = g_0(\gamma_1, \gamma_2, \gamma_3, \gamma_4) \quad (2.8)$$

$$y^{(9)}(a) = f_1\left(x, y, y^{(1)}, y^{(2)}, y^{(3)}, \dots, y^{(8)}\right)|_{x=a} = g_1(\gamma_1, \gamma_2, \gamma_3, \gamma_4) \quad (2.9)$$

$$y^{(10)}(a) = f_2\left(x, y, y^{(1)}, y^{(2)}, y^{(3)}, \dots, y^{(9)}\right)|_{x=a} = g_2(\gamma_1, \gamma_2, \gamma_3, \gamma_4) \quad (2.10)$$

$$y^{(8+n)}(a) = f_n\left(x, y, y^{(1)}, y^{(2)}, y^{(3)}, \dots, y^{(8+(n-1))}\right)|_{x=a} = g_n(\gamma_1, \gamma_2, \gamma_3, \gamma_4) \quad (2.11)$$

The assumed Taylor series approximate solution can be written as

$$y(x) = y(a) + y^{(1)}(a)x + \frac{y^{(2)}(a)x^2}{2!} + \frac{y^{(3)}(a)x^3}{3!} + \cdots + \frac{y^{(8+n)}(a)x^{8+n}}{(8+n)!} \quad (2.12)$$

By substituting $y(a), y^{(1)}(a), y^{(2)}(a), \dots, y^{(8+n)}(a)$ in (2.12) the following result was obtained:

$$y(x) = \alpha_0 + \alpha_1 x + \frac{\alpha_2 x^2}{2!} + \frac{\alpha_3 x^3}{3!} + \frac{\gamma_1 x^4}{4!} + \frac{\gamma_2 x^5}{5!} + \frac{\gamma_3 x^6}{6!} + \frac{\gamma_4 x^7}{7!} + \frac{g_0(\gamma_1, \gamma_2, \gamma_3, \gamma_4)x^8}{8!} + \frac{g_1(\gamma_1, \gamma_2, \gamma_3, \gamma_4)x^9}{9!} + \cdots + \frac{g_n(\gamma_1, \gamma_2, \gamma_3, \gamma_4)x^{8+n}}{(8+n)!} \quad (2.13)$$

To determine the unknown's parameters$\gamma_1, \gamma_2, \gamma_3, \gamma_4$, the boundary conditions at $x = b$ was used.

NUMERICAL RESULTS.

Some examples have been considered to test the suggested method's accuracy. The method's numerical findings further demonstrate its performance.

Example 3.1. Consider the Linear boundary value problem, Khalid et al, (2019)

$$y^{(8)}(x) + xy(x) = -e^x(48 + 15x + x^3) \quad (3.1)$$

Subject to the boundary conditions

$$y(0) = 0, y^{(1)}(0) = 1, y^{(2)}(0) = 0, y^{(3)}(0) = -3$$

$$y(1) = 0, y^{(1)}(1) = -e, y^{(2)}(1) = -4e, y^{(3)}(1) = -9e \quad (3.2)$$

The exact solution for the above example is given by $y(x) = x(1-x)e^{x}$

Assumed that

$$\left.\begin{aligned} y^{(4)}(0) &= \alpha_1 \\ y^{(5)}(0) &= \alpha_2 \\ y^{(6)}(0) &= \alpha_3 \\ y^{(7)}(0) &= \alpha_4 \end{aligned}\right\} \quad (3.3)$$

By successive differentiation of (3.1) and evaluating each derivatives at $x = 0$ the following results were obtained.

$$y(0) = 0, y^{(1)}(0) = 1, y^{(2)}(0) = 0, y^{(3)}(0) = -3, y^{(4)}(0) = \alpha_1, y^{(5)}(0) = \alpha_2, y^{(6)}(0) = \alpha_3, y^{(7)}(0) = \alpha_4, y^{(8)}(0) = -48, y^{(9)}(0) = -63, y^{(10)}(0) = -80, y^{(11)}(0) = -99, y^{(12)}(0) = -120, y^{(13)}(0) = -183 - 5\alpha_1, y^{(14)}(0) = -258 - 6\alpha_2, y^{(15)}(0) = -363 - 7\alpha_3, y^{(16)}(0) = -504 - 8\alpha_4, y^{(17)}(0) = -255, y^{(18)}(0) = -288, y^{(19)}(0) = -323 \quad (3.4)$$

The assumed approximate Taylor series solution of equation (3.1) at $x = 0$ as:

$$y(x) = y(0) + y^{(1)}(0)x + \frac{y^{(2)}(0)x^2}{2!} + \frac{y^{(3)}(0)x^3}{3!} + \frac{y^{(4)}(0)x^4}{4!} + \frac{y^{(5)}(0)x^5}{5!} + \frac{y^{(6)}(0)x^6}{6!} + \frac{y^{(7)}(0)x^7}{7!} + \frac{y^{(8)}(0)x^8}{8!} + \frac{y^{(9)}(0)x^9}{9!} + \frac{y^{(10)}(0)x^{10}}{10!} + \frac{y^{(11)}(0)x^{11}}{11!} + \frac{y^{(12)}(0)x^{12}}{12!} + \frac{y^{(13)}(0)x^{13}}{13!} + \frac{y^{(14)}(0)x^{14}}{14!} + \frac{y^{(15)}(0)x^{15}}{15!} + \frac{y^{(16)}(0)x^{16}}{16!} + \frac{y^{(17)}(0)x^{17}}{17!} + \frac{y^{(18)}(0)x^{18}}{18!} + \frac{y^{(19)}(0)x^{19}}{19!}. \quad (3.5)$$

By substituting (3.4) into (3.5) the following results was obtained.

$$y(x) = x - \frac{x^3}{3!} + \frac{\alpha_1 x^4}{4!} + \frac{\alpha_2 x^5}{5!} + \frac{\alpha_3 x^6}{6!} + \frac{\alpha_4 x^7}{7!} - \frac{48x^8}{8!} - \frac{68x^9}{9!} - \frac{80x^{10}}{10!} - \frac{99x^{11}}{11!} - \frac{120x^{12}}{12!} + \frac{(-183-5\alpha_1)x^{13}}{13!} + \frac{(-258-6\alpha_2)x^{14}}{14!} + \frac{(-363-7\alpha_3)x^{15}}{15!} + \frac{(-504-8\alpha_4)x^{16}}{16!} + \frac{(-255)x^{17}}{17!} + \frac{(-288)x^{18}}{18!} + \frac{(-323)x^{19}}{19!} \quad (3.6)$$

The boundary condition at $x = 1$ was used in (3.6) then, the following system of equation were obtained

$$\left.\begin{aligned} \frac{3192294618517393}{6402373705728000} + \frac{51891839}{1245404160}\alpha_1 + \frac{121080959}{14529715200}\alpha_2 + \frac{259459199}{186810624000}\alpha_3 + \frac{518918399}{2615348736000}\alpha_4 &= 0 \\ \frac{-3273773663057201}{6402373705728000} + \frac{15966719}{95800320}\alpha_1 + \frac{43243199}{1037836800}\alpha_2 + \frac{103783679}{12454041600}\alpha_3 + \frac{227026799}{163459296000}\alpha_4 &= -e \\ -\frac{64472779198147}{20922789888000} + \frac{3991679}{7983360}\alpha_1 + \frac{13305599}{79833600}\alpha_2 + \frac{37065599}{889574400}\alpha_3 + \frac{90810719}{10897286400}\alpha_4 &= -4e \\ -\frac{73359837326771}{20922789888000} + \frac{725759}{725760}\alpha_1 + \frac{3326399}{6652800}\alpha_2 + \frac{11404799}{68428800}\alpha_3 + \frac{32432399}{778377600}\alpha_4 &= -9e \end{aligned}\right\} \quad (3.7)$$

Solving (3.7), the unknowns' parameters can be identified as

$$\alpha_1 = -8.0000000, \alpha_2 = -15.000000, \alpha_3 = -24.00001, \alpha_4 = -35.00000$$

Hence the required approximate solution was

$$y(x) = \frac{1}{2}x^3 - 0.3333333333x^4 - .1250000000x^5 - 0.3333334722x^6 - 0.006944444444x^7 - \frac{1}{840}x^8 - \frac{1}{5760}x^9 - \frac{1}{45360}x^{10} - \frac{1}{403200}x^{11} - \frac{1}{3991680}x^{12} - 2.296443269 * 10^{-8}x^{13} - 1.927085260 * 10^{-9}x^{14} - 1.491196392 * 10^{-10}x^{15} - 1.070602922 * 10^{-11}x^{16} - \frac{1}{1394852659200}x^{17} - \frac{1}{22230464256000}x^{18} - \frac{1}{376610217984000} - x^{19}$$

Table 3.1 shows the comparison between exact and the Taylor series solutions along with the errors obtained by using the proposed method and Cubic B-splines Khalid et al, (2019).

Table 3.1 Numerical result for example 3.1

X	Exact Solution	Taylor series solution	Absolute Error	Absolute Error of CBS Khalid et al, (2019)
0	0	0	0	0
0.1	0.09946538262	0.09946538264	2.0 E-10	4.13E-5
0.2	0.1954244413	0.1954244414	1.0E-10	1.87E-4
0.3	0.2834703497	0.2834703496	1.0E-10	4.43E-4
0.4	0.3580379275	0.3580379275	0	7.48E-4
0.5	0.4121803178	0.4121803176	2.0E-10	9.87E-4
0.6	0.4373085120	0.4373085122	2.0 E-10	1.04E-4
0.7	0.4228880685	0.4228880686	1.0E-10	8.68E-4
0.8	0.3560865485	0.3560865483	2.E-10	5.15E-4
0.9	0.2213642800	0.2213642799	1.0E-10	1.54 E-4
1	0	1.214020131*10^{-10}	1.214020131E-10	0

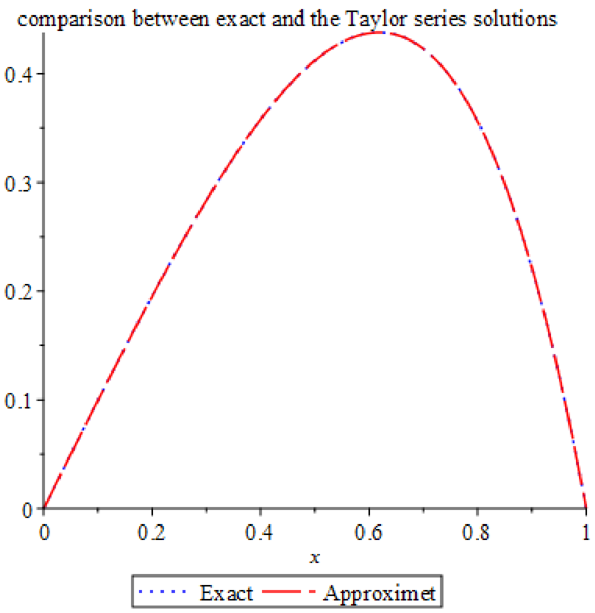

Figure 3.1 Graphical comparison between exact and Taylor series solution example 3.1.

It can be observed that from table 3.1 and figure 3.1 the exact solution and Taylor series solution are in good agreement

Example 3. 2 Consider the Linear boundary value problem. (Elahi et al, 2016)

$$y^{(8)}(x) + y^{(7)}(x) + 2y^{(6)}(x) + 2y^{(5)}(x) + 2y^{(4)}(x) + 2y^{(3)}(x) + 2y^{(2)}(x) + y^{(1)}(x) + y(x) = 14\cos x - 16\sin x - 4x\sin x \quad (3.8)$$

Subject to the boundary conditions

$$y(0) = 0, y^{(1)}(0) = -1, y^{(2)}(0) = 0, y^{(3)}(0) = 7$$

$$y(1) = 0, y^{(1)}(1) = 2\sin 1, y^{(2)}(1) = 4\cos 1 + 2\sin 1, y^{(3)}(1) = 6\cos 1 - 6\sin 1 \quad (3.9)$$

The exact solution for the above example is given by $y(x) = (x^2 - 1)\sin x$

It assumed that

$$\left.\begin{array}{l} y^{(4)}(0) = \gamma_1 \\ y^{(5)}(0) = \gamma_2 \\ y^{(6)}(0) = \gamma_3 \\ y^{(7)}(0) = \gamma_4 \end{array}\right\} \quad (3.10)$$

Following in example (3.1) the Taylor series approximate solution was obtained as:

$$y(x) = -x + \frac{7x^3}{3!} + \frac{\gamma_1 x^4}{4!} + \frac{\gamma_2 x^5}{5!} + \frac{\gamma_3 x^6}{6!} + \frac{\gamma_4 x^7}{7!} + \frac{(1-\gamma_4-2\gamma_3-2\gamma_2-2\gamma_1)x^8}{8!} + \frac{(-\gamma_4-30)x^9}{9!} + \frac{(-1+\gamma_4+2\gamma_3+2\gamma_2+2\gamma_1)x^{10}}{10!} + \frac{(\gamma_4+\gamma_1+68)x^{11}}{11!} + \frac{(22-\gamma_4-2\gamma_3-\gamma_2-2\gamma_1)x^{12}}{12!} + \frac{(\gamma_3-\gamma_4-144)x^{13}}{13!} + \frac{(2\gamma_4+2\gamma_3+2\gamma_2+2\gamma_1-44)x^{14}}{14!} + \frac{(-2\gamma_3-2\gamma_2-2\gamma_1+229)x^{15}}{15!} + \frac{(-2\gamma_4-2\gamma_3-2\gamma_2-2\gamma_1+44)x^{16}}{16!} + \frac{(2\gamma_3+2\gamma_2+2\gamma_1-291)x^{17}}{17!} + \frac{(2\gamma_4+2\gamma_3+2\gamma_2+2\gamma_1-44)x^{18}}{18!} + \frac{(-4\gamma_3-2\gamma_2-2\gamma_1+382)x^{19}}{19!} \quad (3.11)$$

Then we obtained the following system of equation at a boundary point when $x = 1$ from (3.11)

$$\left.\begin{array}{l} \frac{57577614187703}{345582671616000} + \frac{460234677601733}{11058645491712000}\gamma_1 + \frac{40309679192557}{4865804016353280}\gamma_2 + \frac{81492051832721}{60822550204416000}\gamma_3 + \frac{4383161819}{25609494822912}\gamma_4 = 0 \\ \frac{1000159050749623}{400148356608000} + \frac{133069798961219}{800296713216000}\gamma_1 + \frac{52852014908827}{1280474741145600}\gamma_2 + \frac{12711867664913}{1600593426432000}\gamma_3 + \frac{15987863813}{13680285696000}\gamma_4 = 2\sin(1) \\ \frac{59243988196381}{8468748288000} + \frac{3609675962177}{7258927104000}\gamma_1 + \frac{4485443381029}{27360571392000}\gamma_2 + \frac{6924873167087}{177843714048000}\gamma_3 + \frac{70858088881}{10461394944000}\gamma_4 = 4\cos(1) + 2\sin(1) \\ \frac{72896831602759}{10461394944000} + \frac{10291391987783}{10461394944000}\gamma_1 + \frac{1445846824393}{2988969984000}\gamma_2 + \frac{786653145727}{5230697472000}\gamma_3 + \frac{21030461971}{653837184000}\gamma_4 = 6cos(1) - 6sin(1) \end{array}\right\} \quad (3.12)$$

After solving (3.12), the unknowns' parameters can be identified as:

$$\gamma_1 = -0.000008199, \gamma_2 = -20.99985894, \gamma_3 = -0.00101123, \gamma_4 = 43.00311069$$

Hence, the required Taylor Series is

$$y(x) = -x + \frac{7}{6}x^3 3.41625000010^{-7}x^4 - 0.1749988245x^5 - 0.1404486111 * 10^{-5}x^6 + 0.8532363232 * 10^{-2}x^7 - 3.358010913 * 10^{-8}x^8 - 0.201177002 * 10^{-3}x^9 + 3.731123236 * 10^{-10}x^{10} + 0.2780861755 * 10^{-5}x^{11} - 2.532120978 * 10^{-12}x^{12} - 3.003107391 * 10^{-8}x^{13} + 5.121274963 * 10^{-14}x^{14} + 2.072394806 * 10^{-10}x^{15} - 2.133864568 * 10^{-16}x^{16} - 9.362202049 * 10^{-13}x^{17} + 6.960605870 * 10^{-19}x^{18} + 3.312948231 * 10^{-15}x^{19}$$

Table 3.2 shows the comparison between exact and the Taylor series solutions along with the errors obtained by using the proposed method and Legendre Galerkin method (Elahi et al, 2016).

Table 3.2 Numerical result for example 3.2

X	Exact Solution	Taylor series solution	Absolute Error	Absolute Error of Legendre Galerkin method [4]
0	0	0	0	0
0.1	-0.09883508248	-0.09883508250	2.0E-11	5.03751 E-8
0.2	-0.1907225576	-0.1907225578	2.0 E-10	5.1436 E-7
0.3	-0.2689233881	-0.2689233889	8. 0E-10	1.55915 E-6
0.4	-0.3271114075	-0.3271114090	1.5 E-9	2.71487 E-6
0.5	-0.3595691540	-0.3595691558	1.8 E-9	3.26015 E-6
0.6	-0.3613711830	-0.3613711847	1.7 E-9	2.82218 E-6
0.7	-0.3285510205	-0.3285510215	1. 0E-9	1.68491 E-6
0.8	-0.2582481927	-0.2582481931	40. E-10	5.77885 E-7
0.9	-0.1488321128	-0.1488321129	1.0 E-10	5.88442 E-8
1	0	4.709185433*10^{-10}	4.709185433 E-10	

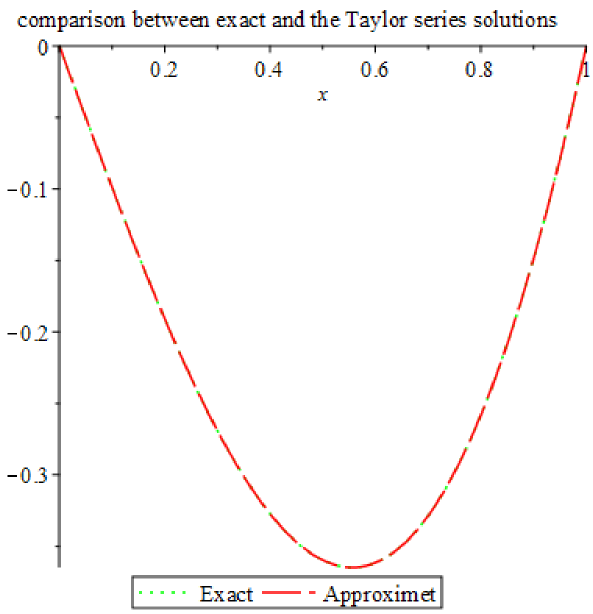

Figure 3.2 Graphical comparison between exact and Taylor series solution example 3.2.

It can be observed that from table 3.2 and figure 3.2 the exact solution and Taylor series solution are in good agreement

Example 3.3 Consider the Nonlinear boundary value problem. (Khalid et al, 2019)

$$y^{(8)}(x) + e^{-x}(y(x))^2 = e^{-x} + e^{-3x} \quad (3.13)$$

Subject to the boundary conditions

$$y(0) = 1, y^{(1)}(0) = 1\,, y^{(2)}(0) = 1\,, y^{(3)}(0) = 1$$

$$y(1) = e^{-1}, y^{(1)}(1) = -e^{-1}\,, y^{(2)}(1) = e^{-1}\,, y^{(3)}(0) = -e^{-1} \quad (3.14)$$

The exact solution for the above example is given by $y(x) = e^{-x}$

Assume that

$$\left.\begin{aligned} y^{(4)}(0) &= \delta_1 \\ y^{(5)}(0) &= \delta_2 \\ y^{(6)}(0) &= \delta_3 \\ y^{(7)}(0) &= \delta_4 \end{aligned}\right\} \quad (3.15)$$

Following in example (3.1) the following Taylor series approximate solution was obtained as:

$$y(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{\delta_1 x^4}{4!} + \frac{\delta_2 x^5}{5!} + \frac{\delta_3 x^6}{6!} + \frac{\delta_4 x^7}{7!} + \frac{x^8}{8!} - \frac{x^9}{9!} + \frac{x^{10}}{10!} - \frac{x^{11}}{11!} - \frac{(-2\delta_1+3)x^{12}}{12!} + \frac{(-20\delta_1-2\delta_2-23)x^{13}}{13!} + \frac{(-120\delta_1+24\delta_2-2\delta_3+167)x^{14}}{14!} + \frac{(560\delta_1-168\delta_2+28\delta_3-2\delta_4-759)x^{15}}{15!} \quad (3.16)$$

Then, the following system of equation was obtained at a boundary point when $x = 1$ from (3.16)

$$\left.\begin{aligned} \frac{6604857841}{19813248000} + \frac{544864211}{13076743680}\delta_1 + \frac{908107181}{108972864000}\delta_2 + \frac{908107199}{653837184000}\delta_3 + \frac{129729599}{653837184000}\delta_4 &= e^{-1} \\ -\frac{14524596773}{29059430400} + \frac{259459037}{1556755200}\delta_1 + \frac{18532799}{444787200}\delta_2 + \frac{1}{120}\delta_3 + \frac{60540479}{43589145600}\delta_4 &= -e^{-1} \\ \frac{1888343}{1556755200} + \frac{24324241}{48648600}\delta_1 + \frac{6177599}{37065600}\delta_2 + \frac{129729601}{3113510400}\delta_3 + \frac{25945919}{3113510400}\delta_4 &= e^{-1} \\ -\frac{52843319}{53222400} + \frac{77759}{77760}\delta_1 + \frac{1663199}{3326400}\delta_2 + \frac{19958401}{119750400}\delta_3 + \frac{9979199}{239500800}\delta_4 &= -e^{-1} \end{aligned}\right\} \quad (3.17)$$

After solving (30), the unknowns' parameters can be identified as:

$\delta_1 = 0.9999918645, \delta_2 = -0.9998624477, \delta_3 = 0.9990551795, \delta_4 = -0.9974235694$

Hence, the required Taylor Series is

$$y(x) = 1 - x + \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3 + 0.04166632769x^4 - 0.008332187064x^5 + 0.001387576638x^6 -$$

$0.0001979015019x^7 + \frac{1}{40320}x^8 - \frac{1}{362880}x^9 + \frac{1}{3628800}x^{10} - \frac{1}{39916800}x^{11} + 2.087709667 * 10^{-9}x^{12} - 6.584226023 * 10^{-9}x^{13} + 2.409563999 * 10^{-10}x^{14} - 8.100430481 * 10^{-13}x^{15}$

Table 3.3 shows the comparison between exact and the Taylor series solutions along with the errors obtained by using the proposed method and Cubic B-splines Khalid et al, (2019).

Table 3.3 Numerical solution of example 3.3

X	Exact Solution	Taylor series solution	Absolute Error	Absolute Error of CBS (Khalid et al, 2019)
0	1	1	0	0
0.1	0.9048374180	0.9048374180	0	3.60 E-6
0.2	0.8187307531	0.8187307529	2.0 E-10	1.21 E-5
0.3	0.7408182207	0.7408182198	9. 0 E-10	2.09 E-5
0.4	0.6703200460	0.6703200446	1.4 E-9	2.59 E-5
0.5	0.6065306597	0 .6065306580	1.7 E-9	2.56 E-5
0.6	0.5488116361	0.5488116344	1.7 E-9	2.07 E-5
0.7	0.4965853038	0.4965853027	1. 1 E-9	1. 33 E-5
0.8	0.4493289641	0.4493289635	6.0 E-10	6.30 E-6
0.9	0.4065696597	0.4065696598	1.0 E-10	1. 59 E-6
1	0.3678794412	0.3678794411	1.0 E-10	0

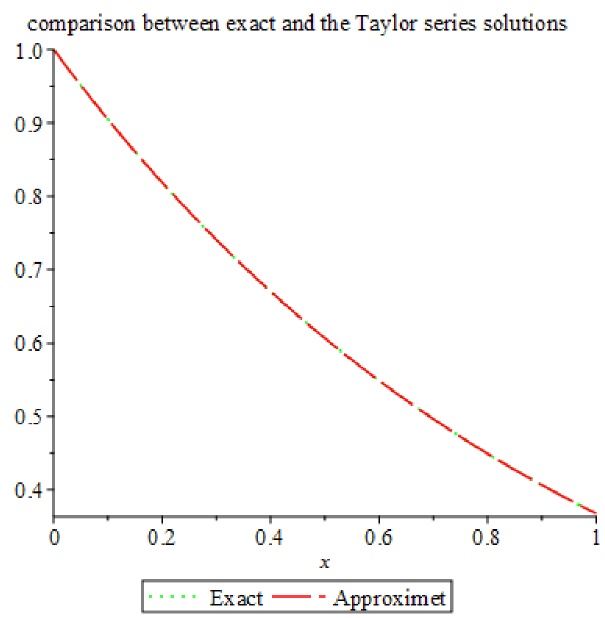

Figure 3.3 Graphical comparison between exact and Taylor series solution example 3.3.

It can be observed that from table 3.3 and figure 3.3 the exact solution and Taylor series solution are in good agreement

Example3. 4 Consider the Nonlinear boundary value problem. (Reddy et al, 2017)

$$y^{(8)}(x) = e^{-x}(y(x))^2 \quad (3.18)$$

Subject to the boundary conditions

$$y(0) = 1, y^{(1)}(0) = 1, y^{(2)}(0) = 1, y^{(3)}(0) = 1$$

$$y(1) = e, y^{(1)}(1) = e, y^{(2)}(1) = e, y^{(3)}(0) = e \quad (3.19)$$

The exact solution for the above example is given by $y(x) = e^x$

Assumed that

$$\left.\begin{aligned} y^{(4)}(0) &= \delta_1 \\ y^{(5)}(0) &= \delta_2 \\ y^{(6)}(0) &= \delta_3 \\ y^{(7)}(0) &= \delta_4 \end{aligned}\right\} \quad (3.20)$$

Following in example (1) the Taylor series approximate solution was obtained as:

$$y(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{\alpha_1 x^4}{4!} + \frac{\alpha_2 x^5}{5!} + \frac{\alpha_3 x^6}{6!} + \frac{\alpha_4 x^7}{7!} + \frac{x^8}{8!} + \frac{x^9}{9!} + \frac{x^{10}}{10!} + \frac{x^{11}}{11!} - \frac{(2\alpha_1 - 1)x^{12}}{12!} + \frac{(2\alpha_2 - 1)x^{13}}{13!} + \frac{(2\alpha_3 - 1)x^{14}}{14!} + \frac{(2\alpha_4 - 1)x^{15}}{15!} \quad (3.21)$$

Then, the following system of equation was obtained at a boundary point when $x = 1$ from (3.21)

$$\left.\begin{aligned} \frac{871792018541}{326918592000} + \frac{9979201}{239500800}\alpha_1 + \frac{25945921}{3113510400}\alpha_2 + \frac{60540481}{43589145600}\alpha_3 + \frac{129729601}{653837184000}\alpha_4 &= e \\ \frac{217965449323}{87178291200} + \frac{3326401}{19958400}\alpha_1 + \frac{9979201}{239500800}\alpha_2 + \frac{25945921}{3113510400}\alpha_3 + \frac{60540481}{43589145600}\alpha_4 &= e \\ \frac{6232047737}{3113510400} + \frac{907201}{1814400}\alpha_1 + \frac{3326401}{19958400}\alpha_2 + \frac{9979201}{239500800}\alpha_3 + \frac{25945921}{3113510400}\alpha_4 &= e \\ \frac{13821829}{13685760} + \frac{181441}{181440}\alpha_1 + \frac{907201}{1814400}\alpha_2 + \frac{3326401}{19958400}\alpha_3 + \frac{9979201}{239500800}\alpha_4 &= e \end{aligned}\right\} \quad (3.22)$$

After solving (3.22), the unknowns' parameters can be identified as:

$\alpha_1 = 0.9999999399, \alpha_2 = 1.000000722, \alpha_3 = 0.9999963139, \alpha_4 = 1.000007510$

Hence, the required Taylor Series is

$$y(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + 0.04166666416x^4 + 0.008333339350x^5 + 0.001388883769x^6 + 0.0001984141885x^7 + \frac{1}{40320}x^8 + \frac{1}{362880}x^9 + \frac{1}{3628800}x^{10} + \frac{1}{39916800}x^{11} + 2.087675448 \times 10^{-9}x^{12} + 1.605906703 \times 10^{-10}x^{13} + 1.147066104 \times 10^{-11}x^{14} + 7.647278592 \times 10^{-13}x^{15}$$

Table 3.4 shows the comparison between exact and the Taylor series solutions along with the errors obtained by using the proposed method and HWCM (Reddy et al, 2017)

Table 3.4 Numerical solution of example 3.4

X	Exact Solution	Taylor series solution	Absolute Error	Absolute Error of (Reddy et al, 2017)
0	1	1	0	
0.1	1.105170918	1.105170918	0	6.6E-12
0.2	1.221402758	1.221402759	1.0E-9	6.9E-11
0.3	1.349858808	1.349858807	1.0E-9	2.1E-10
0.4	1.491824698	1.491824698	0	3.5E-10
0.5	1.648721271	1.648721271	1.0E-9	4.1E-10
0.6	1.822118800	1.822118801	1.0E-9	3.5E-10
0.7	2.013752707	2.013752707	0	2.1E-10
0.8	2.225540928	2.225540927	1.0E-9	7.2E-11
0.9	2.459603112	2.459603111	1.0E-9	8.0E12
1	2.718281828	2.718281829	1.0E-9	

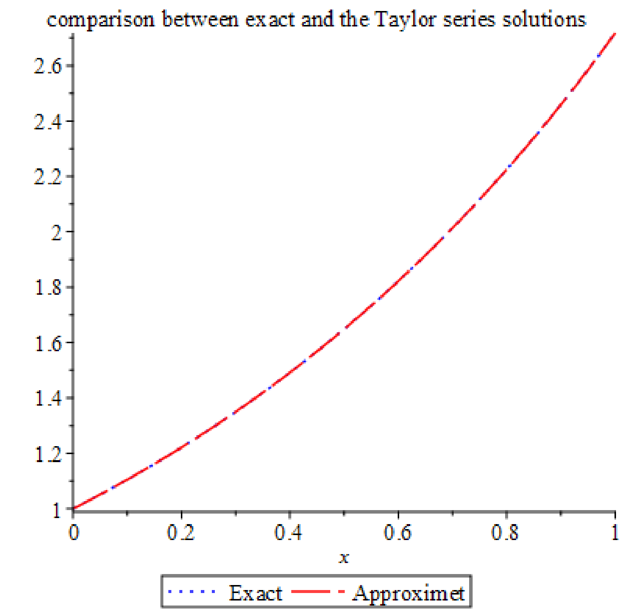

Figure 3.4 Graphical comparison between exact and Taylor series solution example 3.4.

It can be observed that from table 3.2 and figure 3.2 the exact solution and Taylor series solution are in good agreement

CONCLUSION

In this paper Taylor Series Method is employed for the solution of both linear and nonlinear eight order two point boundary value problems. It can be seen from the tables and figures that the numerical results obtained from the proposed method are in good agreement with the exact solution and with some numerical results in the literature, apart it simplicity the method is accurate and reliable.

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