UMYU Scientifica, Vol. 3 NO. 1, March 2024, Pp 048 – 054
https://doi.org/10.56919/usci.2431.005
ISSN: 2955 – 1145 (print); 2955 – 1153 (online)
A periodical of the Faculty of Natural and Applied Sciences, UMYU, Katsina

ORIGINAL RESEARCH ARTICLE

Eleventh Degree Polynomial Series Solution Approach of Special Non-linear Fourth Order Boundary Value Problems.

Nakone Bello[1] and Ibrahim Mutawakilu[1],
[1]Department of Mathematics, Usmanu Danfodiyo University Sokoto. Nigeria.

ARTICLE HISTORY
Received August 02, 2023.
Accepted November 27, 2023.
Published February 22, 2024.

KEYWORDS
Eleventh degree polynomial, fourth order boundary value problems, successive differentiating.

ABSTRACT
In this paper a new eleventh degree polynomial series solution approach is developed for the solution of special non-linear fourth order boundary value problems. The method consist first of obtaining a linear differentials system of twelve equations from the boundary conditions, governing equation and its three successive derivatives which were evaluated at boundary points. Secondly we assume the approximate solution in the form of a polynomial of degree eleventh with twelve unknown coefficients. To determine the unknown coefficients we incorporate the assumed solution into linear differentials systems of twelve equations which results into a linear systems of twelve equations with twelve unknown and which can be solve uniquely. It is clear from the tables and figures that the method is in good agreement with the exact and with some existing results in the literatures. Also it can be seen from example 3.3 that the exact solution is reproduced which is an added advantage of the method.

INTRODUCTION

The numerical solution of boundary value problems plays a crucial role in various fields of science and engineering, where analytical solutions are often difficult or impossible to obtain. In particular, fourth-order nonlinear boundary value problems (BVPs) pose significant challenges due to their complex nature. These problems arise in diverse areas such as fluid mechanics, elasticity theory, and heat transfer; among others (Thenmozhi and Marudai 2021).

To tackle the complexity of such problems, various numerical methods have been developed over the years. Many Scholars have developed numerous numerical methods for highly accurate approximate solutions of Fourth order boundary value problems. For example Hossain and Islam (2014) applied Galerkin weighted residual method for the numerical solutions of the general fourth order linear and nonlinear differential equations. Adeyeye and Omar, (2017). used A $(m+1)$th-step block method by using a modified Taylor series approach to directly solve fourth-order nonlinear boundary value problems. In Wazwaz. (2002). the Authors applied a modified Adomian decomposition method for solving fourth-order boundary value problems. While Kasi Viswanadham et al., (2010) proposed Galerkin method with quintic B-splines as basis functions to find the numerical solution of fourth order boundary value problems. Chen and Cui (2023) studied the existence and uniqueness of solution for fully fourth-order differential equations. The proof will rely on Perov's fixed point theorem in complete generalized metric spaces. Mustafa et al., (2017) presented an iterative collocation numerical approach based on interpolating subdivision schemes for the solution of non-linear fourth order boundary value problems.

Furthermore, Huang, (2021) discuss the existence and approximation solutions for a fourth order nonlinear boundary value problem by using a quasilinearization technique. Singh et al. (2014) applied Green's function and Adomian decomposition method for obtaining approximate Series solutions to fourth-order two-point boundary value problems. Abd-Elhameed et al. (2016) used a novel Galerkin operational matrix of derivatives of some generalized Jacobi polynomials for solving fourth-order linear and nonlinear boundary value problems. Audu et al., (2022) proposed a novel two-step backward differentiation formula in the class of linear multistep schemes with a higher order of accuracy for solving fourth order. Abd-Elhameed and Youssri, (2020) proposed a new explicit solutions for the connection problems between generalizedLucas polynomial sequence and the two polynomials, namely third and fourth kinds of Chebyshev polynomials for numerical solution of certain types fourth-order boundary value problems. Thenmozhi and Marudai, (2021) imported a novel approach which combine Green's function and fixed point iteration to solve the non-linear fourth order boundary value problem.

Correspondence: Ibrahim Mutawakilu. Department of Mathematics, Usmanu Danfodiyo University, Sokoto, Nigeria. ibrahimmutawakkil@gmail.com. Phone Number: +234 703 205 0242.
How to cite: Mutawakilu, I., & Bello, N. (2023). Eleventh Degree Polynomial Series Solution Approach of Special Non-linear Fourth Order Boundary Value Problems. *UMYU Scientifica*, 3(1), 48 – 54. https://doi.org/10.56919/usci.2431.005

In Sohel et al., (2022) the authors presented residual correction procedure for improving the Galerkin for approximate solutions of higher order boundary value problem.

One promising approach is the use of polynomial series solutions, which offer a flexible and efficient way to approximate the unknown function in the problem domain. In this context, the eleventh degree polynomial series solution approach stands out as a powerful tool for solving fourth-order special nonlinear BVPs.

This paper is structured as follows: Section 2 describes the methods. Section 3 discusses the numerical results, while Section 4 summarizes the paper's conclusion.

METHODOLOGY

Given the following nonlinear two-point boundary value problem

$$y^{(4)}(x) = f(x, y) \quad , x \in [0, b] \quad (2.1)$$

Subject to the following boundary conditions,

$$\left.\begin{aligned} y(0) &= \alpha_0 \\ y(b) &= \beta_0 \\ y^{(1)}(0) &= \alpha_1 \\ y^{(1)}(b) &= \beta_1 \end{aligned}\right\} \quad (2.2)$$

Where f is non-linear and αi, βi, i =0, 1, are finite real constant.

Having differentiating (2.1) three times successively the following results were obtained:

$$y^{(4+1)}(x) = f'(x, y) = f_1(x, y, y') \quad (2.3)$$

$$y^{(4+2)}(x) = f''(x, y) = f'_1(x, y, y') = f_2(x, y, y', y'') \quad (2.4)$$

$$y^{(4+3)}(x) = f'''(x, y) = f'_2(x, y, y', y'') = f_3(x, y, y', y'', y''') \quad (2.5)$$

Evaluating, equations (2.1) and (2.3-2,5) at 0 and b together with the (2.2), the linear differential system of equations was obtained as follows:

$$\left.\begin{aligned} y^{(4)}(0) &= f(x, y)\,|_{x=0} \\ y^{(4)}(b) &= f(x, y)\,|_{x=b} \\ y^{(4+1)}(0) &= f_1(x, y, y')|_{x=0} \\ y^{(4+1)}(b) &= f_1(x, y, y')|_{x=b} \\ y^{(4+2)}(0) &= f_2(x, y, y', y'')|_{x=0} \\ y^{(4+2)}(b) &= f_2(x, y, y', y'')|_{x=b} \\ y^{(4+3)}(0) &= f_3(x, y, y', y'', y''')\,|_{x=0} \\ y^{(4+3)}(b) &= f_3(x, y, y', y'', y''')\,|_{x=b} \\ y(0) &= \alpha_0 \\ y(b) &= \beta_0 \\ y^{(1)}(0) &= \alpha_1 \\ y^{(1)}(b) &= \beta_1 \end{aligned}\right\} \quad (2.6)$$

Since there is linear system of twelfth equations it can be assumed that the approximate solution $y(x)$ as follows:

$$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 + a_9x^9 + a_{10}x^{10} + a_{11}x^{11} \quad (2.7)$$

To determine the unknown constants $a_0, a_1, a_3, \dots, a_{11}$ the (2.7) will be incorporated in to (2.6) and system turn to be linear system of twelfth equations with twelfth unknowns $a_0, a_1, a_3, \dots, a_{11}$ and can be solved uniquely.

NUMERICAL RESULTS

To demonstrate the accuracy and applicability of the proposed method, we have solved three nonlinear fourth-order BVPs. All the numerical results obtained by the proposed method are compared with the exact solution and with some results in the literature as shown in the tables 3.1, 3.2 & 3.3.

Example 3.1 Consider the following non-linear boundary value problem. (Singh et al. 2014)

$$y^{(4)}(x) = e^{-x}y(x)^2 \qquad 0 \le x \le 1 \qquad (3,1)$$

Subject to the following boundary conditions,

$$\left.\begin{aligned} y(0) &= 1 \\ y(1) &= e \\ y^{(1)}(0) &= 1 \\ y^{(1)}(1) &= e \end{aligned}\right\} \quad (3.2)$$

The exact solution for the above example is given by $y(x) = e^x$

Having differentiating (3.1) three times successively and the obtained results were evaluated at 0 and 1, the governing equation also evaluated at 0 and 1 together with the boundary conditions, the linear differential system of equations was obtained as follows:

$$\left.\begin{aligned} y^{(4)}(0) &= 1 \\ y^{(4)}(1) &= e \\ y^{(5)}(0) &= 1 \\ y^{(5)}(1) &= e \\ y^{(6)}(0) - 2y^{(2)}(0) &= -1 \\ y^{(6)}(1) - 2y^{(2)}(1) &= -e \\ y^{(7)}(0) - 2y^{(3)}(0) &= -1 \\ y^{(7)}(1) - 2y^{(2)}(1) &= -e \\ y(0) &= 1 \\ y(1) &= e \\ y^{(1)}(0) &= 1 \\ y^{(1)}(1) &= e \end{aligned}\right\} \quad (3.3)$$

Since the above linear system of twelfth equations was obtained. It can be assumed that the approximate solution $y(x)$ can be writing in the form:

$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 + a_9x^9 + a_{10}x^{10} + a_{11}x^{11}$ (3.4)

To determine the unknown coefficients $a_0, a_1, a_3, \dots, a_{11}$ the (3.4) will be incorporated in to (3.3) and system turn to be linear system of twelfth equations with twelfth unknowns $a_0, a_1, a_3, \dots, a_{11}$ and can be writing as:

$$\begin{gathered}
24a_4 = 1 \\
24a_4 + 120a_5 + 360a_6 + 840a_7 + 1680a_8 + 3024a_9 + 5040a_{10} + 7920a_{11} = e \\
120a_5 = 1 \\
120a_5 + 720a_6 + 2520a_7 + 6720a_8 + 15120a_9 + 30240a_{10} + 55440a_{11} = e \\
720a_6 + 4a_2 = -1 \\
660a_6 + 4956a_7 + 20048a_8 + 60336a_9 + 151020a_{10} + 332420a_{11} - 4a_2 - 12a_3 - 24a_4 - 40a_5 = -e \\
5040a_7 - 12a_3 = -1 \\
4620a_7 + 39648a_8 + 180432a_9 + 603360a_{10} + 1661220a_{11} - 12a_3 - 48a_4 - 120a_5 - 240a_6 = -e \\
a_0 = 1 \\
a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11} = e \\
a_1 = 1 \\
a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5 + 6a_6 + 7a_7 + 8a_8 + 9a_9 + 10a_{10} + 11a_{11} = e
\end{gathered} \quad (3.5)$$

After solving (3.5) the following coefficients were obtained:

This algebraic system's solution is given by

$$a_0 = 1, a_1 = 1, a_2 = 0.4999999955, a_3 = 0.1666666722, a_4 = 0.04166666667, a_5 = 0.008333333333, a_6 = 0.001388888864, a_7 = 0.0001984127116, a_8 = 0.00002477829128, a_9 = 0.000002806142, a_{10} = 2.321767482 \times 10^{-7}, a_{11} = 4.18795643 \times 10^{-8}$$

Therefore, the series solution is

$$y(x) = 1 + x + 0.4999999955x^2 + 0.1666666722x^3 + 0.04166666667x^4 + 0.008333333333x^5 + 0.001388888864x^6 + 0.0001984127116x^7 + 0.00002477829128x^8 + 0.000002806142x^9 + 2.321767482 \times 10^{-7}x^{10} + 4.187956430 \times 10^{-8}x^{11}$$

Table 3.1 shows the comparison between exact and numerical solutions along with the errors obtained by using the proposed method and also errors obtained by using Adomian decomposition method with Green's function (Singh et al. 2014). The graphical result between exact and approximate solution is shown in figure3.1.

Table 3.1 Numerical result for example 3.1

X	**Exact Solution**	Numerical solution	**Absolute Error**	(Singh et al. 2014) $\psi_2 - y$
0	1	1	0	0
0.2	1.221402758	1.221402759	1×10^{-9}	3.5093E-8
0.4	1.491824698	1.491824697	1×10^{-9}	8.2634E-8
0.6	1.822118800	1.822118801	1×10^{-9}	8.2376E-8
0.8	2.225540928	2.225540929	1×10^{-9}	3.4819E-8
1	2.718281828	2.718281828	0	0

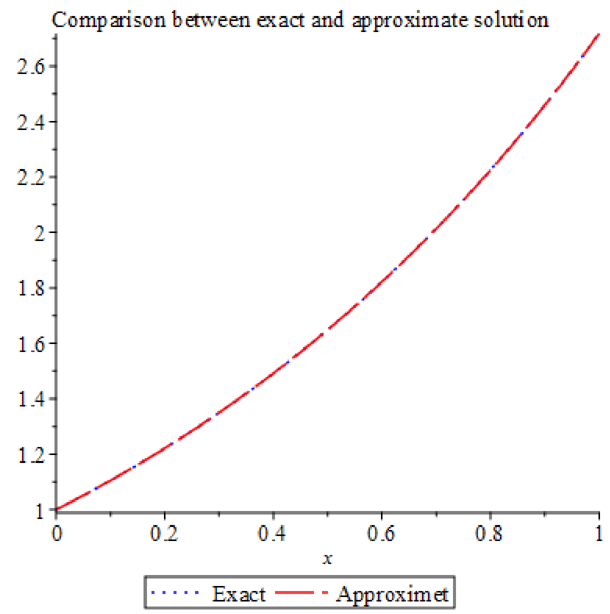

Figure 3.1: Solution graph for example 3.1

Example 3.2 Consider the following non-linear boundary value problem (Singh et al. 2014).

$$y^{(4)}(x) = -6e^{-4y(x)} \qquad 0 \le x \le 4 - e \qquad (3.6)$$

Subject to the following boundary conditions,

$$\left.\begin{aligned} y(0) &= 1 \\ y(4-e) &= \ln 4 \\ y^{(1)}(0) &= 1 \\ y^{(1)}(4-e) &= \tfrac{1}{4} \end{aligned}\right\} \qquad (3.7)$$

The exact solution for the above example is given by $y(x) = \ln(e + x)$

Following in example (3.1) the linear differential system of equations was obtained as follows:

$$\left.\begin{array}{c} y^{(4)}(0) = -6e^{-4} \\ y^{(4)}(4-e) = -\frac{3}{128} \\ y^{(5)}(0) = 24e^{-5} \\ y^{(5)}(4-e) = \frac{3}{128} \\ y^{(6)}(0) - 24y^{(2)}(0)e^{-4} = -96e^{-6} \\ y^{(6)}(4-e) - \frac{3}{32}y^{(2)}(4-e) = -\frac{3}{128} \\ y^{(7)}(0) - 24y^{(3)}(0)e^{-4} + 288y^{(2)}(0)e^{-5} = 384e^{-7} \\ y^{(7)}(4-e) - \frac{3}{32}y^{(3)}(4-e) + \frac{9}{32}y^{(2)}(4-e) = \frac{3}{128} \\ y(0) = 1 \\ y(4-e) = \ln 4 \\ y^{(1)}(0) = 1 \\ y^{(1)}(4-e) = \frac{1}{4} \end{array}\right\} \quad (3.8)$$

Since there is linear system of twelfth equations it can be assumed that the approximate solution $y(x)$ as follows:

$$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 + a_9x^9 + a_{10}x^{10} + a_{11}x^{11} \quad (3.9)$$

To determine the unknown constants $a_0, a_1, a_3, \dots, a_{11}$ the (3.9) will be incorporated in to (3.8) and system turn to be linear system of twelfth equations with twelfth unknowns $a_0, a_1, a_3, \dots, a_{11}$ and can be written as:

$$24a_4 = -6e^{-4}$$

$$27920a_{11}(4-e)^7 + 5040a_{10}(4-e)^6 + 3024a_9(4-e)^5 + 1680a_8(4-e)^4 + 840a_7(4-e)^3 + 360a_6(4-e)^2 + 120a_5(4-e) + 24a_4 = -\frac{3}{128}$$

$$120a_5 = 24e^{-5}$$

$$55440a_{11}(4-e)^6 + 30240a_{10}(4-e)^5 + 5120a_9(4-e)^4 + 6720a_8(4-e)^3 + 2520a_7(4-e)^2 + 720a_6(4-e) + 120a_5 = \frac{3}{128}$$

$$720a_6 - 48a_2e^{-4} = -96e^{-6}$$

$$32640a_{11}(4-e)^5 + 151200a_{10}(4-e)^4 + 60480a_9(4-e)^3 + 20160a_8(4-e)^2 + 5040(4-e)a_7 + 720a_6 - \frac{3}{16}a_2 - \frac{9}{16}a_3(4-e) - \frac{9}{8}a_4(4-e)^2 - \frac{15}{8}a_5(4-e)^3 - \frac{45}{16}a_6(4-e)^4 - \frac{63}{16}a_7(4-e)^5 - \frac{21}{4}a_8(4-e)^6 - \frac{21}{4}a_9(4-e)^7 - \frac{135}{16}a_{10}(4-e)^8 - \frac{165}{16}a_{11}(4-e)^9 = -\frac{12}{512}$$

$$5040a_7 - 144e^{-5}a_3 + 576e^{-4}a_2 = 384e^{-7}$$

$$1663200a_{11}(4-e)^4 + 604800a_{10}(4-e)^3 + 181440a_9(4-e)^2 + 40320(4-e)a_8 + 5040a_7 - \frac{9}{16}a_3 - \frac{9}{4}a_4(4-e) - \frac{45}{8}a_5(4-e)^2 - \frac{45}{4}a_6(4-e)^3 - \frac{315}{16}a_7(4-e)^4 - \frac{63}{2}a_8(4-e)^5 - \frac{189}{4}a_9(4-e)^6 - \frac{135}{2}a_{10}(4-e)^7 - \frac{1485}{16}a_{11}(4-e)^8 + \frac{9}{16}a_2 + \frac{27}{16}a_3(4-e) + \frac{27}{8}a_4(4-e)^2 + \frac{45}{8}a_5(4-e)^3 + \frac{135}{16}a_6(4-e)^4 + \frac{189}{16}a_7(4-e)^5 + \frac{63}{4}a_8(4-e)^6 + \frac{81}{4}a_9(4-e)^7 + \frac{405}{16}a_{10}(4-e)^8 + \frac{495}{16}a_{11}(4-e)^9 = \frac{3}{128}$$

$$a_0 = 1$$

$$a_0 + a_1(4-e) + a_2(4-e)^2 + a_3(4-e)^3 + a_4(4-e)^4 + a_5(4-e)^5 + a_6(4-e)^6 + a_7(4-e)^7 + a_8(4-e)^8 + a_9(4-e)^9 + a_{10}(4-e)^{10} + a_{11}(4-e)^{11} = \ln 4$$

$$a_1 = \frac{1}{e}$$

$$a_1 + 2(4-e)a_2 + 3a_3(4-e)^2 + 4a_4(4-e)^3 + 5a_5(4-e)^4 + 6a_6(4-e)^5 + 7a_7(4-e)^6 + 8a_8(4-e)^7 + 9a_9(4-e)^8 + 10a_{10}(4-e)^9 + 11a_{11}(4-e)^{10} = \frac{1}{4}$$

(3.10)

After solving (3.10) the following coefficients were determined:

$a_0 = 1, a_1 = 0.3678794412, a_2 = -0.06767543773, a_3 = 0.01660287774, a_4 = -0.004578909721, a_5 = 0.001347589400, a_6 = -0.0004131348823, a_7 = 0.0001302786173, a_8 = -0.00004014666628, a_9 = 0.000009344617678, a_{10} = -0.000001065188797, a_{11} = 7.3962794 \times 10^{-9}$

Therefore, the series solution

$$y(x) = 1. + 0.3678794412x - 0.06767543773x^2 + 0.01660287774x^3 - 0.004578909721x^4 + 0.001347589400x^5 - 0.0004131348823x^6 + 0.0001302786173x^7 - 0.00004014666628x^8 + 0.000009344617678x^9 - 0.000001065188797x^{10} + 7.396279400 \times 10^{-9}x^{11}$$

Table 3.2 Numerical result for example 3.2

X	Exact Solution	Numerical solution	Absolute Error	(Singh et al. 2014). $\psi_4 - y(x)$
0	1	0.9999999998	1.0E-10	0
0.2	1.070995028	1.070994774	2.54E-7	4.1012E-6
0.4	1.137282154	1.137281366	7.88E-7	1.2574E-5
0.6	1.199447127	1.199445877	1.250E-6	1.9762E-5
0.8	1.257972753	1.257971429	1.324E-6	2.0714E-5
1	1.313261687	1.313260844	8.43E-6	1.3163-5
1.2	1.365653249	1.365652093	1.156E-6	1.9026E-6
4-e	1.386294361	1.386294362	1.0E-9	0

Table 3.2 shows the comparison between exact and numerical solutions along with the errors obtained by using the proposed method and by using Adomian decomposition method with Green's function (Singh et al. 2014). The graphical result between exact and approximate solutions is shown in figure3.2.

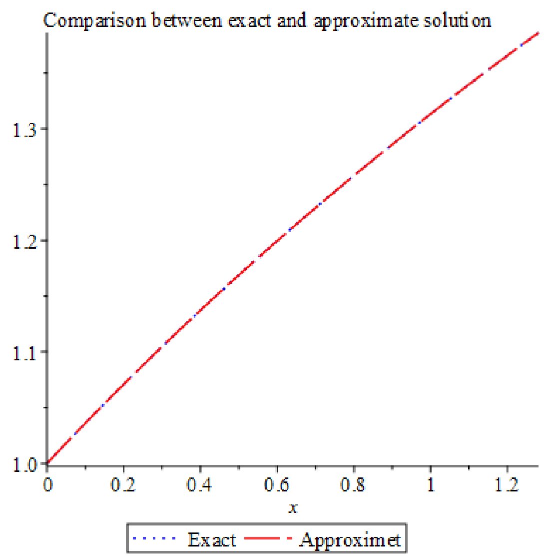

Figure 3.2: Solution graph for example 3.2

Example 3.3 Consider the following non-linear boundary value problem (Mustafa et al, 2017)

$$y^{(4)}(x) = g(x) + y(x)^2 \qquad 0 \le x \le 1 \qquad (3.11)$$

Subject to the following two types of boundary conditions,

$$\left.\begin{aligned} y(0) &= 1 \\ y(1) &= e \\ y^{(1)}(0) &= 1 \\ y^{(1)}(1) &= e \end{aligned}\right\} \qquad (3.12)$$

Where $g(x) = -x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 + 120x - 48$

The exact solution for the above example is given by $y(x) = x^5 - 2x^4 + 2x^2$

Following in example (3.1) the linear differential system of equations was obtained as follows:

$$\left.\begin{aligned} y^{(4)}(0) &= -48 \\ y^{(4)}(1) &= 72 \\ y^{(5)}(0) &= 120 \\ y^{(5)}(1) &= 120 \\ y^{(6)}(0) &= 0 \\ y^{(6)}(1) - 2y^{(2)}(1) &= 0 \\ y^{(7)}(0) &= 0 \\ y^{(7)}(1) - 2y^{(3)}(1) - 6y^{(2)}(1) &= -24 \\ y(0) &= 1 \\ y(1) &= e \\ y^{(1)}(0) &= 1 \\ y^{(1)}(1) &= e \end{aligned}\right\} \qquad (3.13)$$

Since there is linear system of twelfth equations it can be assumed that the approximate solution $y(x)$ as follows:

$$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 + a_9x^9 + a_{10}x^{10} + a_{11}x^{11} \quad (3.14)$$

To determine the unknown constants $a_0, a_1, a_3, \ldots, a_{11}$ the equation (3.14) will be incorporated in to (3.13) and system turn to be linear system of twelfth equations with twelfth unknowns $a_0, a_1, a_3, \ldots, a_{11}$ and can be can be writing as:

$$\begin{gathered}
24a_4 = -48 \\
24a_4 + 120a_5 + 360a_6 + 840a_7 + 1680a_8 + 3024a_9 + 5040a_{10} + 7920a_{11} = 72 \\
120a_5 = 120 \\
120a_5 + 720a_6 + 2520a_7 + 6720a_8 + 15120a_9 + 30240a_{10} + 55440a_{11} = 120 \\
720a_6 = 0 \\
332420a_{11} + 151020a_{10} + 60336a_9 + 20048a_8 + 4956a_7 + 660a_6 - 4a_2 - 12a_3 - 24a_4 - 40a_5 = 0 \\
5040a_7 = 0 \\
1660560a_{11} + 602820a_{10} + 180000a_9 + 39312a_8 + 4368a_7 - 12a_2 - 48a_3 - 120a_4 - 240a_5 - 420a_6 = -24 \\
a_0 = 0 \\
a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11} = 1 \\
a_1 = 0 \\
a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5 + 6a_6 + 7a_7 + 8a_8 + 9a_9 + 10a_{10} + 11a_{11} = 1
\end{gathered} \quad (3.15)$$

After solving (3.15) the following coefficients were obtained:

$$a_0 = 0, a_1 = 0., a_2 = 2., a_3 = 0., a_4 = -2., a_5 = 1., a_6 = 0., a_7 = 0., a_8 = 0., a_9 = 0., a_{10} = 0., a_{11} = 0$$

Therefore, the exact solution is reproduced.

$$y(x) = 2x^2 - 2x^4 + x^5$$

Table 3.3 below shows a comparison between the exact solution and numerical solution, it also shows the error obtained by using the proposed method and by using I.S.S of (Mustafa et al, 2017).The graphical result between exact and approximate solutions is shown in figure 3.3.

Table 3.3 Numerical result for example 3.3

x	**Exact Solution**	**Numerical solution**	**Absolute Error**	**Absolute Error** (Mustafa et al, 2017)
0	0	0	0	0
0.1	0.01981	1.105170918	0	4.095E-4
0.2	0.07712	1.221402759	0	2.5752E-3
0.3	0.16623	1.349858807	0	6.6432E-3
0.4	0.27904	1.491824697	0	1.15595E-2
0.5	0.40625	1.648721270	0	1.56708E-2
0.6	0.53856	1.822118801	0	1.73246E-2
0.7	0.66787	2.013752707	0	1.54706E-2
0.8	0.78848	2.225540929	0	1.02612E-2
0.9	0.89829	2.459603110	0	3.6517E-3
1	1.00000	2.718281828	0	0

It is observed that the numerical results presented on the tables 3.1, 3.2 & 3.3 are in good agreement with the exact solution and with some results obtained from the existing methods in the literatures. It can also seen that from figures 3.1, 3.2 & 3.3 the graphs of the approximate solution and with exact solution overlap and this conforms the effectiveness of the method.

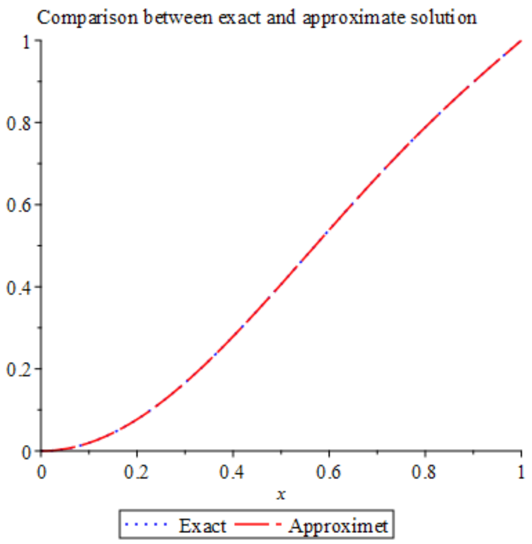

Figure 3.3: Solution graph for example 3.3

CONCLUSION

This paper introduces a new approach for solving special fourth-order nonlinear boundary value problems using an eleventh-degree polynomial series solution approach. By employing this method to some numerical examples, it is fund that the method is accurate and reliable as shows on tables 3.1, 3.2 & 3.3 and figures 3.1, 3.2 & 3.3 Also it can be observed that the exact solution can be reproduced by the method as seen in example 3.3.

REFERENCES

Adeyeye O. and Omar Z. (2017) Solving Nonlinear Fourth-Order Boundary Value Problems Using a Numerical Approach: $(m+1)$th-Step Block Method International Journal of Differential Equation Article ID 4925914, 9 pages **[Crossref]**

Abd-Elhameed W. M.,Ahmed H. M, and Youssri H Youssri Y. H. (2016). A new generalized Jacobi Galerkin operational matrix of derivatives: two algorithms for solving fourth-order boundary value problems. *Advances in Difference Equations* 2016:22

Audu K. J., Garba J. Tiamiyu A. T and Thomas B. A. (2022). Application of Backward Differentiation Formula on Fourth-Order Differential Equations .*Journal of Science And Technology* Vol. 14 No. 2 . 52-65. **[Crossref]**

Abd-Elhameed W. M., Youssri Y. H. (2020). Connection Formulae Between Generalized Lucas Polynomials and Some Jacobi Polynomials: Application to Certain Types of Fourth-Order BVPs. *Int. J. Appl. Comput. Math* 6:45. **[Crossref]**

Ertu¨rk V. S. and Momani S. (2007) Comparing numerical methods for solving fourth-order boundary value problems. *Applied Mathematics and Computation* 188 1963–1968 **[Crossref]**

Hossain B. and Islam S. (2014) Numerical Solutions of General Fourth Order Two Point Boundary Value Problems by Galerkin Method with Legendre Polynomials. Dhaka Univ. J. Sci. 62(2): 103-108.

Huiling Chen and Yujun Cui (2023).Existence and uniqueness of solutions to the nonlinear boundary value problem for fourth-order differential equations with all derivatives. Journal of Inequalities and Applications :23 **[Crossref]**

Mustafa G., Abbas M., Ejaz S.T., Ahmad Izani Md Ismail A.I and Khan F.(2017). A Numerical Approach Based on Subdivision Schemes for Solving Non-Linear Fourth Order Boundary Value Problems. J. Computational Analysis And Applications, Vol. 23, No.4.

Mingzhu Huang (2021) Existence of solutions for fourth-order nonlinear boundary *Advances in Difference* value problems. *Equations*:196 **[Crossref]**

Singh R., Kumar J. and Nelakanti G.(2014) Approximate series solution of fourth-order boundary value problems using decomposition method with Green's function J Math Chem 52:1099–1118. **[Crossref]**

Sohel Md. N., Islam Md. S and Islam Md. S.(2022). Galerkin Residual Correction for Fourth Order BVP Journal of Applied Mathematics and Computation, 6(1), 127-138 **[Crossref]**

Thenmozhi S. and Marudai M. (2021). Solution of nonlinear boundary value problem by Siteration Journal of Applied Mathematics and Computing. 34B15 · 34B27 · 65L10 **[Crossref]**

Viswanadham, K.N.S.K., Krishna, P.M., Koneru, R.S.: (2010) Numerical solutions of fourth order boundary value problems by Galerkin method with Quintic B-splines. Int. J. Nonlinear Sci. **10**, 222–230

Wazwaz A. (2002). The Numerical solution of special fourth order boundary value problems by the modified decomposition method . Intern.J. Math. Vol. 79(3), pp. 345-356. **[Crossref]**