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ORIGINAL RESEARCH ARTICLE

A Single-Step Modified Block Hybrid Method for General Second-Order Ordinary Differential Equations

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ABSTRACT

A multistep collocation approach is used to derive a single-step modified block hybrid method (MBHM) of order five for solving general second-order initial-value problems (IVPs) of ordinary differential equations (ODEs). The new method's basic convergence property is established, and its numerical accuracy is demonstrated using numerical examples from the literature. The new method outperforms similar methods in terms of accuracy, earning it a recommendation as a likely candidate for solving general second-order ODEs.

INTRODUCTION

Consider the general second-order initial value-problem (IVP) in ordinary differential equations (ODEs) of the form

$$y'' = f(x, y, y'), y(a) = \alpha, y'(a) = \beta \qquad (1)$$

on the interval $a \leq x \leq b, \alpha, \beta \in \mathbb{R}$, where f satisfies a Lipschitz condition which guarantees the existence and uniqueness of the solution of (1). Naturally, (1) occurs frequently in the mathematical modelling of ODEs in a variety of applications, including engineering and science studies such as astrophysics, biology, chemical engineering, chemical kinetics, circuit and control theory (Jator, 2010a; Jator, 2010b). According to Jator (2010b), while most direct methods for solving (1) are linear multistep (LMM), multistep collocation (MC), multiderivative, exponentially-fitting and trigonometrically-fitting, and Runge-Kutta-Nystrom methods, implementing some of them requires the use of predictor-corrector (PC) approach, which takes more computer time and thus increases computational burdens. Another approach is to reduce (1) to a system of first-order ODEs and use methods specifically designed for the resulting first-order systems, which increases computational burdens.

While hybrid methods were also used in solving (1), their earlier application became pronounced as they overcame the popular 'Dahlquist barrier theorem', but the introduction of 'off-grid' points, which is a characteristic of hybrid methods, increases additional computer burdens in the PC approach by users.

The preceding establishes the tone for more research by authors such as Jator (2007) and Jator and Li (2009) to overcome the shortcomings among other potential interests. The motivation for this research stems from the works of Adesanya, Alkali and Sunday (2014), Abdelrahim and Omar (2016), and Ogunniran, Tijani, Adedokun and Kareem (2022) as well as the references therein. This particular research is anchored by the benefit of single-step methods, which by themselves are self-starting, and the usage of block methods as a collection of simultaneous integrators without relying on any way to generate starting values. Additionally, the methodology used in the recent studies by Adee, Kumleng and Patrick (2022), Adee and Yunusa (2022) and Singla, Singh, Ramos and Kanwar, (2022), in which block hybrid methods were implemented as a collection of numerical integrators for first-order IVPs of ODEs on non-overlapping subintervals, is employed in this study.

MATERIALS AND METHODS

Development of the single-step modified block hybrid method (MBHM)

We obtain a continuous hybrid linear method, as Adee *et al.* (2022) and Adesanya *et al.* (2014) did, by considering a polynomial of the form :

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$$y(x)=\sum_{j=0}^{6}\alpha_j x^j \qquad (2)$$

Differentiating (2) twice gives

$$y''(x)=\sum_{j=0}^{6} j(j-1)\alpha_j x^{j-2}=f(x,y,y') \qquad (3)$$

Interpolating (2) at $x_{n+r}, r=0,\frac{1}{5},\frac{2}{5},\frac{4}{5}$ and collocating (3) at $x_{n+m}, m=\frac{2}{5},\frac{3}{5}$ gives a system of nonlinear equations of the form

$$AX=U \qquad (4)$$

Where:

$A=[a_1,a_2,a_3,a_4,a_5,a_6,a_7]^T$, $U=[y_n, y_{n+\frac{1}{5}}, y_{n+\frac{2}{5}}, y_{n+\frac{3}{5}}, y_{n+\frac{4}{5}}, f_{n+\frac{2}{5}}, f_{n+\frac{3}{5}}]^T$ and

$$X=\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 \\ 1 & x_{n+\frac{1}{5}} & x_{n+\frac{1}{5}}^2 & x_{n+\frac{1}{5}}^3 & x_{n+\frac{1}{5}}^4 & x_{n+\frac{1}{5}}^5 & x_{n+\frac{1}{5}}^6 \\ 1 & x_{n+\frac{2}{5}} & x_{n+\frac{2}{5}}^2 & x_{n+\frac{2}{5}}^3 & x_{n+\frac{2}{5}}^4 & x_{n+\frac{2}{5}}^5 & x_{n+\frac{2}{5}}^6 \\ 1 & x_{n+\frac{3}{5}} & x_{n+\frac{3}{5}}^2 & x_{n+\frac{3}{5}}^3 & x_{n+\frac{3}{5}}^4 & x_{n+\frac{3}{5}}^5 & x_{n+\frac{3}{5}}^6 \\ 1 & x_{n+\frac{4}{5}} & x_{n+\frac{4}{5}}^2 & x_{n+\frac{4}{5}}^3 & x_{n+\frac{4}{5}}^4 & x_{n+\frac{4}{5}}^5 & x_{n+\frac{4}{5}}^6 \\ 0 & 0 & 2 & 6x_{n+\frac{2}{5}} & 20x_{n+\frac{2}{5}}^2 & 20x_{n+\frac{2}{5}}^3 & 30x_{n+\frac{2}{5}}^4 \\ 0 & 0 & 2 & 6x_{n+\frac{3}{5}} & 12x_{n+\frac{3}{5}}^2 & 12x_{n+\frac{3}{5}}^3 & 12x_{n+\frac{3}{5}}^4 \end{pmatrix}$$

Solving (4) for the unknowns $a_i, i=1(1)7$ and substituting them into (2) gives the continuous hybrid linear method of the form

$$y(x)=\sum_{i=0}^{4}\alpha_{\frac{i}{5}}(x)\,y_{n+\frac{i}{5}}+h^2\left(\sum_{i=2}^{3}\beta_{\frac{i}{5}}(x)\,f_{n+\frac{i}{5}}\right) \qquad (5)$$

Where: $\alpha_{\frac{i}{5}}(x), i=0(\frac{1}{5})\frac{4}{5}, \beta_{\frac{2}{5}}(x), \beta_{\frac{3}{5}}(x)$ are all expressed in terms of $\tau=x-x_n$ as follows:

$$\alpha_0(\tau)=\frac{1}{96h^6}(96h^6-1600h^5\tau+10350h^4\tau^2-33125h^3\tau^3+55625h^2\tau^4-46875h\tau^5+15625\tau^6)$$

$$\alpha_{\frac{1}{5}}(\tau)=\frac{\tau}{6h^6}(648h^5-6750h^4\tau+26625h^3\tau^2-50000h^2\tau^3+45000h\tau^4-15625\tau^5)$$

$$\alpha_{\frac{2}{5}}(\tau)=-\frac{\tau}{16h^6}(2664h^5-30150h^4\tau+123625h^3\tau^2-238125h^2\tau^3+219375h\tau^4-78125\tau^5)$$

$$\alpha_{\frac{3}{5}}(\tau)=\frac{\tau}{6h^6}(424h^5-4950h^4\tau+21125h^3\tau^2-42500h^2\tau^3+41250h\tau^4-15625\tau^5)$$

$$\alpha_{\frac{4}{5}}(\tau)=\frac{\tau}{96h^6}(432h^5-4050h^4\tau+10875h^3\tau^2-4375h^2\tau^3-16875h\tau^4+15625\tau^5)$$

$$\beta_{\frac{2}{5}}(\tau)=-\frac{\tau}{200h^4}(504h^5-5850h^4\tau+24625h^3\tau^2-48125h^2\tau^3+44375h\tau^4-15625\tau^5)$$

$$\beta_3(\tau)=-\frac{\tau}{50h^4}(24h^4-250h^3\tau+875h^2\tau^2-1250h\tau^3+625\tau^4)$$

Evaluating (5) at $\tau=h$ gives the discrete scheme

$$y_{n+1}=y_n-17y_{n+\frac{1}{5}}+46y_{n+\frac{2}{5}}-46y_{n+\frac{3}{5}}+17y_{n+\frac{4}{5}}+\frac{12h^2}{25}f_{n+\frac{2}{5}}-\frac{12h^2}{25}f_{n+\frac{3}{5}} \qquad (6)$$

Differentiating (5) yields

$$y'(x)=\sum_{i=0}^{4}\alpha_{\frac{i}{5}}'(x)y_{n+\frac{i}{5}}+h^2\left(\sum_{i=2}^{3}\beta_{\frac{i}{5}}'(x)f_{n+\frac{i}{5}}\right) \tag{7}$$

where

$$\alpha_0'(\tau)=\frac{1}{96h^6}(-1600h^5+20700h^4\tau-99375h^3\tau^2+222500h^2\tau^3-234375h\tau^4+93750\tau^5)$$
$$\alpha_{\frac{1}{5}}'(\tau)=\frac{1}{6h^6}(648h^5-13500h^4\tau+79875h^3\tau^2-200000h^2\tau^3+225000h\tau^4-93750\tau^5)$$
$$\alpha_{\frac{2}{5}}'(\tau)=-\frac{3}{16h^6}(888h^5-20100h^4\tau+123625h^3\tau^2-317500h^2\tau^3+365625h\tau^4-156250\tau^5)$$
$$\alpha_{\frac{3}{5}}'(\tau)=\frac{1}{6h^6}(424h^5-9900h^4\tau+63375h^3\tau^2-170000h^2\tau^3+206250h\tau^4-93750\tau^5)$$
$$\alpha_{\frac{4}{5}}'(\tau)=\frac{1}{96h^6}(432h^5-8100h^4\tau+32625h^3\tau^2-17500h^2\tau^3-84375h\tau^4+93750\tau^5)$$
$$\beta_{\frac{2}{5}}'(\tau)=-\frac{1}{200h^4}(504h^5-11700h^4\tau+73875h^3-192500h^2\tau^3+221875h\tau^4-93750\tau^5)$$
$$\beta_{\frac{3}{5}}'(\tau)=-\frac{1}{50h^3}(24h^4-500h^3\tau+2625h^2\tau^2-5000h\tau^3+3125\tau^4)$$

Evaluating (7) at $\tau=0,\frac{h}{5},\frac{2h}{5},\frac{3h}{5},\frac{4h}{5}$ and h with $y'=g$ yields

$$-150hg_n=378h^2f_{n+\frac{2}{5}}+72h^2f_{n+\frac{3}{5}}+2500y_n-16200y_{n+\frac{1}{5}}+24975y_{n+\frac{2}{5}}-10600y_{n+\frac{3}{5}}-675y_{n+\frac{4}{5}} \tag{8}$$
$$150hg_{n+\frac{1}{5}}=72h^2f_{n+\frac{2}{5}}+18h^2f_{n+\frac{3}{5}}-3175y_{n+\frac{1}{5}}+5400y_{n+\frac{2}{5}}-2025y_{n+\frac{3}{5}}-200y_{n+\frac{4}{5}} \tag{9}$$
$$-600hg_{n+\frac{2}{5}}=132h^2f_{n+\frac{2}{5}}+48h^2f_{n+\frac{3}{5}}+125y_n-2800y_{n+\frac{1}{5}}+7650y_{n+\frac{2}{5}}-4400y_{n+\frac{3}{5}}-575y_{n+\frac{4}{5}} \tag{10}$$
$$600hg_{n+\frac{3}{5}}=108h^2f_{n+\frac{2}{5}}+72h^2f_{n+\frac{3}{5}}+125y_n-2700y_{n+\frac{1}{5}}+1350y_{n+\frac{2}{5}}+1900y_{n+\frac{3}{5}}-675y_{n+\frac{4}{5}} \tag{11}$$
$$-150hg_{n+\frac{4}{5}}=18h^2f_{n+\frac{2}{5}}+72h^2f_{n+\frac{3}{5}}-200y_{n+\frac{1}{5}}-2025y_{n+\frac{2}{5}}+5400y_{n+\frac{3}{5}}-3175y_{n+\frac{4}{5}} \tag{12}$$
$$150hg_{n+1}=1272h^2f_{n+\frac{2}{5}}-822h^2f_{n+\frac{3}{5}}+2500y_n-43175y_{n+\frac{1}{5}}+104400y_{n+\frac{2}{5}}-90025y_{n+\frac{3}{5}}+26300y_{n+\frac{4}{5}} \tag{13}$$

Further differentiating (7) yields

$$y''(x)=\sum_{i=0}^{4}\alpha_{\frac{i}{5}}''(x)y_{n+\frac{i}{5}}+h^2\left(\sum_{i=2}^{3}\beta_{\frac{i}{5}}''(x)f_{n+\frac{i}{5}}\right) \tag{14}$$

where

$$\alpha_0''(\tau)=\frac{1}{96h^6}(20700h^4-198750h^3\tau+667500h^2\tau^2-937500h\tau^3+468750\tau^4)$$
$$\alpha_{\frac{1}{5}}''(\tau)=\frac{1}{6h^6}(-13500h^4+159750h^3\tau-600000h^2\tau^2+900000h\tau^3-468750\tau^4)$$
$$\alpha_{\frac{2}{5}}''(\tau)=-\frac{3}{16h^6}(-20100h^4+247250h^3\tau-952500h^2\tau^2+1462500h\tau^3-781250\tau^4)$$
$$\alpha_{\frac{3}{5}}''(\tau)=\frac{1}{6h^6}(-9900h^4+126750h^3\tau-510000h^2\tau^2+825000h\tau^3-468750\tau^4)$$
$$\alpha_{\frac{4}{5}}''(\tau)=\frac{1}{96h^6}(-8100h^4+65250h^3\tau-52500h^2\tau^2-337500h\tau^3+468750\tau^4)$$
$$\beta_{\frac{2}{5}}''(\tau)=-\frac{1}{200h^4}(-11700h^4+147750h^3\tau-577500h^2\tau^2+887500h\tau^3-468750\tau^4)$$
$$\beta_{\frac{3}{5}}''(\tau)=-\frac{1}{50h^3}(-500h^3+5250h^2\tau-15000h\tau^2+12500\tau^3)$$

Evaluating (14) at $\tau = \frac{h}{5}, \frac{4h}{5}, h$ gives

$$h^2 f_{n+\frac{1}{5}} = -\frac{1}{8}(44h^2 f_{n+\frac{2}{5}} + 8h^2 f_{n+\frac{3}{5}} - 75y_n - 1200y_{n+\frac{1}{5}} + 2550y_{n+\frac{2}{5}} - 1200y_{n+\frac{3}{5}} - 75y_{n+\frac{4}{5}}) \quad (15)$$

$$h^2 f_{n+\frac{4}{5}} = \frac{1}{8}(28h^2 f_{n+\frac{2}{5}} - 80h^2 f_{n+\frac{3}{5}} + 75y_n - 1200y_{n+\frac{1}{5}} + 4650y_{n+\frac{2}{5}} - 6000y_{n+\frac{3}{5}} + 2475y_{n+\frac{4}{5}}) \quad (16)$$

$$h^2 f_{n+1} = \frac{1}{8}(908h^2 f_{n+\frac{2}{5}} - 360h^2 f_{n+\frac{3}{5}} + 1725y_n - 30000y_{n+\frac{1}{5}} + 66150y_{n+\frac{2}{5}} - 49200y_{n+\frac{3}{5}} + 11325y_{n+\frac{4}{5}}) \quad (17)$$

The modified block method is now formed by combining the discrete methods in (6), (8)-(14), (15)-(17).

Modified block method

The modified block method for the independent evaluation of unknown parameters, as described by Fatunla (1995) and Awoyemi *et al.* (2011), is of the form

$$Ah^{\lambda}Y_m^{(n)} = h^{\lambda}By_{m-i}^{(n)} + h^{\mu-\lambda}CF(Y_m) \quad (18)$$

where

$$Y_m^{(n)} = (y_{n+\frac{1}{5}}, y_{n+\frac{2}{5}}, y_{n+\frac{3}{5}}, y_{n+\frac{4}{5}}, y_{n+1})^T, y_m^{(n)} = (y_{n-\frac{4}{5}}, y_{n-\frac{3}{5}}, y_{n-\frac{2}{5}}, y_{n-\frac{1}{5}}, y_n, g_{n-\frac{4}{5}}, g_{n-\frac{3}{5}}, g_{n-\frac{2}{5}}, g_{n-\frac{1}{5}}, g_n)^T,$$

$F(Y_m) = (f_n, f_{n+\frac{1}{5}}, f_{n+\frac{2}{5}}, f_{n+\frac{3}{5}}, f_{n+\frac{4}{5}}, f_{n+1})$, n is the order of the derivative of (7), μ is the order of the differential equation and λ is the power of h relative to the derivative of the differential equation where A, B and C are constant coefficient matrices from the block method (6), (8)-(14), (15)-(17). Normalizing (18) gives the coefficient matrices as

$$A^1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad B^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{5} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{2}{5} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{3}{5} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{4}{5} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad C^1 = \begin{pmatrix} \frac{19}{288} & \frac{155}{288} & \frac{-5}{9} & \frac{35}{48} & \frac{-5}{18} \\ \frac{367}{36000} & \frac{19}{288} & \frac{-247}{2250} & \frac{631}{6000} & \frac{-58}{1125} \\ \frac{53}{2250} & \frac{409}{2250} & \frac{-56}{225} & \frac{91}{375} & \frac{-134}{1125} \\ \frac{147}{4000} & \frac{1203}{4000} & \frac{-177}{500} & \frac{153}{400} & \frac{-93}{500} \\ \frac{56}{1125} & \frac{472}{1125} & \frac{-512}{1125} & \frac{208}{375} & \frac{-56}{225} \\ \frac{19}{144} & \frac{85}{144} & \frac{-35}{72} & \frac{5}{6} & \frac{-5}{72} \\ \frac{251}{3600} & \frac{1901}{3600} & \frac{-1387}{1800} & \frac{109}{150} & \frac{-63}{1800} \\ \frac{29}{450} & \frac{269}{450} & \frac{-133}{225} & \frac{49}{75} & \frac{-73}{225} \\ \frac{27}{400} & \frac{237}{400} & \frac{-99}{200} & \frac{39}{50} & \frac{-69}{200} \\ \frac{14}{225} & \frac{134}{225} & \frac{-116}{225} & \frac{68}{75} & \frac{-56}{225} \end{pmatrix} \quad (19)$$

Substituting (19) into (18) gives the individual hybrid methods

$$y_{n+1} = y_n + hg_n + \tfrac{h^2}{288}(19 f_{n+1} + 155 f_{n+\frac{1}{5}} - 160 f_{n+\frac{2}{5}} + 210 f_{n+\frac{3}{5}} - 80 f_{n+\frac{4}{5}}) \quad (20)$$

$$y_{n+\frac{1}{5}} = y_n + \tfrac{1}{5}hg_n + \tfrac{h^2}{36000}(367 f_{n+1} + 2375 f_{n+\frac{1}{5}} - 3952 f_{n+\frac{2}{5}} + 3786 f_{n+\frac{3}{5}} - 1856 f_{n+\frac{4}{5}}) \quad (21)$$

$$y_{n+\frac{2}{5}} = y_n + \tfrac{2}{5}hg_n + \tfrac{h^2}{2250}(53 f_{n+1} + 409 f_{n+\frac{1}{5}} - 560 f_{n+\frac{2}{5}} + 546 f_{n+\frac{3}{5}} - 268 f_{n+\frac{4}{5}}) \quad (22)$$

$$y_{n+\frac{3}{5}} = y_n + \tfrac{3}{5}hg_n + \tfrac{3h^2}{4000}(49 f_{n+1} + 401 f_{n+\frac{1}{5}} - 472 f_{n+\frac{2}{5}} + 510 f_{n+\frac{3}{5}} - 248 f_{n+\frac{4}{5}}) \quad (23)$$

$$y_{n+\frac{4}{5}} = y_n + \tfrac{4}{5}hg_n + \tfrac{8h^2}{1125}(7 f_{n+1} + 59 f_{n+\frac{1}{5}} - 64 f_{n+\frac{2}{5}} + 78 f_{n+\frac{3}{5}} - 35 f_{n+\frac{4}{5}}) \quad (24)$$

$$g_{n+1} = g_n + \tfrac{h}{144}(19 f_{n+1} + 85 f_{n+\frac{1}{5}} - 70 f_{n+\frac{2}{5}} + 120 f_{n+\frac{3}{5}} - 10 f_{n+\frac{4}{5}}) \quad (25)$$

$$g_{n+\frac{1}{5}} = g_n + \tfrac{h}{3600}(251 f_{n+1} + 1901 f_{n+\frac{1}{5}} - 2774 f_{n+\frac{2}{5}} + 2616 f_{n+\frac{3}{5}} - 1274 f_{n+\frac{4}{5}}) \quad (26)$$

$$g_{n+\frac{2}{5}} = g_n + \tfrac{h}{450}(29 f_{n+1} + 269 f_{n+\frac{1}{5}} - 266 f_{n+\frac{2}{5}} + 294 f_{n+\frac{3}{5}} - 146 f_{n+\frac{4}{5}}) \quad (27)$$

$$g_{n+\frac{3}{5}} = g_n + \tfrac{3h}{400}(9 f_{n+1} + 79 f_{n+\frac{1}{5}} - 66 f_{n+\frac{2}{5}} + 104 f_{n+\frac{3}{5}} - 46 f_{n+\frac{4}{5}}) \quad (28)$$

$$g_{n+\frac{4}{5}} = g_n + \tfrac{2h}{225}(7 f_{n+1} + 67 f_{n+\frac{1}{5}} - 58 f[n+2/5] + 102 f_{n+\frac{3}{5}} - 28 f_{n+\frac{4}{5}}) \quad (29)$$

The methods (20)-(29) comprise the single-step modified block hybrid method, abbreviated as MBHM in this study.

RESULTS

Analysis of basic properties of the method

To fully understand this section, some useful definitions that can be found in the literature are given below:
Definition 1: The linear difference operator L associated with (18) is defined as

$$L[y(x);h] = h^{\lambda} A Y_m^{(n)} = h^{\lambda} B y_{m-i}^{(n)} + h^{\mu-\lambda} C F(Y_m) \quad (30)$$

where $y(x)$ is an arbitrary test function continuously differentiable on $[a,b]$. Expanding $Y_m^{(n)}$ and $F(Y_m)$ component-wise in the Taylor series and collecting terms in powers of h gives

$$L[y(x);h] = \sum_{i=0}^{\infty} \overrightarrow{C_i} h^{(i)} y^{(i)}(x) \quad (31)$$

where $\overrightarrow{C_i}, i = 0,1,\ldots$ are vectors.

Definition 2: The block method (18) and its associated linear difference operator (30) are said to have order p if $\overrightarrow{C_r} = 0, r \le p+1$ and $\overrightarrow{C_{p+2}} \ne 0$ are called the error constants of the method. The analysis of the MBHM (20)-(29) shows that its order $p = [5,5,5,5,5,5,5,5,5,5]^T$ with the error constant

$$\overrightarrow{C_{p+2}} = \left[\frac{-61}{3150000}, \frac{-1231}{393750000}, \frac{-71}{9843750}, \frac{-123}{10937500}, \frac{-376}{24609375}, \frac{-19}{/900000}, \frac{-19}{900000}, \frac{-14}{703125}, \frac{-51}{2500000}, \frac{-14}{703125}\right]^T$$

.

Definition 3: The block method (18) is said to be zero stable as $h \to 0$ if its first characteristic polynomial

$$\vec{\rho}(z) = \det[zA' - B'] = z^{r-\mu}(z-1)^{\mu} = 0 \quad (32)$$

where r is the order of the matrices A' and B' and the roots $z_s, s = 1(1)10$ of (32) satisfy the condition $|z_s| \le 1$ and those roots with $|z| = 1$ have multiplicity not exceeding the order of the differential equation.

The new method MBHM (20)-(29) satisfies the above conditions since from (18), $r = 10$ and $\mu = 2$. Thus, $\det[zA' - B'] = z^8(z-1)^2 = 0$. The new block method is zero-stable as the roots of $\det[zA' - B'] = z^8(z-1)^2 = 0$ satisfy the above definition, hence convergent as it is both zero-stable and consistent.

Numerical experiments

In this section, we compare the results of some general second-order initial value problems solved with the new method (MBHM) to those obtained using some existing methods in the literature. We evaluate the performance of MBHM in terms of absolute errors, as shown in Tables 1-3, where we have used the notation $a(b) := a \times 10^b$. We use the following code names for the various methods used in the comparison to keep things simple:

MBNM: Eq. (20)-(29) of this research with order $p = 5$.

AAS (2014): Adesanya *et al.* (2014) with order $p = 5$

AO (2016): Abdelrahim and Omar (2016).

OTAK (2022): Ogunniran *et al.* (2022) with order $p = 7$.

Problem 1: (Source: Adesanya *et al.* 2014). Consider the nonlinear initial value problem

$$y'' - x(y')^2 = 0,\ y(0) = 1,\ y'(0) = \frac{1}{2}, h = \frac{1}{100}, 0 \le x \le 1$$

Exact solution: $y(x) = 1 + \frac{1}{2}\ln\left(\frac{2+x}{2-x}\right)$. Table 1 contrasts the findings from AO (2016) with those from MBHM (20)– (29), and the lesser errors found there suggest greater accuracy than in AO (2016), all of order $p = 5$.

Table 1: Comparison of the absolute errors of the MBHM with Adesanya *et al.* (2014) for problem 1.

x	Exact result $y(x)$	MBHM Result $y_n(x)$	Error in MBHM (20)-(29)	Error in AAS (2014)
0.1	1.0500417292784912682	1.0500417292784913335	6.531000(-17)	2.375877(-14)
0.2	1.1003353477310755806	1.1003353477310758422	2.616100(-16)	1.958433(-13)
0.3	1.1511404359364668053	1.1511404359364674275	6.222100(-16)	6.901146(-13)
0.4	1.2027325540540821910	1.2027325540540834025	1.211510(-15)	1.708411(-12)
0.5	1.2554128118829953416	1.2554128118829974847	2.143110(-15)	3.496758(-12)
0.6	1.3095196042031117155	1.3095196042031153355	3.620010(-15)	6.361800(-12)
0.7	1.3654437542713961691	1.3654437542714021577	5.988610(-15)	1.069567(-11)
0.8	1.4236489301936018068	1.4236489301936116953	9.888510(-15)	1.701372(-11)
0.9	1.4847002785940517416	1.4847002785940682823	1.654071(-14)	2.601008(-11)
1.0	1.5493061443340548457	1.5493061443340832461	2.840041(-14)	3.866063(-11)

Problem 2: (Source: Abdelrahim and Omar, 2016). Consider the system of second-order ordinary differential equations

$$y_1'' = -e^{-x}y_2, \quad y_1(0)=1, \ y_1'(0)=0, \ h=0.01$$

$$y_2'' = 2e^{x}y_1', \quad y_2(0)=1, \ y_2'(0)=1$$

Exact solution: $y_1(x)=\cos x, \ y_2(x)=e^x\cos x$. Table 2 compares the results in AO (2016) and MBHM (20)-(29). Again, the smaller errors in the MBHM demonstrate an improvement in accuracy over AO (2016).

Table 2: Comparison of the absolute errors of the MBHM with Abdelrahim and Omar (2016) for problem 2.

x	Exact solution of y_1	MBHM solution of y_1	Error of y_1 in MBHM (20)-(29)	Error of y_1 in AO (2016)
0.2	0.980066577841241630	0.98006657784124126591	-3.6521 $\times 10^{-16}$	3.348466 $\times 10^{-9}$
0.4	0.921060994002884990	0.92106099400288340625	-1.67655 $\times 10^{-15}$	3.276545 $\times 10^{-8}$
0.6	0.825335614909678110	0.82533561490967467453	-3.62271 $\times 10^{-15}$	1.332214 $\times 10^{-7}$
0.8	0.69670670934716505	0.69670670934715917733	-6.24359 $\times 10^{-15}$	3.546280 $\times 10^{-7}$
1.0	0.540302305868139210	0.54030230586813073067	-8.98673 $\times 10^{-15}$	7.355177 $\times 10^{-7}$

Problem 3: (Source: Ogunniran *et al.* 2016). Consider the Linear singular non-homogeneous Lane-Emden equation

$$y''+\frac{8}{x}y'+xy = x^5 - x^4 + 44x^2 - 30x, \ y(0)=0, \ y'(0)=0, h=\frac{1}{32}$$

.

Exact solution: $y(x)=x^4-x^3$. Table 3 compares the outcomes of OTAK (2022) and MBHM (20)- (29). Again, the smaller errors in the MBHM indicate that, despite its order $p=5$, it is more accurate than the higher order $p=7$ in OTAK (2022).

Table 3: Comparison of the absolute errors of the MBHM with Ogunniran *et al*., (2022) for problem 3.

x	Exact solution	MBHM result	Error in MBHM (20)-(29)	Error in OTAK (2022)
0.03125	-0.00002956390381	-0.00002956390380	1.000e-14	7.0000e-14
0.09375	-0.00074672698975	-0.00074672698974	1.000e-13	0.0000
0.15625	-0.00321865081787	-0.00321865081787	0.0000	0.0000
0.21875	-0.00817775726318	-0.00817775726318	0.0000	3.0000e-12
0.28125	-0.01599025726318	-0.01599025726318	0.0000	0.0000
0.34375	-0.02665615081787	-0.02665615081787	0.0000	0.0000
0.40625	-0.03980922698975	-0.03980922698976	0.0000	0.0000

Table 3: Continued

x	Exact solution	MBHM result	Error in MBHM (20)-(29)	Error in OTAK (2022)
0.46875	-0.05471706390381	-0.05471706390381	0.0000	1.0000e-11
0.53125	-0.07028102874756	-0.07028102874756	0.0000	3.0000e-11
0.59375	-0.08503627777100	-0.08503627777099	0.0000	1.0000e-11
0.65625	-0.09715175628662	-0.09715175628663	0.0000	1.0000e-11
0.71875	-0.10443019866943	-0.10443019866943	0.0000	0.0000
0.78125	-0.10430812835693	-0.10430812835694	0.0000	0.0000
0.84375	-0.09385585784912	-0.09385585784912	0.0000	1.0000e-10
0.90625	-0.06977748870850	-0.06977748870850	0.0000	1.1000e-10
0.96875	-0.02841091156006	-0.02841091156006	0.0000	1.0000e-10

CONCLUSION

We developed a new single-step hybrid method of order five in this study and implemented it as a set of numerical integrators using the block approach for the direct solution of general second-order ordinary differential equations. The new block hybrid technique converges because it is consistent and zero-table. Indeed, the block approach yielded the numerical solution at all of the desired points of interest at the same time, and the performance of the new method indicated that the solution was more accurate than similar existing methods in the literature; thus, we recommend it for the direct solution of second-order ordinary differential equations.

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