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ORIGINAL RESEARCH ARTICLE

Development of New Generalized Odd Fréchet-Exponentiated-G Family of Distribution

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ABSTRACT
The study examines the limitations of existing parametric distributional models in accommodating various real-world datasets and proposes an extension termed the New Generalized Odd Fréchet-Exponentiated-G (NGOF-Et-G) family. Building upon prior work, this new distribution model aims to enhance flexibility across datasets by employing the direct substitution method. Mathematical properties including moments, entropy, moment generating function (mgf), and order statistics of the NGOF-Et-G family are analyzed, while parameters are estimated using the maximum likelihood technique. Furthermore, the study introduces the NGOF-Et-Rayleigh and NGOF-Et-Weibull models, evaluating their performance using lifetime datasets. A Monte Carlo simulation is employed to assess the consistency and accuracy of parameter estimation methods, comparing maximum likelihood estimation (MLE) and maximum product spacing (MPS). Results indicate the superiority of MLE in estimating parameters for the introduced distribution, alongside the enhanced flexibility of the new models in fitting positive data compared to existing distributions. In conclusion, the research establishes the potential of the proposed NGOF-Et-G family and its variants as promising alternatives in modelling positive data, offering greater flexibility and improved parameter estimation accuracy, as evidenced by Monte Carlo simulations and real-world dataset applications.

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Development, New-Generalized-Odd-Fréchet-G-Family, Exponentiated, Moments, Hazard, Maximum-Likelihood-Estimation, MPS, Simulation NGOF-Et-G.

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INTRODUCTION

The aim of presenting a novel family of distributions is to address the challenges posed by the current probability distributions, Khalil *et al.* (2021). Real-world occurrences are frequently described using statistical distributions. Because statistical distributions are so useful, much research has been done on their theory, and new distributions are constantly being created, Sadiq *et al.* (2023). There is still much interest in creating more adaptable statistical distributions within the statistics community. Numerous generalized classes of distributions have been created and used to explain a wide range of occurrences, Alzaatreh *et al.* (2013). These distribution families were suggested to create new compound probability distributions that significantly improve the baseline distributions' capacity to model actual occurrences, Khaleel *et al.* (2020). Our study examines the limitations of existing parametric distributional models in accommodating various real-world datasets in connection to flexibility and robustness and proposes an extension termed the New Generalized Odd Fréchet-Exponentiated-G (NGOF-Et-G) family. Building upon prior work, this new distribution model aims to enhance flexibility across datasets by employing the direct substitution method. Extreme value (EV) theory has a prominent function in statistical analysis. The generalized extreme value (GEV) distribution is the most frequently used to describe extreme observations. The Gumbel, Weibull, and Fréchet family of distributions are specific examples of the GEV distribution, Ramos *et al.* (2020). The Fréchet distribution, also viewed as the EV distribution of type II, was introduced by the Western mathematician Maurice René (MR) Fréchet in the 1920s as a maximum value distribution. Abubakar Sadiq *et al.* (2023) explain the GEV distribution and its extensive implementations in various disciplines such as sea currents, natural disasters, horse racing, heavy rainfall, supermarket queues, and wind speeds. Alizadeh *et al.* (2017), statistical models are crucial in describing and forecasting countless real-world events. Several extended and comprehensive distributions have remained broadly employed to model data in different domains over the last few decades. Recent advances in statistical literature have focused on describing innovative families of distributions that can outspread renowned distributions and, at the

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same time, deliver prodigious flexibility in demonstrating observational facts in practice. Therefore, different categories have been proposed for breeding novel distributions by accumulating one or more parameters. Some acknowledged families of distribution were the NGOF-G by Abubakar Sadiq *et al.* (2023), a modified T-X family by Aslam *et al.* (2020), the Odd-Burr generalized family by Alizadeh *et al.* (2017), on generating T-X family by Aljarrah *et al.* (2014), Logistic-X family by Tahir *et al.* (2016), TGOGEG by Reyad. *et al.* (2019), General Linear Model by Sadiq *et al.* (2020), NOBPBX distrbution by Abubakar *et al.* (2023).

According to Abubakar-Sadiq *et al.* (2023), a random variable X is said to have a New Generalized Odd Fréchet-G family of distribution with scale parameter α and shapes parameter β and γ if its CDF (cumulative distribution function) is presented as (for all $x, \alpha, \beta, \gamma, \xi > 0$),

$$F_{NGOF-G}(x;\alpha,\beta,\gamma,\xi) = exp\left\{-\left(\alpha\left(F_{cdf}^{-\gamma}(x;\xi) - 1\right)\right)^{\beta}\right\} \quad (1)$$

Gupta *et al.* (1998) defined the wide-ranging class of exponentiated-G family of distributions. A random variable X is said to have an exponentiated family of distribution with scale parameters if its CDF is given as (for all $x, \delta, \xi > 0$)

$$F_{cdf}(x;\delta,\xi) = [G(x;\xi)]^{\delta}; \forall x, \delta \geq 0 \quad (2)$$

MATERIALS AND METHODS:

The New Family

Using the direct substitution method, putting equation (2) into equation (1), our established family called the New Generalized Odd Fréchet-Exponentiated-G (NGOF-Et-G) family CDF and pdf are set as (for all $x, \alpha, \beta, \gamma, \delta, \xi > 0$)

$$F_{NGOF-Et-G}(x;\alpha,\beta,\gamma,\xi) = exp\left\{-\left(\alpha\left(G^{-\gamma\delta}(x;\xi) - 1\right)\right)^{\beta}\right\} \quad (3)$$

The equivalent pdf of equation (3) is presented by

$$f_{NGOF-Et-G}(x;\alpha,\beta,\gamma,\delta,\xi) = \beta\gamma\alpha^{\beta}\delta g(x;\xi)G^{-(\gamma\delta+1)}(x;\xi)\left(G^{-\gamma\delta}(x;\xi) - 1\right)^{\beta-1} exp\left\{-\left(\alpha\left(G^{-\gamma\delta}(x;\xi) - 1\right)\right)^{\beta}\right\} \quad (4)$$

where $g(x;\xi)$ and $G(x;\xi)$ are the pdf and CDF of the baseline distribution and ξ is the parameter vector. However, a random variable X with density function and distribution function in equations (3) and (4) is denoted by $X \sim NGOF - Et - G\ (\alpha, \beta, \gamma, \delta, \xi)$.

Now, Table 1 offers approximately the distinctive members of the NGOF-Et-G family.

Table 1: Some NGOF-Et-G Distributions

SN	α	γ	δ	$F_{NGOF-Et-G}(x;\alpha,\beta,\gamma,\delta,\xi)$	Special cases reduced distributions.
1	1	1	1	$exp\{-(G^{-1}(x;\xi) - 1)^{\beta}\}$	Odd Frechet-G family; Ulhaq and Elgarhy (2018)
2	1	-	1	$exp\{-(G^{-\gamma}(x;\xi) - 1)^{\beta}\}$	The Generalized Odd Frechet family; Marganpoor *et al.* (2020)
3	-	-	1	$exp\left\{-\left(\alpha(G^{-\gamma}(x;\xi) - 1)\right)^{\beta}\right\}$	New Generalized Odd Frechet-G family; Abubakar Sadiq *et al.* (2023)

However, the hazard function for a random variable X that follows the NGOF-Et-G family is expressed as follows:

$$h_{NGOF-Et-G}(x;\alpha,\beta,\gamma,\delta,\xi) = \beta\gamma\alpha^{\beta}\delta g(x;\xi)G^{-(\gamma\delta+1)}(x;\xi)\left(G^{-\gamma\delta}(x;\xi) - 1\right)^{\beta-1} exp\left\{-\left(\alpha\left(G^{-\gamma\delta}(x;\xi) - 1\right)\right)^{\beta}\right\}\left(1 - exp\left\{-\left(\alpha\left(G^{-\gamma\delta}(x;\xi) - 1\right)\right)^{\beta}\right\}\right)^{-1} \quad (5)$$

To obtain the quantile function of the NGOF-Et-G family, we need to invert the CDF in equation (3). Assuming that the variable U is uniformly distributed on the interval (0,1), we can use this to find the inverse of the CDF and determine the corresponding quantiles.

$$x = \Psi(u) = G^{-1}\left(\frac{\alpha}{\alpha + (-log(u))^{\frac{1}{\beta}}}\right)^{\frac{1}{\gamma\delta}} \quad (6)$$

where G^{-1} is the quantile function of the baseline distribution $G(x;\xi)$. And $0 < u < 1$.

The Special NGOF-Et-G Family of Distribution

Lifetime distributions are crucial in various fields, including survival analysis, biomedical science, engineering, and social sciences. Generally, lifetime refers to the length of human life, the lifespan of a device before its failure, or the survival time of a patient with a severe illness from diagnosis to death. In this article, we introduce two special NGOF-Et-G distributions that may come in handy for applications.

The NGOF-Et-Rayleigh distribution

The NGOF-Et-Rayleigh distribution is defined from equations (3) and (4) by taking $G(x\ ;\ \xi) = 1 - exp\left\{-\left(\frac{\phi}{2}x^2\right)\right\}$ and $g(x\ ;\ \xi\ ) = \phi x\ exp\left\{-\left(\frac{\phi}{2}x^2\right)\right\}$ to be the Rayleigh distribution with positive parameters ϕ and $\xi = \phi$. The CDF and pdf of the NGOF-Et-Rayleigh distribution are given by (for $x > 0$)

$$F_{NGOF-Et-R}(x;\alpha,\beta,\delta,\phi) = exp\left\{-\left(\alpha\left(\left(1 - exp\left\{-\left(\frac{\phi}{2}x^2\right)\right\}\right)^{-\gamma\delta} - 1\right)\right)^{\beta}\right\} \quad (7)$$

$$f_{NGOF-Et-R}(x;\alpha,\beta,\delta,\phi) = \beta\gamma\alpha^{\beta}\delta\left(\phi x\ exp\left\{-\left(\frac{\phi}{2}x^2\right)\right\}\right)\left(1 - exp\left\{-\left(\frac{\phi}{2}x^2\right)\right\}\right)^{-(\gamma\delta+1)} \left(\left(1 - exp\left\{-\left(\frac{\phi}{2}x^2\right)\right\}\right)^{-\gamma\delta} - 1\right)^{\beta-1} exp\left\{-\left(\alpha\left(\left(1 - exp\left\{-\left(\frac{\phi}{2}x^2\right)\right\}\right)^{-\gamma\delta} - 1\right)\right)^{\beta}\right\} \quad (8)$$

for all $x;\alpha,\beta,\gamma,\delta,\phi > 0$.

The NGOF-Et-Weibull distribution

The NGOF-Et-Weibull distribution is defined from equations (3) and (4) by taking $G(x\ ;\ \xi) = 1 - exp\left\{-\left(\frac{x}{\phi}\right)^{\omega}\right\}$ and $g(x\ ;\ \xi) = \omega\phi^{-\omega}x^{\omega-1}\ exp\left\{-\left(\frac{x}{\phi}\right)^{\omega}\right\}$ to be the Weibull distribution with positive parameters ϕ, ω and $\xi = (\omega,\phi)$. The CDF and pdf of the NGOF-Et-Weibull distribution are given by (for $x > 0$)

$$F_{NGOFEtW}(x;\alpha,\beta,\delta,\phi\ ,\ \omega) = exp\left\{-\left(\alpha\left(\left(1 - exp\left\{-\left(\frac{x}{\phi}\right)^{\omega}\right\}\right)^{-\gamma\delta} - 1\right)\right)^{\beta}\right\} \quad (9)$$

$$f_{NGOFEtW}(x;\alpha,\beta,\delta,\phi\ ,\ \omega) = \beta\gamma\alpha^{\beta}\delta\left(\omega\phi^{-\omega}x^{\omega-1}\ exp\left\{-\left(\frac{x}{\phi}\right)^{\omega}\right\}\right)\left(1 - exp\left\{-\left(\frac{x}{\phi}\right)^{\omega}\right\}\right)^{-(\gamma\delta+1)}\left(\left(1 - exp\left\{-\left(\frac{x}{\phi}\right)^{\omega}\right\}\right)^{-\gamma\delta} - 1\right)^{\beta-1} exp\left\{-\left(\alpha\left(\left(1 - exp\left\{-\left(\frac{x}{\phi}\right)^{\omega}\right\}\right)^{-\gamma\delta} - 1\right)\right)^{\beta}\right\} \quad (10)$$

for all $x;\alpha,\beta,\gamma,\delta,\phi,\omega > 0$

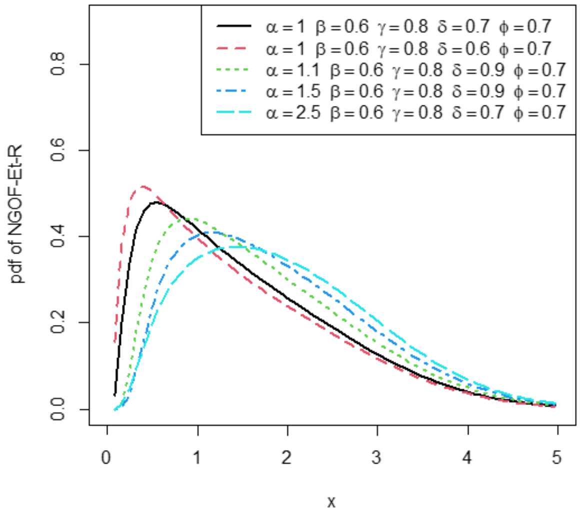

Figure 1: PDF Plot of New Generalized Odd Frechet Exponentiated-Rayleigh Distribution

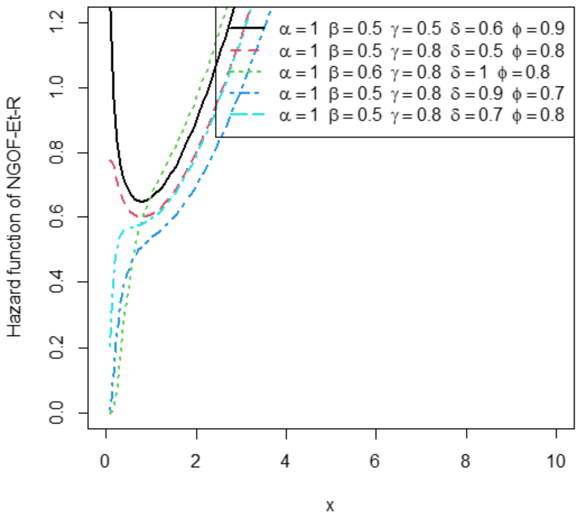

Figure 2: Hazard Plot of New Generalized Odd Frechet-Exponentiated-Rayleigh Distribution

Figures 1 and 3 show the density function of the NGOF-Et-Rayleigh NGOF-Et-Weibull models at different parameter values. The figures display the shapes and behaviour of the distribution and how the parameters interact with one another. For example, the distribution is symmetrical if the parameters have equal values. However, if the values differ, the distribution becomes more positively skewed. Additionally, the greater the difference between the parameter values, the less pronounced the bell shape of the distribution. Furthermore, Figures 2 and 4 show the hazard functions of the NGOF-Et-Rayleigh NGOF-Et-Weibull at various parameter values. The graph displays the modified unimodal and modal shapes of hazard rates at different parameter values.

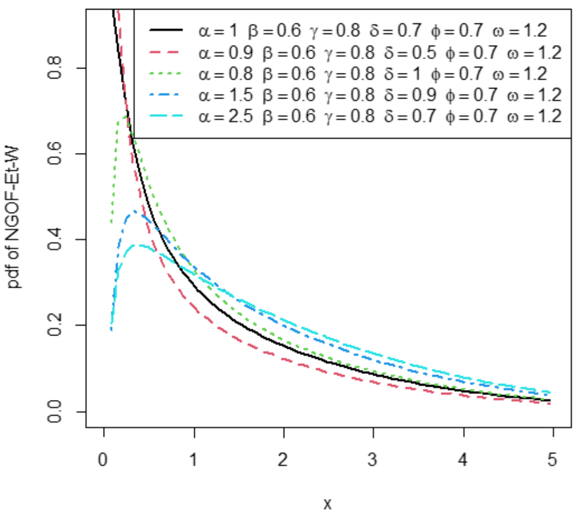

Figure 3: PDF Plot of New Generalized Odd Frechet-Exponentiated-Weibull Distribution

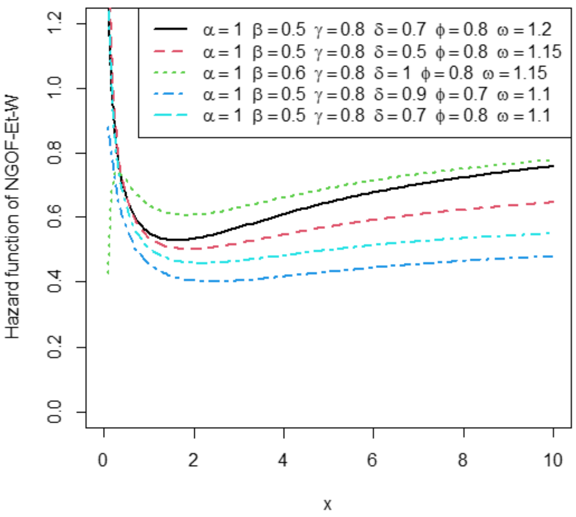

Figure 4: Hazard Plot of New Generalized Odd Frechet-Exponentiated-Weibull Distribution

Useful Expansions

Let's take a closer look at the terms in the NGOF-Et-G family's CDF presented in equations (3). We can use standard mathematical expansions such as the generalized binomial expansion for negative and positive powers, the power series expansion, and more to break down each term.

$$F_{NGOF-Et-G}(x;\alpha,\beta,\gamma,\delta,\xi) = exp\left\{-\left(\alpha\left(G^{-\gamma\delta}(x;\xi)-1\right)\right)^{\beta}\right\} = \sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}}{i!}\alpha^{i\beta}\binom{i\beta}{j}G^{\gamma\delta(j-i\beta)}(x;\xi) \quad (11)$$

Therefore, equation (11) reduces to,

$$F_{NGOF-Et-G}(x;\alpha,\beta,\gamma,\delta,\xi) = \sum_{i,j=0}^{\infty}M_{i,j}G^{\gamma\delta(j-i\beta)}(x;\xi) \quad (12)$$

where $M_{i,j} = \frac{(-1)^{i+j}}{i!}\alpha^{i\beta}\binom{i\beta}{j}$

Differentiating equation (12) w.r.t. x, we have the corresponding pdf presented as:

$$f_{NGOF-Et-G}(x;\alpha,\beta,\gamma,\delta,\xi) = \sum_{i,j=0}^{\infty}M_{i,j}\gamma\delta(j-i\beta)g(x;\xi)G^{\gamma\delta(j-i\beta)-1}(x;\xi) \quad (13)$$

Further simplification of equation (12) is as follows,

$$F_{NGOF-Et-G}(x;\alpha,\beta,\gamma,\delta,\xi) = \sum_{k=0}^{\infty} c_k D_k(x) \quad (14)$$

where $c_k = \sum_{i,j=0}^{\infty}M_{i,j}$ *and* $D_k(x) = G^{\gamma\delta(j-i\beta)}(x;\xi)$

Differentiating equation (14) w.r.t. x, we obtained the corresponding pdf presented as:

$$f_{NGOF-Et-G}(x;\alpha,\beta,\gamma,\delta,\xi) = \sum_{k=0}^{\infty} c_k d_k(x) \quad (15)$$

where $d_k(x) = kg(x;\xi)G^{k-1}(x;\xi)$

MATHEMATICAL PROPERTIES

Moments

We can calculate the rth ordinary moment of a random variable X that follows the NGOF-Et-G family using equation (15). This equation allows us to express the moment in terms of the gamma function and other parameters of the distribution.

$$\mu_r' = E(X^r)$$

$$= \int_0^{\infty} x^r f_{NGOF-Et-G}(x;\alpha,\beta,\gamma,\delta,\xi)dx$$

$$= \int_0^{\infty} x^r \sum_{k=0}^{\infty} c_k d_k(x)\,dx$$

$$= \sum_{k=0}^{\infty} c_k \int_0^{\infty} x^r d_k(x)\,dx$$

$$= \sum_{k=0}^{\infty} c_k E[Z_k^r] \quad (16)$$

where $E[Z_k^r] = \int_0^{\infty} x^r kg(x;\xi)G^{k-1}(x;\xi)\,dx$

Moment-Generating Function

The moment-generating function of a random variable X that follows the New Generalized Odd Frechet-Exponentiated-G (NGOF-Et-G) family by using equation (15) we have,

$$M_X^{NGOF-Et-G}(t) = E(e^{tx})$$

$$= \int_0^{\infty} x^r f_{NGOF-Et-G}(x;\alpha,\beta,\gamma,\delta,\xi)dx$$

$$= \int_0^{\infty} x^r \sum_{k=0}^{\infty} c_k d_k(x)\,dx$$

$$= \sum_{k=0}^{\infty} c_k \int_0^{\infty} x^r d_k(x)\,dx$$

$$= \sum_{k=0}^{\infty} c_k E[Z_k^r] \quad (17)$$

where $E[e^{tZ_k}] = \int_0^{\infty} e^{tx} kg(x;\xi)G^{k-1}(x;\xi)\,dx$

Entropies

However, the entropy of a random variable X measures its unpredictability or uncertainty. For a variable that follows the NGOF-Et-G family, we can calculate its entropy using equation (15). This equation considers the distribution's gamma function and other parameters to determine its entropy, which reflects the level of variability and detail in the potential outcomes of the variable.

$$I_R(\varpi) = \frac{1}{1-\varpi} log\left(\int_0^\infty f^{\varpi}{}_{NGOF-Et-G}(x;\alpha,\beta,\gamma,\delta,\xi)\ dx\right)$$
$$= \frac{1}{1-\varpi} log\left(\int_0^\infty \left(\sum_{k=0}^\infty c_k d_k(x)\right)^{\varpi} dx\right) \quad (18)$$

where $\varpi > 0$ and $\varpi \neq 1$

The nth entropy is defined by

$$I_{nth}(\varpi) = \frac{1}{\varpi-1} log\left(1 - \int_0^\infty f^{\varpi}{}_{NGOF-Et-G}(x;\alpha,\beta,\gamma,\delta,\xi)\ dx\right)$$
$$= \frac{1}{1-\varpi} log\left(1 - \int_0^\infty \left(\sum_{k=0}^\infty c_k d_k(x)\right)^{\varpi} dx\right) \quad (19)$$

where $\varpi > 0$ and $\varpi \neq 1$

Order Statistics

Let's consider a random sample $X_1, X_2, X_3, \dots, X_n$ from the NGOF-Et-G distribution, $X_{i:n}$ representing the ith order statistic. Using equations (14) and (15), we can calculate various statistical properties of the sample, such as its moments and entropy, which provide insight into the distribution and its characteristics.

$$f_{i:n}(x;\alpha,\beta,\gamma,\delta,\xi) = \frac{n!}{[(i-1)!\,(n-i)!]}[f_{NGOF-Et-G}(x;\alpha,\beta,\gamma,\delta,\xi)] [F_{NGOF-Et-G}(x;\alpha,\beta,\gamma,\delta,\xi)]^{i-1}[1-F_{NGOF-Et-G}(x;\alpha,\beta,\gamma,\delta,\xi)]^{n-i}$$

$$= \frac{n!}{[(i-1)!(n-i)!]}\left[\sum_{k=0}^\infty c_k d_k(x)\right]\left[\sum_{k=0}^\infty c_k D_k(x)\right]^{i-1}\left[1-\sum_{k=0}^\infty c_k D_k(x)\right]^{n-i} \quad (20)$$

Estimation

Suppose that $x_1, x_2, x_3, \dots, x_n$ are the observed values from the proposed NGOF-Et-G family with parameters $\alpha, \beta, \gamma, \delta$ and ξ. Suppose that $\Phi = [\alpha, \beta, \gamma, \delta, \xi]^T$ is the $[m \times 1]$ vector of the parameter. The log-likelihood function Φ using equation (4) is expressed by

$$\ell_n = \ell_n(\Phi) = n\,log(\beta) + n\,log(\gamma) + n\beta\,log(\alpha) + n\,log(\delta) + \sum_{i=1}^n log[g(x;\ \xi)] - (\gamma\delta+1)\sum_{i=1}^n log[G(x;\ \xi)] + (\beta-1)\sum_{i=1}^n log\left[G^{-\gamma\delta}(x;\ \xi)-1\right] - \sum_{i=1}^n\left[\alpha\left(G^{-\gamma\delta}(x;\ \xi)-1\right)\right]^\beta \quad (21)$$

Taking the partial derivative of equation (21) w.r.t., the parameters $(\alpha;\ \beta;\ \gamma;\delta;\xi)$ are respectively given as:

$$\frac{\partial\ell_n(\Phi)}{\partial\alpha} = \frac{n\beta}{\alpha} - \sum_{i=1}^n\left(G^{-\gamma\delta}(x;\ \xi)-1\right)^\beta \quad (22)$$

$$\frac{\partial\ell_n(\Phi)}{\partial\beta} = \frac{n}{\beta} + n\,log(\alpha) + \sum_{i=1}^n log\left(G^{-\gamma\delta}(x;\ \xi)-1\right) - \sum_{i=1}^n\left[\alpha\left(G^{-\gamma\delta}(x;\ \xi)-1\right)\right]^\beta ln\left(\alpha\left(G^{-\gamma\delta}(x;\ \xi)-1\right)\right) \quad (23)$$

$$\frac{\partial\ell_n(\Phi)}{\partial\gamma} = \frac{n}{\gamma} - (\delta+1)\sum_{i=1}^n log[G(x;\ \xi)] + (\beta-1)\sum_{i=1}^n \frac{\delta G^{-\gamma\delta}(x;\ \xi)\,ln\left(G(x;\ \xi)\right)}{[G^{-\gamma\delta}(x;\ \xi)-1]} - \sum_{i=1}^n\left[-\delta\alpha\left(G^{-\gamma\delta}(x;\ \xi)-1\right)\right]^\beta ln\left(G(x;\ \xi)\right) \quad (24)$$

$$\frac{\partial\ell_n(\Phi)}{\partial\delta} = \frac{n}{\delta} - (\gamma+1)\sum_{i=1}^n log[G(x;\ \xi)] + (\beta-1)\sum_{i=1}^n \frac{\gamma G^{-\gamma\delta}(x;\ \xi)\,ln\left(G(x;\ \xi)\right)}{[G^{-\gamma\delta}(x;\ \xi)-1]} - \sum_{i=1}^n\left[-\gamma\alpha\left(G^{-\gamma\delta}(x;\ \xi)-1\right)\right]^\beta ln\left(G(x;\ \xi)\right) \quad (25)$$

$$\frac{\partial\ell_n(\Phi)}{\partial\xi} = \frac{g'(x;\ \xi)}{g(x;\ \xi)} - (\gamma\delta+1)\sum_{i=1}^n \frac{G'(x;\xi)}{[G(x;\ \xi)]} + (\beta-1)\sum_{i=1}^n \frac{G'(x;\xi)G(x;\xi)}{(G^{-\gamma\delta}(x;\xi)-1)} + \alpha\beta\gamma\delta\sum_{i=1}^n\left[G'(x;\xi)\left(G^{-\gamma\delta}(x;\ \xi)-1\right)^{\beta-1}\right] \quad (26)$$

The MLEs of the parameters $(\alpha;\ \beta;\ \gamma;\delta;\xi)$, says, $(\hat{\alpha};\ \hat{\beta};\ \hat{\gamma};\hat{\delta};\hat{\xi})$ are the simultaneous solution of equations

(22), (23), (24), (25), and (26) when equated to zero, i.e. $\frac{\partial \ell_n(\Phi)}{\partial \alpha} = 0; \frac{\partial \ell_n(\Phi)}{\partial \beta} = 0; \frac{\partial \ell_n(\Phi)}{\partial \gamma} = 0; \frac{\partial \ell_n(\Phi)}{\partial \delta} = 0; \frac{\partial \ell_n(\Phi)}{\partial \xi}$. These equations are intractable and non-linear and can only be solved using a numerical iterative method.

RESULTS AND DISCUSSIONS

Simulation

Simulations are a popular class of computational algorithms that use replicated random sampling to generate numerical results. The main concept behind Monte Carlo simulations is to use randomness to solve problems that may be theoretically deterministic.

The "results of the Monte Carlo simulation study are presented in Table 2. The study shows that the bias and root mean square error decrease towards zero as the sample size increases". The M.L.E. technique observed and estimated parameter values at different sample sizes and iterative levels, proving its consistency. On the other hand, the M.PS technique has almost unequal actual and estimated parameter values at different sample sizes and iterative levels, proving the least consistency of M.P.S. parameter estimates. Therefore, the M.L.E. technique is the best technique for estimating the parameter of the new generalized Odd Frechet-Exponentiated Rayleigh distribution compared to the M.PS technique.

Table 3 presents "the Monte Carlo Simulation study results. The results indicated that the bias and root mean square error decreases toward zero with increased sample size". However, the actual value of the parameters and the estimated values are almost equal at different sample sizes and iterative levels for the M.L.E technique. This proves the consistency of the MLE parameter estimates. For the M.PS technique, the actual value of the parameters and the estimated values are almost not equal at different sample sizes and iterative levels. This proves the least consistency of the M.P.S parameter estimates. The result also means that the M.L.E technique is the best technique for estimating the parameter of New Generalized Odd Frechet-Exponentiated-Weibull distribution than the M.PS technique.

Table 2: Results of Monte Carlo Simulation from the NGOF-Et-Rayleigh Distribution.

		M.L.E. Techniques			M.P.S. Techniques		
Sample Sizes	Parameters (Actual Values)	Estimates	Bias	RMSE	Estimates	Bias	RMSE
50	α (1.0)	1.0237	0.0237	0.3476	1.0404	0.0404	0.3636
	β (1.0)	1.0148	0.0148	0.1392	0.9585	-0.0415	0.1357
	γ (2.5)	2.5688	0.0688	0.2656	2.5624	0.0624	0.2458
	δ (1.0)	1.0379	0.0379	0.1791	1.0078	0.0078	0.1767
	ϕ (2.0)	2.0476	0.0476	0.1249	2.0286	0.0286	0.1302
100	α (1.0)	1.0243	0.0243	0.2549	1.0438	0.0438	0.2684
	β (1.0)	0.9971	-0.0029	0.0891	0.9678	-0.032	0.0945
	γ (2.5)	2.5684	0.0684	0.1892	2.5618	0.0618	0.1765
	δ (1.0)	1.0217	0.0217	0.1253	0.9954	-0.004	0.1235
	ϕ (2.0)	2.0498	0.0498	0.1043	2.0348	0.0348	0.1044
250	α (1.0)	1.0411	0.0411	0.1800	1.0427	0.0427	0.1679
	β (1.0)	0.9922	-0.007	0.0584	0.9750	-0.025	0.0616
	γ (2.5)	2.5719	0.0719	0.1486	2.5564	0.0564	0.1331
	δ (1.0)	0.9943	-0.005	0.0742	0.9926	-0.007	0.0751
	ϕ (2.0)	2.0471	0.0471	0.0927	2.0426	0.0426	0.0865
500	α (1.0)	1.0353	0.0353	0.1105	1.0379	0.0379	0.1137
	β (1.0)	0.9896	-0.010	0.0405	0.9812	-0.018	0.0427
	γ (2.5)	2.5626	0.0626	0.1013	2.5516	0.0516	0.0889
	δ (1.0)	0.9936	-0.006	0.0444	0.9907	-0.009	0.0476
	ϕ (2.0)	2.0468	0.0468	0.0764	2.0408	0.0408	0.0736
1000	α (1.0)	1.0359	0.0359	0.0883	1.0388	0.0388	0.0929
	β (1.0)	0.9914	-0.008	0.0284	0.9855	-0.0145	0.0305
	γ (2.5)	2.5541	0.0541	0.0855	2.5488	0.0488	0.0837
	δ (1.0)	0.9897	-0.010	0.0333	0.9895	-0.0105	0.0346
	ϕ (2.0)	2.0415	0.0415	0.0687	2.0414	0.0414	0.0693

Table 3: Results of Monte Carlo Simulation from NGOF-Et-Weibull Distribution.

		M.L.E. Techniques			M.P.S. Techniques		
Sample Sizes	Parameters (Actual Values)	Estimates	Bias	RMSE	Estimates	Bias	RMSE
50	α (1.0)	1.0316	0.0316	0.2646	1.0720	0.0720	0.2677
	β (1.0)	0.9988	-0.0012	0.1498	0.9850	-0.0150	0.1513
	γ (2.5)	2.5439	0.0439	0.2497	2.5938	0.0938	0.2571
	δ (1.0)	1.0185	0.0185	0.1469	0.9972	-0.0028	0.1463
	ϕ (2.0)	2.0554	0.0554	0.1098	2.0660	0.06600	0.1222
	ω (3.0)	3.0953	0.0953	0.3578	2.9300	-0.0700	0.3837
100	α (1.0)	1.0320	0.0320	0.1912	1.0623	0.0623	0.2118
	β (1.0)	0.9834	-0.0166	0.1072	0.9834	-0.0166	0.1101
	γ (2.5)	2.5560	0.0560	0.1810	2.5818	0.0818	0.1875
	δ (1.0)	1.0083	0.0083	0.0999	0.9909	-0.0091	0.1026
	ϕ (2.0)	2.0590	0.0590	0.0972	2.0553	0.0553	0.0944
	ω (3.0)	3.0737	0.0737	0.2899	2.9603	-0.0397	0.2854
250	α (1.0)	1.0457	0.0457	0.1207	1.0563	0.0563	0.1262
	β (1.0)	0.9772	-0.0228	0.0729	0.9818	-0.0182	0.0812
	γ (2.5)	2.5469	0.0469	0.1071	2.5573	0.0573	0.1109
	δ (1.0)	0.9928	-0.0072	0.0576	0.9894	-0.0106	0.0629
	ϕ (2.0)	2.0539	0.0539	0.0794	2.0527	0.0527	0.0754
	ω (3.0)	3.0595	0.0595	0.1939	2.9889	-0.0111	0.2110
500	α (1.0)	1.0425	0.0425	0.0858	1.0449	0.0449	0.0901
	β (1.0)	0.9791	-0.0209	0.0530	0.9800	-0.0200	0.0526
	γ (2.5)	2.5400	0.0400	0.0720	2.5443	0.0443	0.0752
	δ (1.0)	0.9914	-0.0086	0.0342	0.9909	-0.0091	0.0375
	ϕ (2.0)	2.0494	0.0494	0.0676	2.0468	0.0468	0.0647
	ω (3.0)	3.0430	0.0430	0.1260	3.0081	0.0081	0.1224
1000	α (1.0)	1.0392	0.0392	0.0686	1.0417	0.0417	0.0717
	β (1.0)	0.9829	-0.0171	0.0383	0.9822	-0.0178	0.0405
	γ (2.5)	2.5280	0.0280	0.0555	2.5309	0.0309	0.0574
	δ (1.0)	0.9923	-0.0077	0.0232	0.9916	-0.0084	0.0258
	ϕ (2.0)	2.0430	0.0430	0.0591	2.0425	0.0425	0.0592
	ω (3.0)	3.0332	0.0332	0.0931	3.0148	0.0148	0.0954

Applications

Here, we evaluated our developed family's flexibility using some existing real-world data sets that serve as a baseline for the Weibull and Rayleigh distributions.

Data set 1

We used the data set that was analyzed by Fulment *et al.* (2023), which represents "the survival times (in years) of a group of patients given chemotherapy treatment. The data are: 0.047, 0.115, 0.121, 0.132, 0.164, 0.197, 0.203, 0.260, 0.282, 0.296, 0.334, 0.395, 0.458, 0.466, 0.501, 0.507, 0.529, 0.534, 0.540, 0.641, 0.644, 0.696, 0.841, 0.863, 1.099, 1.219, 1.271, 1.326, 1.447, 1.485, 1.553, 1.581, 1.589, 2.178, 2.343, 2.416, 2.444, 2.825, 2.830, 3.578, 3.658, 3.743, 3.978, 4.003, 4.033"

Table 4 presents the parameter estimates and goodness of fit measures for the NGOF-Et-Rayleigh distribution, along with other competing models, using a dataset of survival times (in years) for chemotherapy treatment patients. The performance metrics used in this study are

Akaike's Information Criterion (AIC), Bayesian Information Criterion (BIC), and Hannan-Quinn Information Criterion (HQIC). The model with the lowest information or performance metrics is considered the best fit. Based on the data, the NGOFR distribution outperforms other models and best fits this dataset.

Data set 2

The data set "was originally reported by Abubakar Sadiq *et al.* (2023), which represents the Maximum Annual Flood Discharges of North Saskatchewan in units of 1000 cubic feet per second, of the North Saskatchewan River at Edmonton, for 47 years". The data are: 19.885, 20.940, 21.820, 23.700, 24.888, 25.460, 25.760, 26.720, 27.500, 28.100, 28.600, 30.200, 30.380, 31.500, 32.600, 32.680, 34.400, 35.347, 35.700, 38.100, 39.020, 39.200, 40.000, 40.400, 40.400, 42.250, 44.020, 44.730, 44.900, 46.300, 50.330, 51.442, 57.220, 58.700, 58.800, 61.200, 61.740, 65.440, 65.597, 66.000, 74.100, 75.800, 84.100, 106.600, 109.700, 121.970, 121.970, 185.560.

Table 5 presents the parameter estimates and goodness o f fit measures for the NGOF-Et-W distribution, along wi th other competing models, using the dataset for maximu m annual flood discharges. The performance metrics use d in this study are Akaike's Information Criterion (AIC), Bayesian Information Criterion (BIC), and Hannan-Quin n Information Criterion (HQIC). The model with the lo west information or performance metrics is considered t he best fit. Based on the data, the NGOF-Et-W Distribut ion outperforms other models and best fits this dataset.

Table 4: Performance Comparison of the NGOF-Et-R with three others on Dataset 1

Model	Parameter Estimates and Goodness of Fit					
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\delta}$	$\hat{\phi}$	AIC
NGOFEtR	0.1194	0.2972	1.0077	1.1552	1.0729	122.214
GOFR	-	8.3235	0.0287	-	0.0033	28149.8
OFR	-	1.345	-	-	0.5213	197.412
RD	-	-	-	-	0.6025	163.83

Table 5: Performance Comparison of the NGOF-Et-W with three others on Dataset 3

Model	Parameter Estimates and Goodness of Fit							
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\delta}$	$\hat{\phi}$	$\hat{\omega}$	LL	AIC
NGOFEtW	2.6605	4.1128	-5.28	-2.23	2.222	0.1368	215.0648	**438.349**
GOFW	1.3e15	1.4e-01	-	-	5.3e-02	1.3e-01	414.8936	835.78
OFW	2.6391	-	-	-	0.06794	0.64733	215.175	442.129
WD	-	-	-	-	0.00074	1.7724	225.7065	459.413

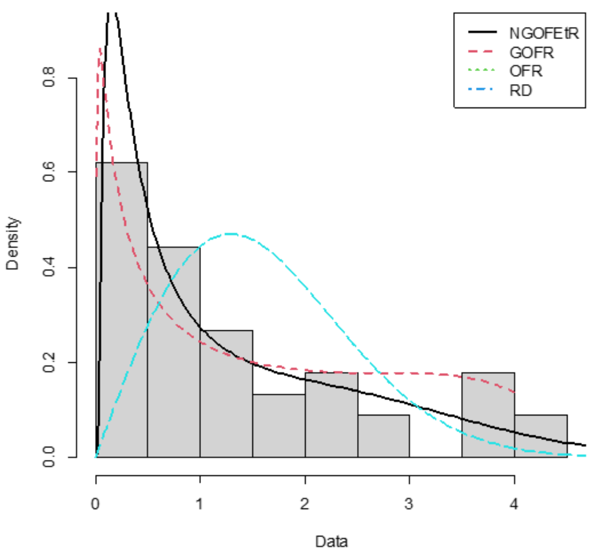

Figure 5: Histogram Plot of the survival times (in years) of a group of patients given chemotherapy treatment

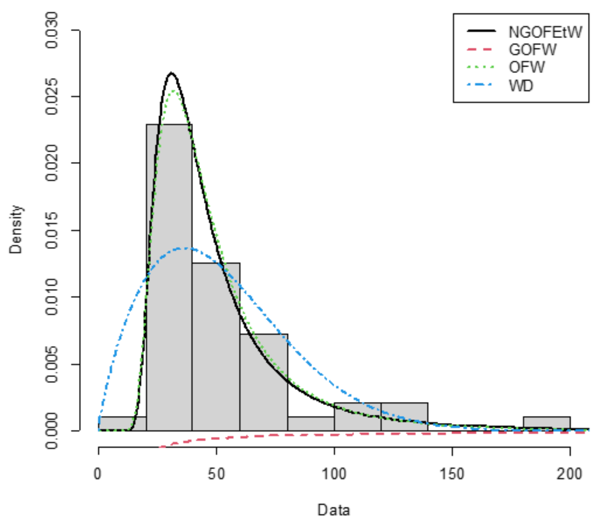

Figure 6: Histogram Plots of the Distribution of Maximum Annual Flood Discharges Data

CONCLUSION

This research paper introduces the NGOF-Et-G family of distributions and explores its statistical properties, including the survival function, hazard function, cumulative hazard function, moments, moment-generating function, entropies, order statistics, and MLE. We also plot the pdf and the hazard rate function to observe the shapes and behaviour of the models at different parameter values. We conduct simulation studies to test the consistency of the MLE and MPS of the parameters. We then apply the NGOF-Et-R distribution to the survival time data of patients who received chemotherapy treatment and the NGOF-Et-W distribution to the data representing Maximum Annual Flood Discharges employing Rayleigh and Weibull as the baseline distribution, respectively. Our analysis shows that the NGOF-Et-R is the "best fit" model for the survival times of patients given chemotherapy treatment data, and the NGOF-Et-W is the "best fit" model for the maximum annual flood discharge data.

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