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ORIGINAL RESEARCH ARTICLE

A model for Optimal Pricing and Ordering Strategies for Perishable Goods with Delayed Deterioration, Two-Stage Demand, and Partial Backorders under Delayed Payment Acceptance

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ABSTRACT

This research developed an economic order quantity model for non-instantaneous deteriorating items with two-phase demand rates, linear holding cost, constant partial backlogging rate and two-level pricing strategies under trade credit policy. It is assumed that the holding cost is linear time-dependent, the unit selling price before deterioration sets in is greater than that after deterioration sets and the demand rate before deterioration sets in is considered as continuous time-dependent quadratic, after which it is considered as constant up to when the inventory is completely exhausted. Shortages are allowed and partially backlogged. The purpose of the model is to determine the optimal time with positive inventory, cycle length and order quantity such that the total profit of the inventory system has a maximum value. The necessary and sufficient conditions for the existence and uniqueness of the optimal solutions have been established. Some numerical examples have been given to illustrate the theoretical result of the model. Sensitivity analysis of some model parameters on the decision variables has been carried out and suggestions towards maximising the total profit were also given. , it is seen that the higher the rate of deterioration (θ), the lower the optimal time with positive inventory (t_1^*), cycle length (T^*), order quantity (EOQ^*) and the total profit $TP(T^*)$ and vice versa. This implies that the retailer needs to take all the necessary measures to avoid or reduce deterioration to maximise higher profit. Based on the results application of the model led to an increase in revenue.

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KEYWORDS

economic order quantity; non-instantaneous deteriorating items; two-phase demand rates; linear holding cost; two-level pricing strategies; constant partial backlogging rate; trade credit policy.

INTRODUCTION

Deterioration refers to damage, spoilage, dryness, vaporisation, etc., that result in a decrease in the usefulness of the commodity. The assumption in the classical EOQ models that items in the stock preserve their original characteristics/conditions forever may not always apply to most physical goods (i.e. which deteriorate with time due to obsolesce, loss of utility, decay, damage, degradation and decrease in their usefulness). Deterioration of goods is an unavoidable phenomenon, and its study plays an essential role in any business organisation's smooth and efficient running. Researchers such as Geetha and Udayakumar (2016), Babangida and Baraya (2020) developed an inventory model with non-instantaneous deterioration under various assumptions.

The classical EOQ model assumes that customers must pay for the goods purchased as soon as it is received. However, in a real market situation, the supplier allows the customers to pay their debt within a specific period, known as the trade credit period. The retailer can accumulate revenues by selling items and by earning interest. The concept of trade credit in the inventory literature was first introduced by Haley and Higgins (1973). Goyal (1985) was the first to propose an EOQ model for non-decaying items with a constant demand rate under permissible delay in payments and assumed that the unit purchasing cost and selling price per unit are the same. Later, Aggarwal and Jaggi (1995) extended Goyal's (1985) model to develop an inventory model for deteriorating items with a constant demand rate under permissible payment delays. Soni and Chauhan (2018) investigated a joint pricing, inventory, and preservation decision-making problem for deteriorating items subject to stochastic demand and promotional effort. The generalised price-dependent stochastic demand, time-proportional deterioration, and partial backlogging rates are used to model the inventory system. The objective is

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to find the optimal pricing, replenishment, and preservation technology investment strategies while maximising the total profit per unit of time.

Mohsen et al. (2018) proposed an economic order quantity model for non-instantaneous deteriorating items under a hybrid payment schedule. This payment schedule comprises a multiple advanced payment scheme and a delayed payment plan. Here, a retailer must prepay a portion of the purchasing cost to his supplier during the order delivery lead time, in several instalments.

Under a trade credit policy, Umakanta et al. (2018) developed an inventory model for deteriorating items with a controllable deterioration rate (using preservation technology). As in practical scenarios, the demand for an item is directly associated with its selling price. Keeping this in mind, it is assumed to be a price-dependent demand.

Bhaula et al. (2019) derived an optimal ordering policy for non-instantaneously deteriorating items under successive price discounts with payment delays. Here, successive price discounts are a strategy to sell almost all the items before decomposition. The cause of implementing this concept in the model is that about 25% of vegetables and fruits in India decayed before selling due to a lack of facilities and awareness of business strategies, although poverty is a vital factor. Thus, we propose to offer successive price discounts of 20% and 40% after selling the stock up to 50% and 90%, respectively, to raise the customer's inflow and rotate the cycle early to avoid more deterioration.

Later researchers such as Babangida and Baraya [(2018), (2019), (2022), (2021a)], Tripathy and Sharma (2022), Sheng and Jinn (2014), Majunder and Kumar (2019) and so on develop inventory models with trade credit policy under various assumptions.

In the classical inventory model, shortages are not allowed. However, sometimes, customers' demands cannot be fulfilled by the supplier from the current stocks. This situation is known as stock out or shortage condition. However, when all the customers are willing to wait for the backorder, the situation is referred to as complete backlogging. Researchers such as Mahato and Mahata (2023), Tiwari et al. (2022), Choudhury et al. (2013), Babangida and Baraya [(2019), (2020)] and so on developed inventory models with complete backlogging under various assumptions.

Moreover, when certain customers constantly wait for the supplier to supply the goods, the situation is called constant partial backlogging. For example, the customers who constantly wait for the backorder might be the supplier's close friends and relatives. Many researchers, such as Baraya and Sani (2013), Bello and Baraya (2018), Babangida and Baraya (2021b), and so on, developed inventory models with constant partial backlogging rates under various assumptions. Furthermore, there are scenarios whereby the customers wait for the backorder based on the time taken before the next replenishment, known as the time-dependent partial backlogging rate. Researchers such as Babangida and Baraya (2022), Geetha and Uthayakumar (2010), Babangida et al. (2023), Umakanta and Chaitanya (2012), Sarkar and Sarkar (2013) developed an inventory model with time-dependent partial backlogging under various assumption.

Babangida and Baraya (2021a) developed an EOQ model for non-instantaneous deteriorating items with two-phase demand rates and two-level pricing strategies under trade credit policy. It is assumed that the unit-selling price before deterioration is greater than after deterioration. In addition, the demand rate before deterioration sets in is assumed to be continuous time-dependent quadratic and is considered constant after deterioration sets in. Holding cost is considered as constant and shortages are not allowed. However, in real-life situations, the holding cost of many items may be dynamic as there is a change in the time value of money and price index. Therefore, the model is extended by considering linear holding cost and constant partial backlogging rate.

The purpose of the model is to determine the optimal time with positive inventory, cycle length and order quantity such that the total profit of the inventory system has a maximum value.

Moreover, if the model is accepted, it will help retailers to increase cash flow, encourage sales, reduce the cost of holding stock, attract new customers, decrease the levels of inventory loss due to deterioration, boost market share or retain customers, increase the cycle length, spread the ordering cost over a long period, reduce the total variable cost of the inventory and generate more revenue.

This paper model considers an EOQ model for non-instantaneous deteriorating items with two phase demand rate, two level pricing strategies, linear holding cost and constant partial backlogging rate under trade credit policy. The demand rate before deterioration sets in is assumed to be time-dependent quadratic, which is considered constant after deterioration sets in. It is also assumed that the unit selling price is different before and after deterioration sets in. The holding cost is assumed to be linear time-dependent. Shortages are allowed with a constant partial backlogging rate.

NOTATIONS AND ASSUMPTIONS

Notation:

The inventory system is developed using the following notations.

A The fixed ordering cost per order

C The purchasing cost per unit time

S_1 Unit selling price during the interval $[0,\ t_d]$

S_2 Unit selling price during the interval $[t_d, T]$, where $S_1 > S_2 > C$

C_b Shortage cost per unit time

I_C The interest charged in stock by the supplier

I_e The interest earned

M The trade credit period (in year for settling account)

θ The constant deterioration rate function

t_d The length of time in which the product exhibit more deterioration

t_1 Length of time in which the inventory has no shortage

T The length of replenishment cycle time

Q_m The maximum inventory level

B_m The backorder level during the shortage period

Q The order quantity during the cycle length i.e.$Q = Q_m + B_m$

C_π Unit cost of lost sales per unit

δ Backlogging parameter

Assumptions

In addition to assumptions 8 and 9, which are not considered in Babangida and Baraya (2021a), this model develops under the following assumptions, which have been adapted from the aforementioned research.

1. The replenishment rate is infinite, i.e., the replenishment rate is instantaneous, and the lead time is zero.
2. During the fixed period,t_d, there is no deterioration, and at the end of this period, the inventory item deteriorates at the rate θ.
3. There is no replacement or repair for deteriorating items.
4. The demand rate before deterioration begins is assumed to be continuous time-dependent quadratic and is given by $a + bt + ct^2$, where $a \geq 0, b \neq 0, c \neq 0c \neq 0$. Here a is the initial demand rate, b is the rate at which the demand rate changes and c is the accelerated change in the demand rate.
5. The demand rate after deterioration sets in is assumed to be constant and is given by d,$d > 0$.
6. During the trade credit period $M(0 < M < 1)$, the account is not settled; generated sales revenue is deposited in an interest-bearing account. At the end of the period, the retailer pays off all units bought and starts to pay the capital opportunity cost for the items in stock. No interest is earned after the trade credit period.
7. The unit selling price is not the same as the unit purchasing cost. It is assumed that the unit selling price before deterioration sets in is greater than that after deterioration sets in $(S_1 > S_2 > C)$.
8. Shortages are allowed and partially backlogged.
9. Holding cost $C_1(t)$ per unit time is linear time-dependent and is assumed to be $C_1(t) = h_1 + h_2t$; where $h_1 > 0$and $h_2 > 0$.

FORMULATION OF THE MODEL

Q_m units of items are ordered at the beginning of the cycle (i.e., at time $t = 0$). During the interval $[0, t_d]$, the inventory level is depleting gradually due to market demand only and the demand rate is assumed to be time dependent quadratic. At time interval $[t_d, t_1]$, the inventory level is depleting due to the combined effects of customer demand and deterioration, and the demand rate reduces to d. At time $t = t_1$, the inventory level depletes to zero. Shortages occur at the time interval $[t_1, T]$ and partially backlogged at the rate δ, the behaviour of the inventory system is described in Figure 1 below:

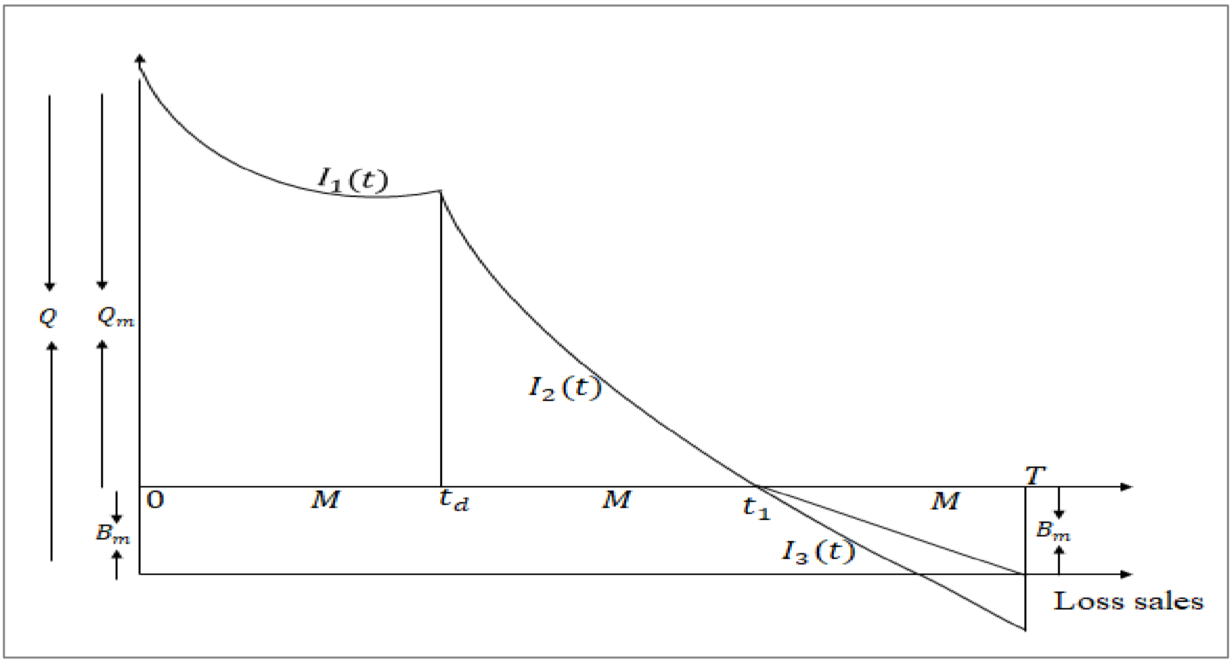

Figure 1: Graphical representation of the inventory system

During the time interval $[0,\ t_d]$, the change of inventory at any time t is represented by the following differential equations

$$\frac{dI_1(t)}{dt} = -(a + bt + ct^2), \qquad 0 \le t \le t_d \tag{1}$$

with boundary conditions $I_1(0) = Q_m$ and $I_1(t_d) = Q_d$.

Graphical representation of the inventory system

$$\frac{dI_2(t)}{dt} + \theta I_2(t) = -d, \qquad t_d \le t \le t_1 \tag{2}$$

With boundary condition $I_2(t_1) = 0$ at $t = t_1$ and $I_2(t_d) = Q_d$ at $t = t_d$

$$\frac{dI_3(t)}{dt} = -\delta d, \qquad t_1 \le t \le T \tag{3}$$

With the boundary condition $I_3(t_1) = 0$ at $t = t_1$ and $I_3(t_1) = 0$ at $t = t_1$.

The solution of equations (1), (2) and (3) are respectively given by

$$I_1(t) = a(t_d - t) + \frac{b}{2}(t_d^2 - t^2) + \frac{c}{3}(t_d^3 - t^3) + Q_d \qquad 0 \le t \le t_d \tag{4}$$

$$I_2(t) = \frac{d}{\theta}\left(e^{\theta(t_1 - t)} - 1\right), \qquad t_d \le t \le t_1 \tag{5}$$

$$I_3(t) = \delta d(t_1 - t) \qquad t_1 \le t \le T \tag{6}$$

From Figure 1, using the condition $I_1(0) = Q_m$ in equation (4), the maximum stock level is given by

$$Q_m = \frac{d}{\theta}\left(e^{\theta(t_1 - t_d)} - 1\right) + \left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3}\right) \tag{7}$$

Similarly, the value of Q_d can be derived at $t = t_d$, then it follows from equation (5) that

$$Q_d = \frac{d}{\theta}\left(e^{\theta(t_1 - t_d)} - 1\right) \tag{8}$$

The maximum backordered inventory B_m is obtained at $t = T$, and then from equation (6), it follows that

$$B_m = d\delta(T - t_1) \tag{9}$$

Therefore, the order size Q during the period $[0,\ T]$ is obtained as the sum of maximum inventory level Q_m and maximum backordered inventory B_m

$$Q = \frac{d}{\theta}\left(e^{\theta(t_1 - t_d)} - 1\right) + \left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3}\right) + \delta d(T - t_1) \tag{10}$$

(i) *The total demand during the period* $[t_d,\ t_1]$ *is given by*

$$D_M = \int_{t_d}^{t_1} d\,dt = d(t_1 - t_d) \tag{11}$$

(ii) *The total number of deteriorated items per cycle is given by*

$$D_P = \frac{d}{\theta}\left[e^{\theta(t_1 - t_d)} - 1 - \theta(t_1 - t_d)\right] \tag{12}$$

(iii) *Total number of items sold*

$$SN = Q - D_P = \left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3}\right) + d(t_1 - t_d) + \delta d(T - t_1) \tag{13}$$

(iv) Sale revenue (SR)

$$SR = S_1\left[\int_0^{t_d}(a + bt + ct^2)dt\right] + S_2\left[\int_{t_d}^{t_1} ddt + \int_{t_1}^{T}\delta ddt\right]$$

$$= S_1\left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3}\right) + S_2 d(t_1 - t_d) + S_2\delta d(T - t_1) \tag{14}$$

(v) Purchasing cost *(PC)*

$$PC = CQ = C\left[\frac{d}{\theta}\left(e^{\theta(t_1 - t_d)} - 1\right) + \left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3}\right) + \delta d(T - t_1)\right] \tag{15}$$

(iv) *The fixed ordering cost per order is given by A*

(v) *The inventory holding cost for the entire cycle is given by*

$$C_H = \int_0^{t_d} (h_1 + h_2 t) I_1(t) dt + \int_{t_d}^{t_1} (h_1 + h_2 t) I_2(t) dt \tag{16}$$

Substituting equations (4) and (5) into equation (16)

$$C_H = h_1 \left[\frac{dt_d}{\theta} e^{\theta(t_1 - t_d)} + \frac{a}{2} t_d^2 + \frac{b}{3} t_d^3 + \frac{c}{4} t_d^4 + \frac{d}{\theta^2} e^{\theta(t_1 - t_d)} - \frac{d}{\theta^2} - \frac{dt_1}{\theta} \right] + h_2 \left[\frac{dt_d^2}{2\theta} e^{\theta(t_1 - t_d)} + \frac{a}{6} t_d^3 + \frac{b}{8} t_d^4 + \frac{c}{10} t_d^5 + \frac{dt_d}{\theta^2} e^{\theta(t_1 - t_d)} - \frac{dt_1}{\theta^2} - \frac{d}{\theta^3} + \frac{d}{\theta^3} e^{\theta(t_1 - t_d)} - \frac{dt_1^2}{2\theta} \right] \tag{17}$$

(vi) *The backordered cost per cycle is given by*

$$SC = C_b \int_{t_1}^{T} -I_3(t) dt = \frac{C_b \delta d}{2} (T - t_1)^2 \tag{18}$$

(vii) The opportunity cost per cycle due to lost sales is given by

$$LC = C_\pi \int_{t_1}^{T} d(1 - \delta) dt = C_\pi d(1 - \delta)(T - t_1) \tag{19}$$

(vii) The total profit per unit time for a replenishment cycle (denoted by $TP(t_{1,}T)$ is given by

$$TP(t_{1,}T) = \begin{cases} TP_1(t_{1,}T) & 0 < M \le t_d \\ TP_2(t_{1,}T) & t_d < M \le t_1 \\ TP_3(t_{1,}T) & M > t_1 \end{cases} \tag{20}$$

where $TP_1(t_{1,}T)$, $TP_2(t_{1,}T)$, and $TP_3(t_{1,}T)$ are discussed for three different cases follows.

Case 1: ($\mathbf{0} < M \le \boldsymbol{t_d}$)

The interest payable

This is the period before deterioration sets in, and payment for goods is settled with the capital opportunity cost rate I_c for the items in stock. Therefore, the interest payable is given below.

$$I_{P1} = CI_c \left[\int_M^{t_d} I_1(t) dt + \int_{t_d}^{t_1} I_2(t) dt \right] = CI_c \left[\frac{d(t_d - M)}{\theta} \left(e^{\theta(t_1 - t_d)} - 1 \right) + \frac{a}{2} (t_d - M)^2 + \frac{b}{6} (2t_d + M)(t_d - M)^2 + \frac{c}{12} (3t_d^2 + 2t_d M + M^2)(t_d - M)^2 + \frac{d}{\theta^2} \left(e^{\theta(t_1 - t_d)} - 1 - \theta(t_1 - t_d) \right) \right] \tag{21}$$

The interest earned.

In this case, the retailer can earn interest on revenue generated from the sales up to the trade credit period M. Although the retailer has to settle the accounts at period M, for that, he has to arrange money at some specified rate of interest to get his remaining stocks financed for the period M tot_d. The interest earned is

$$I_{E1} = S_1 I_e \left[\int_0^M (a + bt + ct^2) t dt \right] = S_1 I_e \left(a\frac{M^2}{2} + b\frac{M^3}{3} + c\frac{M^4}{4} \right) \tag{22}$$

The total profit per unit time for case 1 $(0 < M \le t_d)$ is

$TP_1(t_1, T) = \frac{1}{T}${Sales Revenue - Purchasing cost - Ordering cost - inventory holding cost - backordered cost - lost sales cost- interest payable during the permissible delay period + interest earned during the cycle}

$$= \frac{1}{T}\Bigg\{(S_1 - C)\left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3}\right) + S_2 d(t_1 - t_d) + (S_2 - C)\delta d(T - t_1) - C\left[\frac{d}{\theta}\left(e^{\theta(t_1 - t_d)} - 1\right)\right] - A$$
$$- h_1\left[\frac{dt_d}{\theta}e^{\theta(t_1 - t_d)} + \frac{a}{2}t_d^2 + \frac{b}{3}t_d^3 + \frac{c}{4}t_d^4 + \frac{d}{\theta^2}e^{\theta(t_1 - t_d)} - \frac{d}{\theta^2} - \frac{dt_1}{\theta}\right]$$
$$- h_2\left[\frac{dt_d^2}{2\theta}e^{\theta(t_1 - t_d)} + \frac{a}{6}t_d^3 + \frac{b}{8}t_d^4 + \frac{c}{10}t_d^5 + \frac{dt_d}{\theta^2}e^{\theta(t_1 - t_d)} - \frac{dt_1}{\theta^2} - \frac{d}{\theta^3} + \frac{d}{\theta^3}e^{\theta(t_1 - t_d)} - \frac{dt_1^2}{2\theta}\right]$$
$$- \frac{C_b \delta d}{2}(T - t_1)^2 - C_\pi d(1 - \delta)(T - t_1)$$
$$- cI_c\Bigg[\frac{d(t_d - M)}{\theta}\left(e^{\theta(t_1 - t_d)} - 1\right) + \frac{a}{2}(t_d - M)^2 + \frac{b}{6}(2t_d + M)(t_d - M)^2$$
$$+ \frac{c}{12}(3t_d^2 + 2t_d M + M^2)(t_d - M)^2 + \frac{d}{\theta^2}\left(e^{\theta(t_1 - t_d)} - 1 - \theta(t_1 - t_d)\right)\Bigg]$$
$$+ S_1 I_e\left(a\frac{M^2}{2} + b\frac{M^3}{3} + c\frac{M^4}{4}\right)\Bigg\} \quad (23)$$

Case 2: ($t_d < M \leq t_1$)

The interest payable

This is when the endpoint of the credit period is greater than the period with no deterioration but shorter than or equal to the length of the period with positive inventory stock of the items. The interest payable is

$$I_{P2} = cI_c\left[\int_M^{t_1} I_2(t)dt\right]$$
$$= cI_c\left[\frac{d}{\theta^2}\left(e^{\theta(t_1 - M)} - 1 - \theta(t_1 - M)\right)\right] \quad (24)$$

The interest earned

In this case, the retailer can earn interest on revenue generated from the sales up to the trade credit period M. Although the retailer has to settle the accounts at period M, for that, he has to arrange money at some specified rate of interest in order to get his remaining stocks financed for the period M tot_1. The interest earned is

$$I_{E2} = S_1 I_e\left[\int_0^{t_d}(a + bt + ct^2)tdt\right] + S_2 I_e\left[\int_{t_d}^{M} dtdt\right]$$
$$= S_1 I_e\left(a\frac{t_d^2}{2} + b\frac{t_d^3}{3} + c\frac{t_d^4}{4}\right) + S_2 I_e\left(\frac{dM^2}{2} - \frac{dt_d^2}{2}\right) \quad (25)$$

The total profit per unit time for case 2($t_d < M \leq t_1$) is

$TP_2(t_1, T) = \frac{1}{T}${Sales Revenue - Purchasing cost - Ordering cost - inventory holding cost - backordered cost - lost sales cost- interest payable during the permissible delay period + interest earned during the cycle}

$$= \frac{1}{T}\Bigg\{(S_1 - C)\left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3}\right) + S_2 d(t_1 - t_d) + (S_2 - C)\delta d(T - t_1) - C\left[\frac{d}{\theta}\left(e^{\theta(t_1 - t_d)} - 1\right)\right] - A$$
$$- h_1\left[\frac{dt_d}{\theta}e^{\theta(t_1 - t_d)} + \frac{a}{2}t_d^2 + \frac{b}{3}t_d^3 + \frac{c}{4}t_d^4 + \frac{d}{\theta^2}e^{\theta(t_1 - t_d)} - \frac{d}{\theta^2} - \frac{dt_1}{\theta}\right]$$
$$- h_2\left[\frac{dt_d^2}{2\theta}e^{\theta(t_1 - t_d)} + \frac{a}{6}t_d^3 + \frac{b}{8}t_d^4 + \frac{c}{10}t_d^5 + \frac{dt_d}{\theta^2}e^{\theta(t_1 - t_d)} - \frac{dt_1}{\theta^2} - \frac{d}{\theta^3} + \frac{d}{\theta^3}e^{\theta(t_1 - t_d)} - \frac{dt_1^2}{2\theta}\right]$$
$$- \frac{C_b \delta d}{2}(T - t_1)^2 - C_\pi d(1 - \delta)(T - t_1) - cI_c\left[\frac{d}{\theta^2}\left(e^{\theta(t_1 - M)} - 1 - \theta(t_1 - M)\right)\right]$$
$$+ S_1 I_e\left(a\frac{t_d^2}{2} + b\frac{t_d^3}{3} + c\frac{t_d^4}{4}\right) + S_2 I_e\left(\frac{dM^2}{2} - \frac{dt_d^2}{2}\right)\Bigg\} \quad (26)$$

Case 3: ($M > t_1$)

The interest payable

In this case, the period of delay in payment is greater than period with positive inventory. In this case the retailer pays no interest. Therefore,$I_{P3} = 0$.

The interest earned

In this case, the period of delay in payment (M) is greater than period with positive inventory(t_1). In this case the retailer earns interest on the sales revenue up to the permissible delay period and no interest is payable during the period for the item kept in stock. Interest earned for the time period $[0, T]$

$$I_{E3} = S_1 I_e\left[\int_0^{t_d}(a + bt + ct^2)tdt + (M - t_1)\int_0^{t_d}(a + bt + ct^2)dt\right] + S_2 I_e\left[\int_{t_d}^{t_1} dtdt + (M - t_1)\int_{t_d}^{t_1} ddt\right]$$

$$= S_1 I_e \left[\left(a\frac{t_d^2}{2} + b\frac{t_d^3}{3} + c\frac{t_d^4}{4} \right) + (M - t_1)\left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3} \right) \right] + S_2 I_e \left[-\frac{d}{2}(t_1 - t_d)^2 + Md(t_1 - t_d) \right] \quad (27)$$

The total profit per unit time for case 3 $(M > t_1)$ is

$TP_3(t_1, T) = \frac{1}{T}${Sales Revenue - Purchasing cost - Ordering cost - inventory holding cost - backordered cost - lost sales cost + interest earned during the cycle}

$$= \frac{1}{T}\Bigg\{ (S_1 - C)\left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3} \right) + S_2 d(t_1 - t_d) + (S_2 - C)\delta d(T - t_1) - C\left[\frac{d}{\theta}\left(e^{\theta(t_1 - t_d)} - 1 \right) \right] - A$$
$$- h_1 \left[\frac{dt_d}{\theta} e^{\theta(t_1 - t_d)} + \frac{a}{2}t_d^2 + \frac{b}{3}t_d^3 + \frac{c}{4}t_d^4 + \frac{d}{\theta^2} e^{\theta(t_1 - t_d)} - \frac{d}{\theta^2} - \frac{dt_1}{\theta} \right]$$
$$- h_2 \left[\frac{dt_d^2}{2\theta} e^{\theta(t_1 - t_d)} + \frac{a}{6}t_d^3 + \frac{b}{8}t_d^4 + \frac{c}{10}t_d^5 + \frac{dt_d}{\theta^2} e^{\theta(t_1 - t_d)} - \frac{dt_1}{\theta^2} - \frac{d}{\theta^3} + \frac{d}{\theta^3} e^{\theta(t_1 - t_d)} - \frac{dt_1^2}{2\theta} \right]$$
$$- \frac{C_b \delta d}{2}(T - t_1)^2 - C_\pi d(1 - \delta)(T - t_1)$$
$$+ S_1 I_e \left[\left(a\frac{t_d^2}{2} + b\frac{t_d^3}{3} + c\frac{t_d^4}{4} \right) + (M - t_1)\left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3} \right) \right]$$
$$+ S_2 I_e \left[-\frac{d}{2}(t_1 - t_d)^2 + Md(t_1 - t_d) \right] \Bigg\} \quad (28)$$

Since $0 < \theta < 1$, by utilising a quadratic approximation for the exponential terms in equations (23), (26) and (28) to obtain

$$TP_1(t_1, T) = \frac{d}{T}\left\{ -\frac{1}{2}X_1 t_1^2 + Y_1 t_1 - W_1 - \frac{C_b \delta T^2}{2} + C_b \delta t_1 T + (S_2 - C)\delta T - C_\pi (1 - \delta)T \right\} \quad (29)$$

Where

$$X_1 = \left[h_1(t_d\theta + 1) + h_2\left(\tfrac{t_d\theta}{2} + 1 \right) t_d + C\theta + C_b\delta + cI_c(\theta(t_d - M) + 1) \right],$$

$$Y_1 = \left[(S_2 - C)(1 - \delta) + h_1 t_d^2 \theta + \frac{h_2}{2}(1 + t_d\theta)t_d^2 + Ct_d\theta + C_\pi(1 - \delta) + cI_c(M + (t_d - M)\theta t_d) \right]$$

and

$$W_1 = -\frac{1}{d}\Bigg[(S_1 - C)\left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3} \right) - (S_2 - C)dt_d - \frac{Cd\theta t_d^2}{2} - A - h_1\left(\frac{a}{2}t_d^2 + \frac{b}{3}t_d^3 + \frac{c}{4}t_d^4 - \frac{dt_d^2}{2} + \frac{dt_d^3\theta}{2} \right)$$
$$- h_2\left(\frac{a}{6}t_d^3 + \frac{b}{8}t_d^4 + \frac{c}{10}t_d^5 + \frac{dt_d^4\theta}{4} \right)$$
$$- CI_c\Bigg(\frac{a}{2}(t_d - M)^2 + \frac{b}{6}(2t_d + M)(t_d - M)^2 + \frac{c}{12}(3t_d^2 + 2t_d M + M^2)(t_d - M)^2 + dMt_d$$
$$- \frac{dt_d^2}{2} + \frac{d}{2}(t_d - M)\theta t_d^2 \Bigg) + S_1 I_e\left(a\frac{M^2}{2} + b\frac{M^3}{3} + c\frac{M^4}{4} \right) \Bigg]$$

Similarly,

$$TP_2(t_1, T) = \frac{d}{T}\left\{ -\frac{1}{2}X_2 t_1^2 + Y_2 t_1 - W_2 - \frac{C_b \delta T^2}{2} + C_b \delta t_1 T + (S_2 - C)\delta T - C_\pi (1 - \delta)T \right\} \quad (30)$$

Where

$$X_2 = \left[h_1(t_d\theta + 1) + h_2\left(\tfrac{t_d\theta}{2} + 1 \right) t_d + C\theta + C_b\delta + CI_c \right],$$

$$\left[h_1 t_d^2 \theta + \frac{h_2}{2}(1 + t_d\theta)t_d^2 + Ct_d\theta + C_\pi(1 - \delta) + cI_c M \right]$$

$$Y_2 = \left[(S_2 - C)(1 - \delta) + h_1 t_d^2 \theta + \frac{h_2}{2}(1 + t_d\theta)t_d^2 + Ct_d\theta + C_\pi(1 - \delta) + cI_c M \right]$$

and

$$W_2 = -\frac{1}{d}\Bigg[(S_1 - C)\left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3} \right) - (S_2 - C)dt_d - \frac{Cd\theta t_d^2}{2} - A - h_1\left(\frac{a}{2}t_d^2 + \frac{b}{3}t_d^3 + \frac{c}{4}t_d^4 - \frac{dt_d^2}{2} + \frac{dt_d^3\theta}{2} \right)$$
$$- h_2\left(\frac{a}{6}t_d^3 + \frac{b}{8}t_d^4 + \frac{c}{10}t_d^5 + \frac{dt_d^4\theta}{4} \right) - CI_c\frac{d}{2}M^2 + S_1 I_e\left(a\frac{t_d^2}{2} + b\frac{t_d^3}{3} + c\frac{t_d^4}{4} \right)$$
$$+ S_2 I_e\left(\frac{dM^2}{2} - \frac{dt_d^2}{2} \right) \Bigg]$$

And

$$TP_3(t_1,T) = \frac{d}{T}\left\{-\frac{1}{2}X_3t_1^2 + Y_3t_1 - W_3 - \frac{C_b\delta T^2}{2} + C_b\delta t_1 T + (S_2 - C)\delta T - C_\pi(1-\delta)T\right\} \tag{31}$$

Where

$$X_3 = \left[h_1(t_d\theta + 1) + h_2\left(\frac{t_d\theta}{2} + 1\right)t_d + C\theta + C_b\delta + S_2 I_e\right],$$

$$Y_3 = \left[(S_2 - C)(1-\delta) + h_1 t_d^2\theta + \frac{h_2}{2}(1 + t_d\theta)t_d^2 + Ct_d\theta + C_\pi(1-\delta) - \frac{1}{d}\left\{S_1 I_e\left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3}\right)\right\} + S_2 I_e t_d + S_2 I_e M\right]$$

and

$$W_3 = -\frac{1}{d}\left[(S_1 - C)\left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3}\right) - (S_2 - C)dt_d - \frac{Cd\theta t_d^2}{2} - A - h_1\left(\frac{a}{2}t_d^2 + \frac{b}{3}t_d^3 + \frac{c}{4}t_d^4 - \frac{dt_d^2}{2} + \frac{dt_d^3\theta}{2}\right) - h_2\left(\frac{a}{6}t_d^3 + \frac{b}{8}t_d^4 + \frac{c}{10}t_d^5 + \frac{dt_d^4\theta}{4}\right) + S_1 I_e\left[\left(a\frac{t_d^2}{2} + b\frac{t_d^3}{3} + c\frac{t_d^4}{4}\right) + \left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3}\right)M\right] - S_2 I_e\frac{d}{2}t_d^2 - S_2 I_e M d t_d\right]$$

OPTIMAL DECISION

This section determines the optimal ordering policies that maximise the total profit per unit time. The necessary and sufficient conditions for the existence and uniqueness of optimal solutions have been established. The necessary conditions for the total profit per unit time $TP_i(t_1,T)$ to be maximum are $\frac{\partial TP_i(t_1,T)}{\partial t_1} = 0$ and $\frac{\partial TP_i(t_1,T)}{\partial T} = 0$ for $i = 1,2,3$. The value of (t_1,T) obtained from $\frac{\partial TP_i(t_1,T)}{\partial t_1} = 0$ and $\frac{\partial TP_i(t_1,T)}{\partial T} = 0$ and for which the sufficient condition $\left\{\left(\frac{\partial^2 TP_i(t_1,T)}{\partial t_1^2}\right)\left(\frac{\partial^2 TP_i(t_1,T)}{\partial T^2}\right) - \left(\frac{\partial^2 TP_i(t_1,T)}{\partial t_1\,\partial T}\right)^2\right\} > 0$ is satisfied gives a maximum value for the total profit per unit time $TP_i(t_1,T)$.

For case 1 $(0 < M \le t_d)$

The necessary condition for the total profit $TP_1(t_1,T)$ in equation (29) to be the maximum are $\frac{\partial TP_1(t_1,T)}{\partial t_1} = 0$ and $\frac{\partial TP_1(t_1,T)}{\partial T} = 0$, which give

$$\frac{\partial TP_1(t_1,T)}{\partial t_1} = \frac{d}{T}\{-X_1t_1 + Y_1 + C_b\delta T\}$$

Setting $\frac{\partial TP_1(t_1,T)}{\partial t_1} = 0$ gives

$$\{-X_1t_1 + Y_1 + C_b\delta T\} = 0 \tag{32}$$

and

$$T = \frac{1}{C_b\delta}(X_1t_1 - Y_1) \tag{33}$$

Since $(t_d - M) \ge 0, (t_1 - t_d) > 0, t_1 - M > 0$

It should be noted that

$$(X_1t_1 - Y_1) = \left[(C - S_2)(1-\delta) + h_1(t_d\theta(t_1 - t_d) + t_1) + h_2\left(t_1 - \frac{t_d}{2}\right)t_d + \frac{h_2 t_d\theta}{2}(t_1 - t_d)t_d + C\theta(t_1 - t_d) + C_\pi(\delta - 1) + C_b\delta t_1 + cI_c\big((t_1 - M) + \theta(t_d - M)(t_1 - t_d)\big)\right] > 0$$

Provided

$$\left[C + S_2\delta + h_1(t_d\theta(t_1 - t_d) + t_1) + h_2\left(t_1 - \frac{t_d}{2}\right)t_d + \frac{h_2 t_d\theta}{2}(t_1 - t_d)t_d + C\theta(t_1 - t_d) + C_\pi\delta + C_b\delta t_1 + cI_c\big((t_1 - M) + \theta(t_d - M)(t_1 - t_d)\big)\right] > (S_2 + C\,\delta + C_\pi)$$

Similarly,

$$\frac{\partial TP_1(t_1,T)}{\partial T} = -\frac{d}{T^2}\left\{-\frac{1}{2}X_1t_1^2 + Y_1t_1 - W_1 + \frac{C_b\delta T^2}{2}\right\} \tag{34}$$

Setting $\frac{\partial TP_1(t_1,T)}{\partial T} = 0$ to obtain

$$-\frac{d}{T^2}\left\{-\frac{1}{2}X_1t_1^2 + Y_1t_1 - W_1 + \frac{C_b\delta T^2}{2}\right\} = 0 \tag{35}$$

Substituting T from equation (33) into equation (35) yields

$$\{X_1(C_b\delta - X_1)t_1^2 - 2Y_1(C_b\delta - X_1)t_1 - (Y_1^2 - 2C_b\delta W_1)\} = 0 \tag{36}$$

Let $\Delta_1 = X_1(C_b\delta - X_1)t_d^2 - 2Y_1(C_b\delta - X_1)t_d - (Y_1^2 - 2C_b\delta W_1)$, then the following result is obtained.

Lemma 1

(i) If $\Delta_1 \geq 0$, then the solution of $t_1 \in [t_d, \infty)$ (say t_{11}^*) which satisfies equation (36) not only exists but also is unique.
See the proof in Appendix 1a

(ii) If $\Delta_1 < 0$, then the solution of $t_1 \in [t_d, \infty)$ which satisfies equation (36) does not exist.
See the proof in Appendix 1b

Therefore, the value of t_1 (denoted by t_{11}^*) can be found from equation (36) and is given by

$$t_{11}^* = \frac{Y_1}{X_1} + \frac{1}{X_1}\sqrt{\frac{(2X_1W_1 - Y_1^2)C_b\delta}{(X_1 - C_b\delta)}} \tag{37}$$

Once the value of t_{11}^* is obtained, then the value of T (denoted by T_1^*) can be found from (33) and is given by

$$T_1^* = \frac{1}{C_b\delta}(X_1t_{11}^* - Y_1) \tag{38}$$

Equations (37) and (38) give the optimal values of t_{11}^* and T_1^* for the profit function in equation (29) only if Y_1 satisfies the inequality given in equation (39)

$$2X_1W_1 > Y_1^2 \tag{39}$$

Theorem 1

(i) If $\Delta_1 \geq 0$, then the total profit $TP_1(t_1, T)$ is concave and reaches its global maximum at the point (t_{11}^*, T_1^*), where (t_{11}^*, T_1^*) is the point which satisfies equations (36) and (32), if all principal minors are positive definite i.e., if

$$\left(\frac{\partial^2 TP_1(t_1,\ T)}{\partial t_1^2}\bigg|_{(t_{11}^*,\ T_1^*)}\right) < 0, \left(\frac{\partial^2 TP_1(t_1,\ T)}{\partial T^2}\bigg|_{(t_{11}^*,\ T_1^*)}\right) < 0$$

and

$$\begin{vmatrix} \frac{\partial^2 TP_1(t_1,\ T)}{\partial t_1^2}\bigg|_{(t_{11}^*,\ T_1^*)} & \frac{\partial^2 TP_1(t_1,\ T)}{\partial t_1 \partial T}\bigg|_{(t_{11}^*,\ T_1^*)} \\ \frac{\partial^2 TP_1(t_1,\ T)}{\partial t_1 \partial T}\bigg|_{(t_{11}^*,\ T_1^*)} & \left(\frac{\partial^2 TP_1(t_1,\ T)}{\partial T^2}\bigg|_{(t_{11}^*,\ T_1^*)}\right) \end{vmatrix} > 0.$$

See the proof in Appendix 1c

(ii) If $\Delta_1 < 0$, then the total profit $TP_1(t_1, T)$ has a maximum value at the point (t_{11}^*, T_1^*) where $t_{11}^* = t_d$ and $T_1^* = \frac{1}{C_b\delta}(X_1 t_d - Y_1)$

See the proof in Appendix 1d

For case 2 ($t_d < M \leq t_1$)

The necessary condition for the total profit $TP_1(t_1, T)$ in equation (39) to be the maximum are $\frac{\partial TP_2(t_1,T)}{\partial t_1} = 0$ and $\frac{\partial TP_2(t_1,T)}{\partial T} = 0$, which give

$$\frac{\partial TP_2(t_1,T)}{\partial t_1} = \frac{d}{T}\{-X_2t_1 + Y_2 + C_b\delta T\}$$

Setting $\frac{\partial TP_2(t_1,T)}{\partial t_1} = 0$ gives

$$\{-X_2t_1 + Y_2 + C_b\delta T\} = 0 \tag{40}$$

and

$$T = \frac{1}{C_b\delta}(X_2t_1 - Y_2) \tag{41}$$

Since $(t_1 - t_d) > 0, (t_1 - M) \geq 0$, it should be noted that

$$(X_2t_1 - Y_2 = \Big[(C - S_2)(1-\delta) + h_1(t_d\theta(t_1 - t_d) + t_1) + h_2\left(t_1 - \frac{t_d}{2}\right)t_d + \frac{h_2t_d\theta}{2}(t_1 - t_d)t_d + C\theta(t_1 - t_d) + C_\pi(\delta - 1) + C_b\delta t_1 + cI_c(t_1 - M)\Big] > 0$$

Provided

$$\Big[C + S_2\delta + h_1(t_d\theta(t_1 - t_d) + t_1) + h_2\left(t_1 - \frac{t_d}{2}\right)t_d + \frac{h_2t_d\theta}{2}(t_1 - t_d)t_d + C\theta(t_1 - t_d) + C_\pi\delta + C_b\delta t_1 + cI_c(t_1 - M)\Big] > (S_2 + C\ \delta + C_\pi)$$

Similarly,

$$\frac{\partial TP_2(t_1,T)}{\partial T} = -\frac{d}{T^2}\left\{-\frac{1}{2}X_2t_1^2 + Y_2t_1 - W_2 + \frac{C_b\delta T^2}{2}\right\} \tag{42}$$

Setting $\frac{\partial TP_2(t_1,T)}{\partial T} = 0$ to obtain

$$-\frac{d}{T^2}\left\{-\frac{1}{2}X_2t_1^2 + Y_2t_1 - W_2 + \frac{C_b\delta T^2}{2}\right\} = 0 \tag{43}$$

Substituting T from equation (41) into equation (43) yields

$$\{X_2(C_b\delta - X_2)t_1^2 - 2Y_2(C_b\delta - X_2)t_1 - (Y_2^2 - 2C_b\delta W_2)\} = 0 \tag{44}$$

Let $\Delta_2 = X_2(C_b\delta - X_2)M^2 - 2Y_2(C_b\delta - X_2)M(Y_2^2 - 2C_b\delta W_2)$, then the following result is obtained.

Lemma 2

(i) If $\Delta_2 \geq 0$, then the solution of $t_1 \in [M,\infty)$ (say t_{12}^*) which satisfies equation (44) not only exists but also is unique.
The proof is similar to Appendix 1a, hence is omitted

(ii) If $\Delta_2 < 0$, then the solution of $t_1 \in [M,\infty)$ which satisfies equation (44) does not exist.
The proof is similar to Appendix 1b, hence is omitted

Therefore, the value of t_1 (denoted by t_{12}^*) can be found from equation (44) and is given by

$$t_{12}^* = \frac{Y_2}{X_2} + \frac{1}{X_2}\sqrt{\frac{(2X_2W_2 - Y_2^2)C_b\delta}{(X_2 - C_b\delta)}} \tag{45}$$

Once the value of t_{12}^* is obtained, then the value of T (denoted by T_2^*) can be found from (41) and is given by

$$T_2^* = \frac{1}{C_b\delta}(X_2t_{12}^* - Y_2) \tag{46}$$

Equations (45) and (46) give the optimal values of t_{12}^* and T_2^* for the profit function in equation (30) only if Y_2 satisfies the inequality given in equation (47)

$$2X_2W_2 > Y_2^2 \tag{47}$$

Theorem 2

(i) If $\Delta_2 \geq 0$, then the total profit $TP_2(t_1,T)$ is concave and reaches its global maximum at the point (t_{12}^*, T_2^*), where (t_{12}^*, T_2^*) is the point which satisfies equations (44) and (40), if all principal minors are positive definite i.e., if

$$\left(\left.\frac{\partial^2 TP_2(t_1,\ T)}{\partial t_1^2}\right|_{(t_{12}^*,\ T_2^*)}\right) < 0, \left(\left.\frac{\partial^2 TP_2(t_1,\ T)}{\partial T^2}\right|_{(t_{12},\ T_2^*)}\right) < 0$$

and

$$\begin{vmatrix} \left.\frac{\partial^2 TP_2(t_1,\ T)}{\partial t_1^2}\right|_{(t_{12}^*,\ T_2^*)} & \left.\frac{\partial^2 TP_2(t_1,\ T)}{\partial t_1\partial T}\right|_{(t_{12}^*,\ T_2^*)} \\ \left.\frac{\partial^2 TP_2(t_1,\ T)}{\partial t_1\partial T}\right|_{(t_{12}^*,\ T_2^*)} & \left(\left.\frac{\partial^2 TP_2(t_1,\ T)}{\partial T^2}\right|_{(t_{12}^*,\ T_2^*)}\right) \end{vmatrix} > 0.$$

The proof is similar to Appendix 1c, hence is omitted

(ii) If $\Delta_2 < 0$, then the total profit $TP_2(t_1,T)$ has a maximum value at the point (t_{12}^*, T_2^*) where $t_{12}^* = t_d$ and $T_2^* = \frac{1}{C_b\delta}(X_2t_d - Y_2)$

The proof is similar to Appendix 1d, hence is omitted

For case 3 (M> t_1)

The necessary condition for the total profit $TP_3(t_1,T)$ in equation (40) to be the maximum are $\frac{\partial TP_3(t_1,T)}{\partial t_1} = 0$ and $\frac{\partial TP_3(t_1,T)}{\partial T} = 0$, which gives

$$\frac{\partial TP_3(t_1,T)}{\partial t_1} = \frac{d}{T}\{-X_3t_1 + Y_3 + C_b\delta T\}$$

Setting $\frac{\partial TP_3(t_1,T)}{\partial t_1} = 0$ gives

$$\{-X_3t_1 + Y_3 + C_b\delta T\} = 0 \tag{48}$$

and

$$T = \frac{1}{C_b\delta}(X_3t_1 - Y_3) \tag{49}$$

Since $(t_1 - t_d) > 0$, it should be noted that

$$(X_3t_1 - Y_3) = \left[(C - S_2)(1-\delta) + h_1(t_d\theta(t_1 - t_d) + t_1) + h_2\left(t_1 - \frac{t_d}{2}\right)t_d + \frac{h_2 t_d\theta}{2}(t_1 - t_d)t_d + C\theta(t_1 - t_d) + C_\pi(\delta - 1) + C_b\delta t_1 + \frac{1}{d}\left\{S_1 I_e\left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3}\right)\right\} - S_2 I_e(t_d + M)\right] > 0$$

Provided

$$\left[C + S_2\delta + h_1(t_d\theta(t_1 - t_d) + t_1) + h_2\left(t_1 - \frac{t_d}{2}\right)t_d + \frac{h_2 t_d\theta}{2}(t_1 - t_d)t_d + C\theta(t_1 - t_d) + C_\pi\delta + C_b\delta t_1 + \frac{1}{d}\left\{S_1 I_e\left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3}\right)\right\}\right] > (S_2 + C\,\delta + C_\pi) + S_2 I_e(t_d + M)$$

Similarly,

$$\frac{\partial TP_3(t_1,T)}{\partial T} = -\frac{d}{T^2}\left\{-\frac{1}{2}X_3t_1^2 + Y_3t_1 - W_3 + \frac{C_b\delta T^2}{2}\right\} \quad (50)$$

Setting $\frac{\partial TP_3(t_1,T)}{\partial T} = 0$ to obtain

$$-\frac{d}{T^2}\left\{-\frac{1}{2}X_3t_1^2 + Y_3t_1 - W_3 + \frac{C_b\delta T^2}{2}\right\} = 0 \quad (51)$$

Substituting T from equation (49) into equation (51) yields

$$\{X_3(C_b\delta - X_3)t_1^2 - 2Y_3(C_b\delta - X_3)t_1 - (Y_3^2 - 2C_b\delta W_3)\} = 0 \quad (52)$$

Let $\Delta_{3a} = X_3(C_b\delta - X_3)t_d^2 - 2Y_3(C_b\delta - X_3)t_d - (Y_3^2 - 2C_b\delta W_3)$

and $\Delta_{3b} = X_3(C_b\delta - X_3)M^2 - 2Y_3(C_b\delta - X_3)M - (Y_3^2 - 2C_b\delta W_3)$, then the following result is obtained.

Lemma 3

(i) If $\Delta_{3b} \le 0 \le \Delta_{3a}$, then the solution of $t_1 \in [t_d, M]$ (say t_{13}^*) which satisfies equation (52) not only exists but also is unique.

The proof is similar to Appendix 1a, hence is omitted.

(ii) If $\Delta_{3a} < 0$, then the solution of $t_1 \in [t_d, M]$ which satisfies equation (52) does not exist.

The proof is similar to Appendix 1b, hence is omitted.

Therefore, the value of t_1 (denoted by t_{13}^*) can be found from equation (52) and is given by

$$t_{13}^* = \frac{Y_3}{X_3} + \frac{1}{X_3}\sqrt{\frac{(2X_3W_3 - Y_3^2)C_b\delta}{(X_3 - C_b\delta)}} \quad (53)$$

Once the value of t_{13}^* is obtained, then the value of T (denoted by T_3^*) can be found from (49) and is given by

$$T_3^* = \frac{1}{C_b\delta}(X_3t_{13}^* - Y_3) \quad (54)$$

Equations (53) and (54) give the optimal values of t_{13}^* and T_3^* for the profit function in equation (31) only if Y_3 satisfies the inequality given in equation (55)

$$2X_3W_3 > Y_3^2 \quad (55)$$

Theorem 3

(i) If $\Delta_{3a} \ge 0$, then the total profit $TP_3(t_1,T)$ is concave and reaches its global maximum at the point (t_{13}^*, T_3^*), where (t_{13}^*, T_3^*) is the point which satisfies equations (52) and (48), if all principal minors are positive definite i.e., if

$$\left(\left.\frac{\partial^2 TP_3(t_1,\ T)}{\partial t_1^2}\right|_{(t_{13}^*,\ T_3^*)}\right) < 0, \left(\left.\frac{\partial^2 TP_3(t_1,\ T)}{\partial T^2}\right|_{(t_{13},\ T_3^*)}\right) < 0$$

and

$$\begin{vmatrix} \left.\frac{\partial^2 TP_3(t_1,\ T)}{\partial t_1^2}\right|_{(t_{13}^*,\ T_3^*)} & \left.\frac{\partial^2 TP_3(t_1,\ T)}{\partial t_1\partial T}\right|_{(t_{13}^*,\ T_3^*)} \\ \left.\frac{\partial^2 TP_3(t_1,\ T)}{\partial t_1\partial T}\right|_{(t_{13}^*,\ T_3^*)} & \left(\left.\frac{\partial^2 TP_3(t_1,\ T)}{\partial T^2}\right|_{(t_{13}^*,\ T_3^*)}\right) \end{vmatrix} > 0.$$

The proof is similar to Appendix 1c, hence is omitted

(ii) If $\Delta_{3a} < 0$, then the total profit $TP_3(t_1,T)$ has a maximum value at the point (t_{13}^*, T_3^*) where $t_{13}^* = M$ and $T_3^* = \frac{1}{C_b\delta}(X_3M - Y_3)$.

The proof is similar to Appendix 1d, hence is omitted

(iii) If $\Delta_{3b} > 0$, then the total profit $TP_3(t_1, T)$ has a maximum value at the point (t_{13}^*, T_3^*) where $t_{13}^* = t_d$ and $T_3^* = \frac{1}{C_b \delta}(X_3 t_d - Y_3)$.

The proof is similar to Appendix 1d, hence is omitted

After obtaining the optimal values of t_1^* and T^*, the optimal Economic Order Quantity (denoted by EOQ^*) can be computed as follows:

EOQ^* =Total demand before deterioration sets in+total demand after deterioration sets in+total number of deteriorated items + the total number of items back-ordered

$$= at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3} + \frac{d}{\theta}\left(e^{\theta(t_1^* - t_d)} - 1\right) + d\delta(T^* - t_1^*) \quad (56)$$

Note: It is obvious when $t_d = t_1 = M$ that $TP_1(t_1, T) = TP_2(t_1, T) = TP_3(t_1, T)$. When $t_d = M$, $TP_1(t_1, T) = TP_2(t_1, T)$. When $t_1 = M$, $TP_2(M, T) = TP_3(M, T)$. Hence, the profit function $TP(t_1, T)$ is continuous and well-defined.

NUMERICAL RESULTS

Example 6.1 ($M \leq t_d$)

The following parameters are adopted from Babangida and Baraya (2021a) in addition to h_1, δ, C_π and C_b which are not considered in their work. The parameters and their values are as follows:

Table 1: parameters and their values

Parameter(s)	Value(s)
A	$250/order
h_1	$2 unit/year
h_2	$15 unit/year
θ	0.01 unit/year
a	180 unit
b	30 unit
c	15 unit
d	120 unit
t_d	0.1354 year
M	0.0888 year
I_c	0.1
I_e	0.08
C_b	$30
δ	0.85
C_π	1

It is seen that $M \leq t_d$, Δ_1=33.9202 > 0, $2X_1W_1$ =44.9517,$Y_1^2 = 1.4200$ and $2X_1W_1 > Y_1^2$. Substituting the above values in equation (37), (47), (29) and (56), The result is obtained in the table below

Table 2: Optimal Solutions for case 1

Parameters	Values
t_{11}^*	0.4863 (177 days)
T_1^*	0.5479 (199 days)
$TP_1(t_{11}^*, T_1^*)$	$303.2293
EOQ_1^*	73.1284 unit.

Example 6.2 ($M > t_d$)

The values of the parameters are same as in example 6.1 [as in Babangida and Baraya (2021)] except that $M = 0.1523$. It is seen that $M > t_d$, Δ_2= 32.7438 > 0, $2X_2W_2 = 44.3728$,$B_2^2 = 1.6559$ and $2X_2W_2 > Y_2^2$. Substituting the above values in equation (45), (46), (30) and (56). The result is obtained in the table below

Table 3: Optimal Solutions for case 2

Parameters	Values
t_{12}^*	0.4851 (177 days)
T_2^*	0.5428 (198 days)
$TP_2(t_{12}^*, T_2^*)$	\$315.4550
EOQ_2^*	72.5857 unit.

Example 6.3 ($M > t_1$)

The values of the parameters are same as in example 6.1 except that $M = 0.36$. It is seen that $M > t_d, \Delta_{3a} = 16.2308 > 0, \Delta_{3b} = -0.1650 < 0$ $2X_3W_3 = 23.9601, Y_3^2 = 2.0736$ and $2X_3W_3 > Y_3^2$. Substituting the above values in equation (53), (54), (31) and (56).The result is obtained in the table below.

Table 4: Optimal Solutions for case 3

Parameters	Values
t_{13}^*	0.3585 (130 days)
T_3^*	0.3834 (139 days)
$TP_3(t_{13}^*, T_3^*)$	\$386.9494
EOQ_3^*	54.0035 unit.

Table 5: Comparison table

Comparison of our model with Babangida and Bature (2021a)			
Models	Average total profit per unit for case 1	Average total profit per unit for case 2	Average total profit per unit for case 3
Babangida and Baraya (2021)	\$4.1341	\$4.3176	-
Proposed Model	\$4.1465	\$4.3460	\$7.1653

It is clearly seen from the table above that the average total profit for case 1 and case 2 of the proposed model is greater than that of Babangida and Baraya (2021). Hence the proposed model is more optimal than Babangida and Baraya (2021).

SENSITIVITY ANALYSIS

The sensitivity analysis of some model parameters is performed by changing each of the parameters from $-6\%, -3\%, +6\%$ to $+3\%$ taking one parameter at a time and keeping the remaining parameters unchanged. The effects of these changes of parameters on cycle length, optimal time with positive inventory, total profit and economic order quantity per cycle are discussed and summarised in table 6, 7 and 8 below:

Table 6: Effect of changes of some model parameters from $-6\%, -3\%, +3\%$ to $+6\%$ on decision variables for example 6.1

Parameter	% change in Parameter	% change in t_{11}^*	% change in T_{11}^*	% change in EOQ_1^*	% change in $TP_1(t_{11}^*, T_1^*)$
θ	-6%	0.0802	0.0637	0.0524	0.0413
	-3%	0.0401	0.0318	0.0262	0.0206
	3%	-0.0400	-0.0318	-0.0262	-0.0206
	6%	-0.0800	-0.0636	-0.0523	-0.0413
C	-6%	-2.8843	-4.3652	-3.6894	40.2585
	-3%	-1.3988	-2.1381	-1.8054	20.0960
	3%	1.3186	2.0558	1.7327	-20.0341
	6%	2.5627	4.0350	3.3979	-40.0105
S_1	-6%	18.5309	20.1113	17.6472	-20.2750
	-3%	9.6879	10.5141	9.2242	-10.5997
	3%	-10.8341	-11.7581	-10.3113	11.8538
	6%	-23.3539	-25.3456	-22.2212	25.5520
S_2	-6%	-12.1440	-11.8915	-10.5731	-34.2018
	-3%	-5.8899	-5.7481	-5.1140	-17.3001
	3%	5.5853	5.4176	4.8253	17.6334
	6%	10.9108	10.5531	9.4044	35.5510
I_c	-6%	0.8314	0.6436	0.5938	0.5209
	-3%	0.4132	0.3198	0.2950	0.2596
	3%	-0.4083	-0.3157	-0.2913	-0.2578
	6%	-0.8118	-0.6275	-0.5790	-0.5139
I_e	-6%	0.0473	0.0513	0.0450	-0.0518
	-3%	0.0237	0.0257	0.0225	-0.0259
	3%	-0.0237	-0.0257	-0.0225	0.0259
	6%	-0.0473	-0.0514	-0.0451	0.0518
A	-6%	-8.6580	-9.3964	-8.2405	9.4729
	-3%	-4.2223	-4.5824	-4.0191	4.6198
	3%	4.0372	4.3815	3.8435	-4.4171
	6%	7.9115	8.5863	7.5326	-8.6562
C_b	-6%	-0.2032	0.4955	0.3538	0.2223
	-3%	-0.0988	0.2404	0.1716	0.1081
	3%	0.0935	-0.2269	-0.1620	-0.1024
	6%	0.1823	-0.4415	-0.3151	-0.1994
C_π	-6%	-0.0367	0.0245	0.0143	0.0402
	-3%	-0.0183	0.0123	0.0071	0.0201
	3%	0.0183	-0.0123	-0.0072	-0.0200
	6%	0.0366	-0.0247	-0.0144	-0.0400
δ	-6%	1.0029	-0.5101	-0.7191	-1.0973
	-3%	0.5146	-0.2191	-0.3470	-0.5630
	3%	-0.5386	0.1537	0.3241	0.5893
	6%	-1.0994	0.2480	0.6273	1.2028

Table 7: Effect of changes of some model parameters from $-6\%, -3\%, +3\%$ to $+6\%$ on decision variables for example 6.2

Parameter	% change in Parameter	% change in t_{12}^{*}	% change in T_{12}^{*}	% change in EOQ_{2}^{*}	% change in $TP_1(t_{12}^{*}, T_{2}^{*})$
θ	-6%	0.0796	0.0636	0.0522	0.0397
	-3%	0.0398	0.0318	0.0261	0.0198
	3%	-0.0397	-0.0317	-0.0261	-0.0198
	6%	-0.0794	-0.0634	-0.0521	-0.0396
C	-6%	-3.0386	-4.5250	-3.8256	38.6280
	-3%	-1.4742	-2.2159	-1.8718	19.2810
	3%	1.3910	2.1299	1.7960	-19.2198
	6%	2.7047	4.1800	3.5216	-38.3826
S_1	-6%	18.7746	20.5200	17.9709	-19.6975
	-3%	9.8231	10.7363	9.4009	-10.3060
	3%	-11.0133	-12.0371	-10.5354	11.5547
	6%	-23.7951	-26.0072	-22.7568	24.9648
S_2	-6%	-12.2319	-12.0686	-10.7089	-32.8152
	-3%	-5.9296	-5.8307	-5.1770	-16.6031
	3%	5.6188	5.4910	4.8809	16.9291
	6%	10.9731	10.6928	9.5098	34.1358
I_c	-6%	0.7625	0.6146	0.5627	0.3521
	-3%	0.3789	0.3053	0.2796	0.1754
	3%	-0.3743	-0.3014	-0.2760	-0.17340
	6%	-0.7440	-0.5991	-0.5486	-0.3466
I_e	-6%	0.1275	0.1393	0.1219	-0.1337
	-3%	0.0637	0.0697	0.0610	-0.0669
	3%	-0.06378	-0.0697	-0.0610	0.0669
	6%	-0.1276	-0.1395	-0.1221	0.1339
A	-6%	-8.7696	-9.5848	-8.3895	9.2007
	-3%	-4.2745	-4.6719	-4.0896	4.4846
	3%	4.0835	4.4632	3.9076	-4.2843
	6%	7.9996	8.7433	7.6555	-8.3929
C_b	-6%	-0.1799	0.4795	0.3436	0.1887
	-3%	-0.0874	0.2326	0.1667	0.0917
	3%	0.0828	-0.2120	-0.1573	-0.0869
	6%	0.1613	-0.4272	-0.3060	-0.1692
C_π	-6%	-0.0348	0.0270	0.0163	0.0365
	-3%	-0.0174	0.0135	0.0082	0.0182
	3%	0.0173	-0.0136	-0.0082	-0.0182
	6%	0.0346	-0.0272	-0.0165	-0.0363
δ	-6%	0.9540	-0.6185	-0.7734	-1.0009
	-3%	0.4912	-0.2709	-0.3731	-0.5154
	3%	-0.5171	0.2013	0.3483	0.5426
	6%	-1.0581	0.3394	0.6739	1.1101

Table 8: Effect of changes of some model parameters from $-6\%, -3\%, +3\%$ to $+6\%$ on decision variables for example 6.3

Parameter	% change in Parameter	% change in t_{13}^{*}	% change in T_{13}^{*}	% change in EOQ_3^{*}	% change in $TP_3(t_{13}^{*}, T_3^{*})$
θ	-6%	0.0706	0.0599	0.0486	0.02589
	-3%	0.0351	0.0299	0.0243	0.0129
	3%	-0.0352	-0.0299	-0.0243	-0.0129
	6%	-0.0704	-0.0597	-0.0485	-0.0259
C	-6%	-7.9762	-10.5496	-8.6064	31.6647
	-3%	-3.8471	-5.1138	-4.1697	15.8053
	3%	3.6065	4.8389	3.9418	-15.7481
	6%	7.0041	9.4394	7.6861	-31.4389
S_1	-6%	35.1471	40.0715	33.2997	-18.3728
	-3%	19.0213	21.6967	18.0246	-9.6900
	3%	-24.4804	-27.9599	-23.2087	10.6559
	6%	-70.4044	-80.5217	-66.7806	-5.6315
S_2	-6%	-24.3021	-25.8205	-21.6378	-26.4717
	-3%	-11.1971	-11.8389	-9.92978	-13.2972
	3%	9.9040	10.3924	8.7287	13.5293
	6%	18.8523	19.7199	16.5725	27.2920
I_c	-6%	0	0	0	0
	-3%	0	0	0	0
	3%	0	0	0	0
	6%	0	0	0	0
I_e	-6%	3.4180	3.7915	3.1605	-1.3616
	-3%	1.7102	1.9001	1.5835	-0.6861
	3%	-1.7135	-1.9099	-1.5910	0.6964
	6%	-3.4315	-3.8311	-3.1905	1.4033
A	-6%	-17.4680	-20.0407	-16.6274	8.1955
	-3%	-8.2492	-9.4641	-7.8533	4.0380
	3%	7.5333	8.6428	7.1735	-3.8806
	6%	14.5099	16.6470	13.8183	-7.6021
C_b	-6%	-0.0631	0.3408	0.2392	0.0818
	-3%	-0.0306	0.1653	0.1160	0.0397
	3%	0.0289	-0.1559	-0.1094	-0.0376
	6%	0.0563	-0.3032	-0.2128	-0.0731
C_π	-6%	-0.0283	0.0596	0.0397	0.4738
	-3%	-0.0141	0.0298	0.0199	0.2370
	3%	0.0140	-0.0299	-0.0120	-0.2372
	6%	0.0278	-0.0560	-0.0400	-0.4745
δ	-6%	0.7273	-2.0713	-1.5741	-3.3551
	-3%	0.3949	-0.9057	-0.7172	-1.6010
	3%	-0.4866	0.7752	0.6670	1.6621
	6%	-1.0436	1.4006	1.2729	3.3254

RESULTS AND DISCUSSION

The following managerial insights are obtained based on the results shown in Tables 6, 7 and 8.

(i) It is obviously seen that the higher the rate of deterioration (θ), the lower the optimal time with positive inventory (t_1^*), cycle length (T^*), order quantity (EOQ^*) and the total profit $TP(T^*)$ and vice versa. This implies that the retailer needs to take all the necessary measures to avoid or reduce deterioration to maximise higher profit.

(ii) It is visibly seen that as the unit purchasing cost (C) increases, the total profit $TP(T^*)$ decreases while the optimal time with positive inventory (t_1^*), cycle length (T^*) and order quantity (EOQ^*) increase and vice versa. This result reveals that when the unit purchasing cost increases, the retailer will order smaller quantity to enjoy the benefits of permissible delay in payments more frequently, which will consequently shorten the cycle length.

(iii) It is apparently seen that as the unit selling price before deterioration sets in (S_1) increases, the optimal time with positive inventory (t_1^*), cycle length (T^*) and order quantity (EOQ^*) decrease while the total profit $TP(T^*)$ increases and vice versa. This implies that as the selling price increases the retailer will order less quantity to enjoy the benefits of trade credit more frequently.

(iv) It is evidently seen that as the unit selling price after deterioration sets in (S_2) increases, the optimal time with positive inventory (t_1^*), cycle length (T^*), order quantity (EOQ^*) and the total profit $TP(T^*)$ increase and vice versa. This implies that as the selling price increases, the retailer maximises higher profit.

(v) It is obviously seen that the lower the interest charged (I_c) the higher the optimal time with positive inventory (t_1^*), cycle length (T^*), order quantity (EOQ^*) and total profit $TP(T^*)$ and vice versa. This implies that when the interest charged is high the retailer is expected to order less quantity of inventory to enjoy the benefits of trade credit more frequently. As for case 3 $(M > t_1)$, any increase or decrease in the interest charged does not affect the optimal time with positive inventory (t_1^*), cycle length (T^*), order quantity (EOQ^*) and total profit $TP(T^*)$, this is because the interest charged in this case is zero.

(vi) It is clearly seen that as the interest earned (I_e) is increasing, the total profit $TP(T^*)$ is also increasing while the optimal time with positive inventory (t_1^*), cycle length (T^*) and order quantity (EOQ^*) are decreasing and vice versa. This implies that when the interest earned is high the retailer should order less quantity of inventory to enjoy the benefits of trade credit more frequently.

(vii) It is obviously seen that as the ordering cost (A) is increasing the total profit $TP(T^*)$ is decreasing while the optimal time with positive inventory (t_1^*), cycle length (T^*) and order quantity (EOQ^*) increase. This implies that the retailer should order large quantity when the ordering cost per order is high.

(viii) It is clearly seen that as the shortage cost (C_b) increases the total profit $TP(T^*)$, the economic order quantity (EOQ^*), the optimal cycle length (T^*) decreases while the time with positive inventory increases.

(ix) It is evidently seen that as the unit cost of lost sales per unit (C_π) increases the optimal time with positive inventory (t_1^*) also increases while cycle length (T^*), order quantity (EOQ^*) and the total profit $TP(T^*)$ decrease.This implies that the retailer should order less quantity when the unit cost of lost sales is high.

(x) It is clearly seen that as the backlogging parameter is increasing, the cycle length (T^*), order quantity (EOQ^*) and the total profit $TP(T^*)$ are also increasing while the optimal time with positive inventory (t_1^*) is decreasing and vice versa. This implies that when the backlogging parameter is increasing, the retailer should order large quantity to get large profit.

CONCLUSION

This research developed an economic order quantity model for non-instantaneous deteriorating items with two phase demand rates ,linear holding cost, complete backlogging rate and two-level pricing strategies under trade credit policy. The purpose of the model is to determine the optimal time with positive inventory, cycle length and order quantity such that the total profit of the inventory system has a maximum value. Some numerical examples have been given to illustrate the theoretical result of the model. Sensitivity analysis of some model parameters on the decision variables has been carried out and suggestions towards maximising the total profit were also given. The retailer can maximise the total profit by ordering less quantity and shorten the cycle length if the rate of deterioration, unit purchasing cost, interest charged, ordering cost, backlogging parameter and shortage cost increase and unit selling price before deterioration start, unit selling price after deterioration start interest earned and unit cost of lost sales per unit decrease. The model can be used in inventory control and management of items such as food items (e.g., beans, maise, corns, millet), electronics (e.g., mobile phones, computers), automobiles, fashionable items, etc. The proposed model can be extended by considering factors such as variable deterioration, inflation and time value of money, quantity discount, order size dependent trade credit, etc.

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APPENDIX 1a:

proof of lemma 1(i)

From equation (45), a new function $F_1(t_1)$ is defined as follows

$$F_1(t_1) = \{X_1(C_b\delta - X_1)t_1^2 - 2Y_1(C_b\delta - X_1)t_1(Y_1^2 - 2C_b\delta W_1)\}, \quad t_1 \in [t_d, \infty) \qquad (66)$$

Taking the first-order derivative of $F_1(t_1)$ with respect to $t_1 \in [t_d, \infty)$, it follows that

$$\frac{F_1(t_1)}{dt_1} = \{2X_1(C_b\delta - X_1)t_1 - 2Y_1(C_b\delta - X_1)\}$$
$$= 2(C_b\delta - X_1)(X_1t_1 - Y_1) < 0$$

Because $(X_1t_1 - Y_1) > 0$

and

$$(C_b\delta - X_1) = C_b\delta - \left[h_1(t_d\theta + 1) + h_2\left(1 + \frac{t_d\theta}{2}\right)t_d + C\theta + C_b\delta + cI_c(\theta(t_d - M) + 1)\right]$$
$$= -\left[h_1(t_d\theta + 1) + h_2\left(1 + \frac{t_d\theta}{2}\right)t_d + C\theta + cI_c(\theta(t_d - M) + 1)\right] < 0$$

Hence $F_1(t_1)$ is a strictly decreasing function of t_1 in the interval $[t_d, \infty)$. Moreover, $\lim_{t_1 \to \infty} F_1(t_1) = -\infty$ and $F_1(t_d) = \Delta_1 \geq 0$. Therefore, by applying intermediate value theorem, there exists a unique t_1 say $t_{11}^* \in [t_d, \infty)$ such that $F_1(t_{11}^*) = 0$. Hence t_{11}^* is the unique solution of equation (45).

APPENDIX 1b:

proof of lemma 1(ii)

If $\Delta_1 < 0$, then from equation (46), $F_1(t_1) < 0$. Since $F_1(t_1)$ is a strictly decreasing function of $t_1 \in [t_d, \infty)$ and $F_1(t_1) < 0$ for all $T \in [t_d, \infty)$. Therefore, a value of $T \in [t_d, \infty)$ such that $F_1(t_1) = 0$ cannot found. This completes the proof.

APPENDIX 1c: proof of Theorem 1(i)

When $\Delta_1 \geq 0$, it is seen that t_{11}^* and T_1^* are the unique solutions of equations (44) and (40) respectively from Lemma l(i). Taking the second derivative of $TP_1(t_1, T)$ with respect to t_1 and T, and then finding the values of these functions at the point (t_{11}^*, T_1^*), it follows that

$$\left.\frac{\partial^2 TP_1(t_1, T)}{\partial t_1^2}\right|_{(t_{11}^*, T_1^*)} = -\frac{d}{T_1^*}X_1 < 0$$

$$\left.\frac{\partial^2 TP_1(t_1, T)}{\partial t_1 \partial T}\right|_{(t_{11}^*, T_1^*)} = -\frac{d}{T_1^{*2}}\{-X_1t_{11}^* + Y_1 + C_b\delta T_1^*\} + \frac{d}{T_1^*}\{C_b\delta\}$$
$$= \frac{d}{T_1^*}C_b\delta$$

$$\left.\frac{\partial^2 TP_1(t_1, T)}{\partial T^2}\right|_{(t_{11}^*, T_1^*)} = \frac{2d}{T_1^{*3}}\left\{-\frac{1}{2}X_1t_{11}^{*2} + Y_1t_{11}^* - W_1 + \frac{C_b\delta T_1^{*2}}{2}\right\} - \frac{d}{T_1^{*2}}\{C_b\delta T_1^*\}$$
$$= -\frac{d}{T_1^*}C_b\delta < 0$$

and

$$\left(\left.\frac{\partial^2 TP_1(t_1, T)}{\partial t_1^2}\right|_{(t_{11}^*, T_1^*)}\right)\left(\left.\frac{\partial^2 TP_1(t_1, T)}{\partial T^2}\right|_{(t_{11}^*, T_1^*)}\right) - \left(\left.\frac{\partial^2 TP_1(t_1, T)}{\partial t_1 \partial T}\right|_{(t_{11}^*, T_1^*)}\right)^2$$
$$= \left(-\frac{d}{T_1^*}X_1\right)\left(-\frac{d}{T_1^*}C_b\delta\right) - \left(\frac{d}{T_1^*}C_b\delta\right)^2$$
$$= \frac{d^2}{T_1^{*2}}X_1C_b\delta - \frac{d^2}{T_1^{*2}}C_b^2\delta^2$$

$$= \frac{d^2 C_b \delta}{T_1^{*2}} (X_1 - C_b \delta)$$
$$= \frac{d^2 C_b \delta}{T_1^{*2}} \left(\left[h_1 (t_d \theta + 1) + h_2 \left(1 + \frac{t_d \theta}{2} \right) t_d + C\theta + C_b \delta + cI_c (\theta (t_d - M) + 1) \right] - C_b \delta \right)$$

$$= \frac{d^2 C_b}{T_1^{*2}} \left(\left[h_1 (t_d \theta + 1) + h_2 \left(1 + \frac{t_d \theta}{2} \right) t_d + C\theta + cI_c (\theta (t_d - M) + 1) \right] \right) > 0 \quad (67)$$

It is therefore conclude from (48) and Lemma 1 that $TP_1(t_{11}^*, \ T_1^*)$ is concave and $(t_{11}^*, \ T_1^*)$ is the global maximum point of $TP_1(t_1, \ T)$. Hence the values of t_1 and T in (45) and (46) are optimal.

APPENDIX 1d:

proof of Theorem 1(ii)

When $\Delta_1 < 0$, then $F_1(t_1) < 0$ for all $t_1 \in [t_d, \ \infty)$. Therefore, $\frac{\partial TP_1(t_1, \ T)}{\partial T} = \frac{F_1(t_1)}{T^2} < 0$ for all $t_1 \in [t_d, \infty)$ which implies $TP_1(t_1, \ T)$ is a strictly decreasing function of t_1. Therefore, $TP_1(t_1, \ T)$ has a maximum value when t_1 is minimum. Therefore, $TP_1(t_1, \ T)$ has a maximum value at the point $(t_{11}^*, \ T_1^*)$ where $t_{11}^* = t_d$ and $T_1^* = \frac{1}{C_b \delta} (X_1 t_d - Y_1)$. This completes the proof.