UMYU Scientifica, Vol. 3 NO. 1, March 2024, Pp 001 – 014
https://doi.org/10.56919/usci.2431.001
ISSN: 2955 – 1145 (print); 2955 – 1153 (online)
A periodical of the Faculty of Natural and Applied Sciences, UMYU, Katsina

ORIGINAL RESEARCH ARTICLE

A New Fixed Coefficient Diagonally Implicit Block Backward Differentiation Formula for Solving Stiff Initial Value Problems

Yusuf Hamza[1]*, Musa Hamisu[2] and Alhassan Buhari[3].
[1]Department of Mathematics, Faculty of Physical Sciences, Federal University Dutsin-Ma, Katsina State, Nigeria
[2]Department of Mathematics and Statistics, Faculty of Natural and Applied Sciences, Umaru Musa Yar'adua University, Katsina, Nigeria.
[3]Department of Mathematics and Statistics, College of Natural and Applied Sciences, Al-Qalam University, Katsina, Nigeria

ABSTRACT

Stiff initial value problems in ordinary differential equations occur when solution components evolve at varying rates, posing challenges for traditional computational methods. Specialized techniques are crucial for maintaining accuracy and stability during rapid transitions, emphasizing their significance in developing reliable numerical algorithms across scientific and engineering applications. This study aims to develop a new fixed coefficient 3-point diagonally implicit block backward differentiation formula for the numerical solution of first order stiff initial value problems. The method is constructed by integrating a triangular matrix into the coefficient matrix of an existing extended 3-point super class of block BDF for solving stiff initial value problems. The selection of a fixed coefficient within the interval accompanies this integration $(-1,1)$ to ensure optimal stability. The method is found to order five. Stability analysis indicates that the method is consistent, zero-stable, and almost A-stable, validating its applicability to stiff initial value problems. Implementation of the method involves Newton's iteration, and a code in the C programming language is devised to demonstrate its effectiveness. Comparative examination of numerical outcomes with the existing 3BBDF and 3ESBBDF methods highlights the proposed method's enhanced accuracy and reduced computation time.

ARTICLE HISTORY

Received August 15, 2023.
Accepted December 24, 2023.
Published February 16, 2024.

KEYWORDS

Fixed coefficient, Diagonally implicit block method, Stiff IVP, Stability analysis

INTRODUCTION

In this research, we explore the general form of the first-order stiff initial value problems (IVPs) presented as follows:

$$y' = f(x, y), \;\; y(a) = y_0, \qquad a \le x \le b \tag{1}$$

It is postulated that the function $f(x, y)$ adheres to the Lipschitz conditions (Alhassan et al., 2023). Despite various attempts by researchers to apply diverse analytical approaches to solve equation (1), solutions to specific initial value problems (IVPs) have proven to be intricate or beyond analytical resolution. Hence, the necessity of advocating for numerical methods becomes apparent. If the solution of IVPs (1) using a specified numerical method becomes unstable when a large number of step lengths are chosen due to its physical property of causing rapid variation in the solution, then it is called Stiff IVPs. Stiff IVPs are commonly encountered in Chemical kinetics, electric circuits, string variations, control systems, and more (Musa et al., 2022).

When employing a designated numerical method for solving IVPs (1), instability emerges when opting for an extensive number of step lengths, attributed to its inherent characteristic of inducing swift variation in the solution. This phenomenon is termed Stiff IVPs and is frequently encountered in chemical kinetics, electric circuits, string variation, control systems, and other domains (Musa et al., 2022).

Numerical methods for tackling stiff IVPs can be classified as either block or non-block, and both can be explicit or implicit. Implicit Linear Multistep Methods (LMM) have demonstrated superior effectiveness in addressing stiff IVPs compared to their explicit counterparts. Instances of non-block implicit methods are documented in [Cash, 1980; Curtiss & Hirschfelder, 1952; Dalquist, 1974; Alexander, 1977], while examples of block implicit methods are outlined in [Musa et al., 2012; Ibrahim et al., 2007a; Musa et al., 2022; Suleiman et al., 2014; Musa & Muhammad, 2019; Alhassan et al., 2023; Bala et al., 2022]. Ibrahim et al. (2007b & 2019) identified

Correspondence: Yusuf Hamza. Department of Mathematics, Faculty of Physical Sciences, Federal University Dutsin-Ma, Katsina State, Nigeria. ✉ yhamza@fudutsinma.edu.ng.
How to cite: Yusuf, H., Hamisu, M., & Buhari, A. (2024). A New Fixed Coefficient Diagonally Implicit Block Backward Differentiation Formula for Solving Stiff Initial Value Problems. *UMYU Scientifica*, 3(1), 1 – 14. https://doi.org/10.56919/usci.2431.001

an implicit fixed coefficient block method for solving stiff IVPs.

This research proposes a new fixed coefficient diagonally implicit method grounded in the block backward differentiation formula (NFDIBBDF) for addressing stiff initial value problems. The subsequent sections will explore the derivation of the method, stability analysis, and implementation, highlighting the potential of our innovative numerical approach in overcoming this pivotal challenge.

METHODOLOGY

This section contains the derivation of the proposed method using Taylor's series and the derivation of the predictor method, which will predict the initial approximation for stiff initial value problems.

Derivation of (NFDIBBDF) Method

Consider the extended 3-point super class of block backward differentiation formula for solving stiff initial value problems developed by Musa et al. (2019) of the form:

$$\sum_{j=0}^{5} \alpha_{j,i}\, y_{n+j-2} = h\beta_{k,i}(f_{n+k} - \rho f_{n+k-2}), \quad k = 1,2,3 \qquad (2)$$

In this research, we will modify (2) by introducing a triangular matrix in the coefficient matrix of the method (2) and choosing the value of $\rho = \frac{3}{5}$ from the interval $(-1,1)$ which gives better stability region required for solving stiff IVPs. The new formula would compute the approximated solution values y_{n+1}, y_{n+2} and y_{n+3} simultaneously in a block using three previous values y_{n-2}, y_{n-1} and y_n with constant step size h.

Definition 1: A New Fixed Coefficient Diagonally Implicit Block Backward Differentiation Formula (NFDIBBDF) is defined as:

$$\sum_{j=0}^{2+k} \alpha_{j,i}\, y_{n+j-2} = h\beta_{k,i}\left(f_{n+k} - \frac{3}{5} f_{n+k-2}\right), \quad k = i = 1,2,3 \qquad (3)$$

To derive the first point y_{n+1}, we define the linear operator of (3) associated with the first point as:

$$L_1[y(x_n), h]: \alpha_{0,1} y_{n-2} + \alpha_{1,1} y_{n-1} + \alpha_{2,1} y_n + \alpha_{3,1} y_{n+1} - h\beta_{1,1}\left(f_{n+1} - \frac{3}{5} f_{n-1}\right) = 0, \qquad (4)$$

By expanding (4) as Taylor's series about any point x_n and after collecting like terms, we get:

$$C_{0,1} y(x_n) + C_{1,1} h y'(x_n) + C_{2,1} h^2 y''(x_n) + C_{3,1} h^3 y'''(x_n) + \cdots = 0 \qquad (5)$$

where,

$$\left.\begin{aligned} C_{0,1} &= \alpha_{0,1} + \alpha_{1,1} + \alpha_{2,1} + \alpha_{3,1} = 0 \\ C_{1,1} &= -2\alpha_{0,1} - \alpha_{1,1} + \alpha_{3,1} - \tfrac{2}{5}\beta_{1,1} = 0 \\ C_{2,1} &= 2\alpha_{0,1} + \tfrac{1}{2}\alpha_{1,1} + \tfrac{1}{2}\alpha_{3,1} - \tfrac{8}{5}\beta_{1,1} = 0 \\ C_{3,1} &= -\tfrac{4}{3}\alpha_{0,1} - \tfrac{1}{6}\alpha_{1,1} + \tfrac{1}{6}\alpha_{3,1} - \tfrac{1}{5}\beta_{1,1} = 0 \end{aligned}\right\} \qquad (6)$$

Solving the system of simultaneous equations in (6) for the values of $\alpha_{j,i}$ and $\beta_{j,i}$by normalizing the coefficient of y_{n+1} to one and substituting the values obtained in (4) yields the formula for the first point as:

$$y_{n+1} = \frac{2}{29} y_{n-2} - \frac{27}{29} y_{n-1} + \frac{54}{29} y_n + \frac{15}{29} h f_{n+1} - \frac{9}{29} h f_{n-1} \qquad (7)$$

Similarly, to derive the second point y_{n+2}, we defined the linear operator associated with the second point as:

$$L_2[y(x_n), h]: \alpha_{0,2} y_{n-2} + \alpha_{1,2} y_{n-1} + \alpha_{2,2} y_n + \alpha_{3,2} y_{n+1} + \alpha_{4,2} y_{n+2} - h\beta_{2,2}\left(f_{n+2} - \frac{3}{5} f_n\right) = 0, \qquad (8)$$

The corresponding approximate relationship for the equation (8) is given by:

$$\alpha_{0,2} y(x_n - 2h) + \alpha_{1,2} y(x_n - h) + \alpha_{2,2} y(x_n) + \alpha_{3,2} y(x_n + h) + \alpha_{4,2} y(x_n + 2h)$$

$$-h\beta_{2,2}\left(y'(x_n + 2h) - \frac{3}{5} y'(x_n)\right) = 0 \qquad (9)$$

Again, by expanding (9) as Taylor's series about any point x_n and collecting like terms, we get:

$$C_{0,2} y(x_n) + C_{1,2} h y'(x_n) + C_{2,2} h^2 y''(x_n) + C_{3,2} h^3 y'''(x_n) + +C_{4,2} h^4 y^{iv}(x_n) + \cdots = 0 \qquad (10)$$

where,

$$\left.\begin{aligned} C_{0,2} &= \alpha_{0,2} + \alpha_{1,2} + \alpha_{2,2} + \alpha_{3,2} + \alpha_{4,2} = 0 \\ C_{1,2} &= -2\alpha_{0,2} - \alpha_{1,2} + \alpha_{3,2} + 2\alpha_{4,2} - \tfrac{2}{5}\beta_{2,2} = 0 \\ C_{2,2} &= 2\alpha_{0,2} + \tfrac{1}{2}\alpha_{1,2} + \tfrac{1}{2}\alpha_{3,2} + 2\alpha_{4,2} - 2\beta_{2,2} = 0 \\ C_{3,2} &= -\tfrac{4}{3}\alpha_{0,2} - \tfrac{1}{6}\alpha_{1,2} + \tfrac{1}{6}\alpha_{3,2} + \tfrac{4}{3}\alpha_{4,2} - \beta_{2,2} = 0 \\ C_{4,2} &= \tfrac{2}{3}\alpha_{0,2} + \tfrac{1}{24}\alpha_{1,2} + \tfrac{1}{24}\alpha_{3,2} + \tfrac{2}{3}\alpha_{4,2} - \tfrac{4}{3}\beta_{2,2} = 0 \end{aligned}\right\} \qquad (11)$$

The coefficient of y_{n+2} is similarly normalized to 1, by adopting the same procedure as in the derivation of the first point, we obtain the following formula for the second point:

$$y_{n+2} = -\frac{3}{32} y_{n-2} + \frac{7}{16} y_{n-1} - \frac{45}{32} y_n + \frac{33}{16} y_{n+1} + \frac{15}{32} h f_{n+2} - \frac{9}{32} h f_n \qquad (12)$$

In obtaining the third point formula, a similar procedure is applied as in the derivation of the

first and second point formulae yields:

$$y_{n+3} = \frac{27}{347}y_{n-2} - \frac{165}{347}y_{n-1} + \frac{410}{347}y_n - \frac{720}{347}y_{n+1} + \frac{795}{347}y_{n+2} + \frac{150}{347}hf_{n+3} - \frac{90}{347}hf_{n+1} \quad (13)$$

Thus, by combining the formulae in (7), (12), and (13), we have obtained a New 3-point fixed coefficient diagonally implicit block backward differentiation formula (NFDIBBDF) as:

$$\left.\begin{aligned} y_{n+1} &= \frac{2}{29}y_{n-2} - \frac{27}{29}y_{n-1} + \frac{54}{29}y_n + \frac{15}{29}hf_{n+1} - \frac{9}{29}hf_{n-1} \\ y_{n+2} &= -\frac{3}{32}y_{n-2} + \frac{7}{16}y_{n-1} - \frac{45}{32}y_n + \frac{33}{16}y_{n+1} + \frac{15}{32}hf_{n+2} - \frac{9}{32}hf_n \\ y_{n+3} &= \frac{27}{347}y_{n-2} - \frac{165}{347}y_{n-1} + \frac{410}{347}y_n - \frac{720}{347}y_{n+1} + \frac{795}{347}y_{n+2} + \frac{150}{347}hf_{n+3} - \frac{90}{347}hf_{n+1} \end{aligned}\right\} \quad (14)$$

It is therefore derived and established that the method is of order five as in Bala et al. (2022), with error constants given as:

$$C_6 = \begin{pmatrix} -\frac{57}{290} \\ -\frac{1}{5} \\ -\frac{47}{694} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Derivation of the Predictor of (NFDIBBDF) Method

The prediction of initial approximation through the explicit block predictor method involves deriving the method by employing Taylor's expansion of the following formula:

$$\sum_{j=0}^{6} \gamma_{j,i}\, y_{n+j-3} = 0 \quad k = 1,2,3 \quad (15)$$

The derivation of formula (15) involves setting the coefficient $\beta_{k,i} = 0$ in the right-hand side of the general K-step linear multistep method. To determine the coefficient of the first, second, and third points for the predictor method (15), we introduce the linear operator associated with (15) as:

$$\gamma_{0,i}y_{n-3} + \gamma_{1,i}y_{n-2} + \gamma_{2,i}y_{n-1} + \gamma_{3,i}y_n + \gamma_{4,i}y_{n+1} + \gamma_{5,i}y_{n+2} + \gamma_{6,i}y_{n+3} = 0, \quad (16)$$

First Point: $k = 1$.

By setting the coefficients $\gamma_{5,1} = \gamma_{6,1} = 0$ in (16), the linear operator (16) becomes:

$$\gamma_{0,1}y_{n-3} + \gamma_{1,1}y_{n-2} + \gamma_{2,1}y_{n-1} + \gamma_{3,1}y_n + \gamma_{4,1}y_{n+1} = 0, \quad (17)$$

The Taylors series expansion about the point x_n leads to the following system of simultaneous linear equations as:

$$\left.\begin{aligned} C_{0,1} &= \gamma_{0,1} + \gamma_{1,1} + \gamma_{2,1} + \gamma_{3,1} + \gamma_{4,1} = 0 \\ C_{1,1} &= -3\gamma_{0,1} - 2\gamma_{1,1} - \gamma_{2,1} + \gamma_{4,1} = 0 \\ C_{2,1} &= \frac{9}{2}\gamma_{0,1} + 2\gamma_{1,1} + \frac{1}{2}\gamma_{2,1} + \frac{1}{2}\gamma_{4,1} = 0 \\ C_{3,1} &= -\frac{9}{2}\gamma_{0,1} - \frac{4}{3}\gamma_{1,1} - \frac{1}{6}\gamma_{2,1} + \frac{1}{6}\gamma_{4,1} = 0 \end{aligned}\right\} \quad (18)$$

Solving these set of equations in the Maple18 environment after setting $\gamma_{4,1} = 1$, we obtain the coefficient for the first point given in Table 1 below:

Table 1: Coefficient of the first point

$\gamma_{0,1}$	$\gamma_{1,1}$	$\gamma_{2,1}$	$\gamma_{3,1}$	$\gamma_{4,1}$
1	−4	6	−4	1

By substituting these obtained coefficients in equation (17), the first point formula is therefore obtained as:

$$y_{n-3} - 4y_{n-2} + 6y_{n-1} - 4y_n + y_{n+1} = 0 \quad (19)$$

Which is equivalent to

$$y_{n+1} = -y_{n-3} + 4y_{n-2} - 6y_{n-1} + 4y_n \quad (20)$$

Second Point: $k = 2$.

Likewise, when the coefficients $\gamma_{4,2}$ and $\gamma_{6,2}$ are set to zero in equation (16), the linear operator (16) transforms into:

$$y_{n-3} + \gamma_{1,2}y_{n-2} + \gamma_{2,2}y_{n-1} + \gamma_{3,2}y_n + \gamma_{5,2}y_{n+2} = 0, \quad (21)$$

The following system of simultaneous linear equations is derived by expanding the Taylor series around the point x_n, resulting in:

$$\left.\begin{aligned} C_{0,2} &= \gamma_{0,2} + \gamma_{1,2} + \gamma_{2,2} + \gamma_{3,2} + \gamma_{5,2} = 0 \\ C_{1,2} &= -3\gamma_{0,1} - 2\gamma_{1,1} - \gamma_{2,1} + 2\gamma_{5,2} = 0 \\ C_{2,2} &= \frac{9}{2}\gamma_{0,1} + 2\gamma_{1,1} + \frac{1}{2}\gamma_{2,1} + 2\gamma_{5,2} = 0 \\ C_{3,2} &= -\frac{9}{2}\gamma_{0,2} - \frac{4}{3}\gamma_{1,2} - \frac{1}{6}\gamma_{2,2} + \frac{4}{3}\gamma_{5,2} = 0 \end{aligned}\right\} \quad (22)$$

After setting $\gamma_{5,2} = 1$ and solving this set of equations in the Maple 18 environment, we acquire the coefficients for the second point as presented in Table 2 below:

Table 2: Coefficient of the second point

$\gamma_{0,2}$	$\gamma_{1,2}$	$\gamma_{2,2}$	$\gamma_{3,2}$	$\gamma_{5,2}$
4	−15	20	−10	1

By substituting these acquired coefficients into equation (17), we derive the formula for the second point as follows

$$4y_{n-3} - 15y_{n-2} + 20y_{n-1} - 10y_n + y_{n+2} = 0 \quad (23)$$

This is, therefore, equivalent to

$$y_{n+2} = -4y_{n-3} + 15y_{n-2} - 20y_{n-1} + 10y_n \quad (24)$$

Third Point: $k = 3$

To obtain the formula for the third point, we apply the same procedure as in the derivation of the first and second point formulas, resulting in:

$$y_{n+3} = -10y_{n-3} + 36y_{n-2} - 45y_{n-1} + 20y_n \quad (25)$$

Hence, the 3-point explicit block predictor method is therefore given by

$$\left.\begin{aligned} y_{n+1} &= -y_{n-3} + 4y_{n-2} - 6y_{n-1} + 4y_n \\ y_{n+2} &= -4y_{n-3} + 15y_{n-2} - 20y_{n-1} + 10y_n \\ y_{n+3} &= -10y_{n-3} + 36y_{n-2} - 45y_{n-1} + 20y_n \end{aligned}\right\} \quad (26)$$

STABILITY OF THE (NFDIBBDF) METHOD

The stability of implicit numerical methods for stiff initial value problems is essential for preventing numerical instabilities and obtaining accurate and reliable solutions. Zero and A-stability are key criteria in assessing the robustness of this method (14), allowing for efficient simulations of problems with disparate timescales (Cash, 2015).

Definition 2 (Zero stability): A block method (14) is said to be zero stable if all the roots of first characteristics polynomial $\rho(\xi)$ have modulus less than or equal to one and if every root with modulus one is simple (Lambert, 1973).

Definition 3 (A- stability): A block method (14) is said to be A-stable if the stability region covers the entire negative half plane (Lambert, 1991).

A-stability, short for absolute stability, is a desirable property of numerical methods for solving ordinary differential equations (ODEs). Specifically, it pertains to implicit methods used in the numerical integration of ODEs. A-stability ensures that the numerical method remains stable over a wide range of problem characteristics and time step sizes (Lambert, 1973).

For an implicit numerical method to be A-stable, its stability region in the complex plane should include the entire left-half plane (Abasi et al., 2014). In other words, the method should be unconditionally stable, regardless of the eigenvalues of the underlying differential equation, and without imposing stringent restrictions on the size of the time step.

A-stability is crucial when dealing with stiff ODEs, where the variables have significant differences in timescales. Stiff problems can be challenging for numerical methods, and A-stable methods provide robustness by allowing for larger time steps without sacrificing stability (Cash, 1980).

Implicit methods involve solving algebraic equations at each time step and often possess A-stability. This property makes them particularly well-suited for stiff problems, as they can efficiently handle the numerical integration without being overly sensitive to the choice of time step (Suleiman et al., 2014).

Hence, in this section, we will examine the zero-stability and A-stability of our method. This analysis will be conducted using a first-order scalar differential equation represented as $y' = \lambda y$. By applying this equation to the expressions in formula (14), we obtain:

$$\left.\begin{aligned} y_{n+1} &= \frac{2}{29}y_{n-2} - \frac{27}{29}y_{n-1} + \frac{54}{29}y_n + \frac{15}{29}h\lambda y_{n+1} - \frac{9}{29}h\lambda y_{n-1} \\ y_{n+2} &= -\frac{3}{32}y_{n-2} + \frac{7}{16}y_{n-1} - \frac{45}{32}y_n + \frac{33}{16}y_{n+1} + \frac{15}{32}h\lambda y_{n+2} - \frac{9}{32}h\lambda y_n \\ y_{n+3} &= \frac{27}{347}y_{n-2} - \frac{165}{347}y_{n-1} + \frac{410}{347}y_n - \frac{720}{347}y_{n+1} + \frac{795}{347}y_{n+2} + \frac{150}{347}h\lambda y_{n+3} - \frac{90}{347}h\lambda y_{n+1} \end{aligned}\right\} \quad (27)$$

Rearranging and collecting the like terms of equation (27) leads to

$$\left.\begin{aligned} \left(1 - \frac{15}{29}h\lambda\right)y_{n+1} &= \frac{2}{29}y_{n-2} + \left(-\frac{27}{29} - \frac{9}{29}h\lambda\right)y_{n-1} + \frac{54}{29}y_n \\ \left(1 - \frac{15}{32}h\lambda\right)y_{n+2} - \frac{33}{16}y_{n+1} &= -\frac{3}{32}y_{n-2} + \frac{7}{16}y_{n-1} + \left(-\frac{45}{32} - \frac{9}{32}h\lambda\right)y_n \\ \left(1 - \frac{150}{347}h\lambda\right)y_{n+3} - \frac{795}{347}y_{n+2} + \left(\frac{720}{347} + \frac{90}{347}h\lambda\right)y_{n+1} &= \frac{27}{347}y_{n-2} - \frac{165}{347}y_{n-1} + \frac{410}{347}y_n \end{aligned}\right\} \quad (28)$$

The matrix formulation of these equations is written as:

$$\begin{bmatrix} \left(1-\frac{15}{29}h\lambda\right) & 0 & 0 \\ -\frac{33}{16} & \left(1-\frac{15}{32}h\lambda\right) & 0 \\ \left(\frac{720}{347}+\frac{90}{347}h\lambda\right) & -\frac{795}{347} & \left(1-\frac{150}{347}\right) \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \end{bmatrix} = \begin{bmatrix} \frac{2}{29} & \left(-\frac{27}{29}-\frac{9}{29}h\lambda\right) & \frac{54}{29} \\ -\frac{3}{32} & \frac{7}{16} & \left(-\frac{45}{32}-\frac{9}{32}h\lambda\right) \\ \frac{27}{347} & -\frac{165}{347} & \frac{410}{347} \end{bmatrix} \begin{bmatrix} y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} \quad (29)$$

Putting $\bar{h} = h\lambda$ in matrix equation (29), we have

$$\begin{bmatrix} \left(1-\frac{15}{29}\bar{h}\right) & 0 & 0 \\ -\frac{33}{16} & \left(1-\frac{15}{32}\bar{h}\right) & 0 \\ \left(\frac{720}{347}+\frac{90}{347}\bar{h}\right) & -\frac{795}{347} & \left(1-\frac{150}{347}\bar{h}\right) \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \end{bmatrix} = \begin{bmatrix} \frac{2}{29} & \left(-\frac{27}{29}-\frac{9}{29}\bar{h}\right) & \frac{54}{29} \\ -\frac{3}{32} & \frac{7}{16} & \left(-\frac{45}{32}-\frac{9}{32}\bar{h}\right) \\ \frac{27}{347} & -\frac{165}{347} & \frac{410}{347} \end{bmatrix} \begin{bmatrix} y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} \quad (30)$$

If m is the number of block and r is the number of points in the block, then $n = mr$, where $r = 3$ and $n = 3m$. By (Bala et al., 2022), we let

$$Y_m = \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \end{bmatrix} = \begin{bmatrix} y_{3m+1} \\ y_{3m+2} \\ y_{3m+3} \end{bmatrix} Y_{m-1} = \begin{bmatrix} y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} = \begin{bmatrix} y_{3m-2} \\ y_{3m-1} \\ y_{3m} \end{bmatrix} = \begin{bmatrix} y_{3(m-1)+1} \\ y_{3(m-1)+2} \\ y_{3(m-1)+3} \end{bmatrix},$$

Equation (30) can also be expressed in the following form:

$$C_0 Y_m = C_1 Y_{m-1} \quad (31)$$

where,

$$C_0 = \begin{bmatrix} \left(1-\frac{15}{29}\bar{h}\right) & 0 & 0 \\ -\frac{33}{16} & \left(1-\frac{15}{32}\bar{h}\right) & 0 \\ \left(\frac{720}{347}+\frac{90}{347}\bar{h}\right) & -\frac{795}{347} & \left(1-\frac{150}{347}\bar{h}\right) \end{bmatrix}, C_1 = \begin{bmatrix} \frac{2}{29} & \left(-\frac{27}{29}-\frac{9}{29}\bar{h}\right) & \frac{54}{29} \\ -\frac{3}{32} & \frac{7}{16} & \left(-\frac{45}{32}-\frac{9}{32}\bar{h}\right) \\ \frac{27}{347} & -\frac{165}{347} & \frac{410}{347} \end{bmatrix}$$

The characteristic polynomial of the method is obtained by evaluating

$$\pi(u, \bar{h}) = det(C_0 u - C_1) = 0, \quad (32)$$

we obtain:

$$\pi(u,\bar{h}) = \left| \begin{bmatrix} \left(1-\frac{15}{29}\bar{h}\right) & 0 & 0 \\ -\frac{33}{16} & \left(1-\frac{15}{32}\bar{h}\right) & 0 \\ \left(\frac{720}{347}+\frac{90}{347}\bar{h}\right) & -\frac{795}{347} & \left(1-\frac{150}{347}\bar{h}\right) \end{bmatrix} u - \begin{bmatrix} \frac{2}{29} & \left(-\frac{27}{29}-\frac{9}{29}\bar{h}\right) & \frac{54}{29} \\ -\frac{3}{32} & \frac{7}{16} & \left(-\frac{45}{32}-\frac{9}{32}\bar{h}\right) \\ \frac{27}{347} & -\frac{165}{347} & \frac{410}{347} \end{bmatrix} \right| = 0,$$

$$= \frac{8289}{11104}u\bar{h} - \frac{476961}{322016}u^2 - \frac{144531}{161008}u^2\bar{h} - \frac{397575}{322016}u^2\bar{h}^2 - \frac{456705}{322016}u^3\bar{h} + \frac{7425}{11104}u^3\bar{h}^2 - \frac{16875}{161008}u^3\bar{h}^3 - \frac{108}{10063}h$$
$$+ \frac{9843}{20126}u - \frac{2543}{322016} + \frac{3645}{161008}u\bar{h}^3 + \frac{142965}{322016}u\bar{h}^2 - \frac{2187}{322016}\bar{h}^2 + u^3 = 0 \quad (33)$$

For absolute stability of the method, the stability region is obtained by substituting$u = e^{i\theta}$, into (33). The graph of stability region for the method is given below:

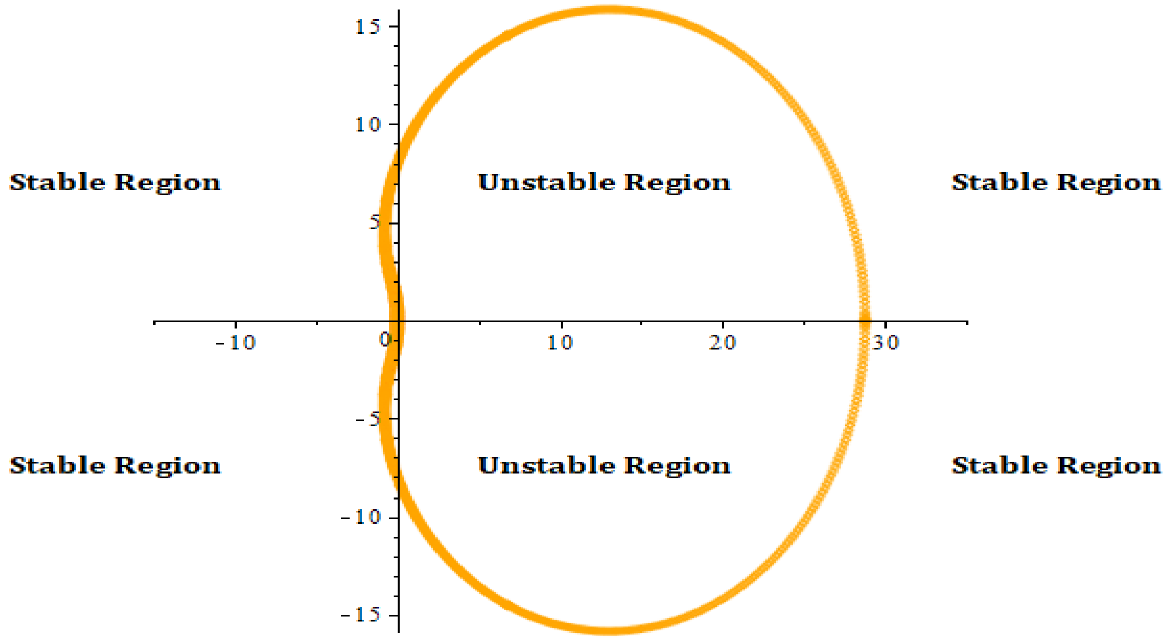

Figure 1: Stability region of the method.

Following the definition of A-stability, the method (14) is nearly A-stable, as its stability region encompasses the entire negative half-plane. Hence, the method is suitable for the numerical integration of stiff ordinary differential equations.

For zero stability, we set $\bar{h} = 0$ in equation (33) to obtain

$$\pi(u,0) = \det\left[\begin{bmatrix} 1 & 0 & 0 \\ -\frac{33}{16} & 1 & 0 \\ \frac{720}{347} & -\frac{795}{347} & 1 \end{bmatrix} u - \begin{bmatrix} \frac{2}{29} & -\frac{27}{29} & \frac{54}{29} \\ -\frac{3}{32} & \frac{7}{16} & -\frac{45}{32} \\ \frac{27}{347} & -\frac{165}{347} & \frac{410}{347} \end{bmatrix}\right] = 0, \tag{34}$$

Evaluating the above determinant leads to the first characteristic polynomial as:

$$-\frac{476961}{322016}u^2 + \frac{9843}{20126}u - \frac{2543}{322016} + u^3 = 0 \tag{35}$$

By solving the cubic equation (35), we obtained the roots of the first characteristic polynomial as:

u=0.0170138731, u=0.4641578699, u=1

And whose modulus are; 0.0170138731, 0.4641578699and 1. Hence, from definition (3), we conclude that the method (14) is zero stable.

IMPLEMENTATION OF THE (NFDIBBDF) METHOD

The NFDIBBDF method is implemented by applying the idea of Newton's iteration. We start by writing the formula (14) in the form:

$$\left.\begin{aligned} F_1 &= y_{n+1} - \frac{15}{29}hf_{n+1} + \frac{9}{29}hf_{n-1} - \varepsilon_1 \\ F_2 &= y_{n+2} - \frac{33}{16}y_{n+1} - \frac{15}{32}hf_{n+2} + \frac{9}{32}hf_n - \varepsilon_2 \\ F_2 &= y_{n+3} + \frac{720}{347}y_{n+1} - \frac{795}{347}y_{n+2} - \frac{150}{347}hf_{n+3} + \frac{90}{347}hf_{n+1} - \varepsilon_3 \end{aligned}\right\} \quad (36)$$

Where $\varepsilon_1, \varepsilon_2$ and ε_3 are the back values defined as:

$$\left.\begin{aligned} \varepsilon_1 &= \frac{2}{29}y_{n-2} - \frac{27}{29}y_{n-1} + \frac{54}{29}y_n \\ \varepsilon_2 &= -\frac{3}{32}y_{n-2} + \frac{7}{16}y_{n-1} - \frac{45}{32}y_n \\ \varepsilon_3 &= \frac{27}{347}y_{n-2} - \frac{165}{347}y_{n-1} + \frac{410}{347}y_n \end{aligned}\right\} \quad (37)$$

Definition 4: Let y_i and $y(x_i)$ be the approximate and exact solution of the system of first order stiff IVP (1), respectively. Then, the absolute error in the $(i)^{th}$ iteration is defined as;

$$(error_i)_t = |(y_i)_t - y(x_i)_t| \quad (38)$$

The maximum error is defined as;

$$MAXE = \underbrace{max}_{1 \le i \le T}\left(\underbrace{max(error_i)_t}_{1 \le i \le N}\right) \quad (39)$$

Where T denotes the total number of steps and N denotes the number of the equations.

Then, let$y_{n+1}^{(i+1)}$ denote the $(i+1)^{th}$ iteration

$$e_{n+j}^{(i+1)} = y_{n+1}^{(i+1)} - y_{n+1}^{(i)}, \qquad j = 1,2,3 \quad (40)$$

Applying the Newton's iteration, we get:

$$y_{n+j}^{(i+1)} = y_{n+j}^{(i)} - \left(F_j'(y_{n+j}^{(i)})\right)^{-1}\left(F_j(y_{n+j}^{(i)})\right), \; j = 1,2,3 \quad (41)$$

This implies:

$$\left(F_j'(y_{n+j}^{(i)})\right)e_{n+j}^{(i+1)} = -\left(F_j(y_{n+j}^{(i)})\right), \; j = 1,2,3 \quad (42)$$

The matrix representation (42) is given by:

$$\underbrace{\begin{bmatrix} \left(1 - \frac{15}{29}h\frac{\partial f_{n+1}}{\partial y_{n+1}}\right) & 0 & 0 \\ -\frac{33}{16} & \left(1 - \frac{15}{32}h\frac{\partial f_{n+2}}{\partial y_{n+2}}\right) & 0 \\ \left(\frac{720}{347} + \frac{90}{347}h\frac{\partial f_{n+1}}{\partial y_{n+1}}\right) & -\frac{795}{347} & \left(1 - \frac{150}{347}h\frac{\partial f_{n+3}}{\partial y_{n+3}}\right) \end{bmatrix}}_{JACOBIAN}\begin{bmatrix} e_{n+1}^{(i+1)} \\ e_{n+2}^{(i+1)} \\ e_{n+3}^{(i+1)} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ \frac{33}{16} & -1 & 0 \\ -\frac{720}{347} & \frac{795}{347} & -1 \end{bmatrix}\begin{bmatrix} y_{n+1}^{(i)} \\ y_{n+2}^{(i)} \\ y_{n+3}^{(i)} \end{bmatrix} +$$

$$h\begin{bmatrix} 0 & -\frac{9}{29} & 0 \\ 0 & 0 & -\frac{9}{32} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} f_{n-2}^{(i)} \\ f_{n-1}^{(i)} \\ f_n^{(i)} \end{bmatrix} + h\begin{bmatrix} \frac{15}{29} & 0 & 0 \\ 0 & \frac{15}{32} & 0 \\ -\frac{90}{347} & 0 & \frac{150}{347} \end{bmatrix}\begin{bmatrix} f_{n+1}^{(i)} \\ f_{n+2}^{(i)} \\ f_{n+3}^{(i)} \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} \quad (43)$$

The implementation of the equation (43) will be carried out using a C programming language code.

TEST PROBLEMS USED

The following stiff initial value problems show the performance of the method developed.

Problem 1:

$$y' = -9y, \qquad y(0) = e, \qquad 0 \le x \le 1$$

Exact Solution: $y(x) = e^{(1-9x)}$

Source: (Musa *et al*, 2012)

Problem 2:

$$y' = 5e^{5x}(y-1)^2 + 1, \; y(0) = -1, \qquad 0 \le x \le 1$$

Exact Solution: $y(x) = x - e^{-5x}$

Source: (Lee *et al.*, 2002)

Problem 3:

$$y' = -20y + 20sinx + cosx, \qquad y(0) = 1, \quad 0 \le x \le 2$$

Exact Solution: $y(x) = sinx + e^{-20x}$

Source: (Musa *et al.*, 2015)

Problem 4:

$$y'_1 = -20y_1 - 19y_2, \qquad y_1(0) = 2$$

$$0 \le x \le 20$$

$$y'_2 = -19y_1 - 20y_2, \qquad y_2(0) = 0$$

Exact Solution: $y_1(x) = e^{-39x} + e^{-x}$

$y_2(x) = e^{-39x} - e^{-x}$

Eigenvalues: -1 and -39

Source: (Musa *et al*, 2014)

Problem 5:

$$y'_1 = 198y_1 + 199y_2, \qquad y_1(0) = 1$$

$$0 \le x \le 10$$

$$y'_2 = -398y_1 - 399y_2, \qquad y_2(0) = -1$$

Exact Solution: $y_1(x) = e^{-x}$

$y_2(x) = -e^{-x}$

Eigenvalues: -1 and -200

Source: (Musa *et al.*, 2014)

NUMERICAL COMPUTATIONS

The three (3) selected test problems will now be solved numerically using the 3-point BBDF, 3-point Extended SBBDF, and NFDIBBDF methods. This is to compare the efficiency and accuracy of the methods; the maximum absolute errors obtained from different step lengths H are given in each problem. The tables below also show the number of steps taken to solve each problem and computational time. For easy referencing, the existing 3-point block backward differentiation formula developed by Ibrahim et al. (2007a) is given by:

$$\left.\begin{aligned} y_{n+1} &= \frac{1}{10}y_{n-2} - \frac{3}{4}y_{n-1} + 3y_n - \frac{3}{2}y_{n+2} + \frac{3}{20}y_{n+3} + 3hf_{n+1} \\ y_{n+2} &= -\frac{3}{65}y_{n-2} + \frac{4}{13}y_{n-1} - \frac{12}{13}y_n + \frac{24}{13}y_{n+1} - \frac{12}{65}y_{n+3} + \frac{12}{13}hf_{n+2} \\ y_{n+3} &= \frac{12}{137}y_{n-2} - \frac{75}{137}y_{n-1} + \frac{200}{137}y_n - \frac{300}{137}y_{n+1} + \frac{300}{137}y_{n+2} + \frac{60}{137}hf_{n+1} \end{aligned}\right\} \quad (44)$$

However, the extended 3-point super class of block backward differentiation formula developed by Musa et al. (2019) is given by:

$$\left.\begin{aligned} y_{n+1} &= -\frac{29}{70}y_{n-2} - \frac{37}{28}y_{n-1} + \frac{9}{7}y_n + \frac{23}{14}y_{n+2} - \frac{27}{140}y_{n+3} - \frac{15}{7}hf_{n+1} - \frac{12}{7}hf_{n-1} \\ y_{n+2} &= -\frac{27}{265}y_{n-2} + \frac{44}{53}y_{n-1} - \frac{44}{53}y_n + \frac{72}{53}y_{n+1} - \frac{68}{265}y_{n+3} + \frac{60}{53}hf_{n+2} + \frac{48}{53}hf_n \\ y_{n+3} &= \frac{68}{673}y_{n-2} - \frac{435}{673}y_{n-1} + \frac{1240}{673}y_n - \frac{1580}{673}y_{n+1} + \frac{1380}{673}y_{n+2} + \frac{300}{673}hf_{n+3} + \frac{240}{673}hf_{n+1} \end{aligned}\right\} \quad (45)$$

Tables 3 4, 5, 6, through 7 below give the numerical results. The following notations are used in the tables:

H: Step length/size

METHOD: Methods used

NS: Number of steps

3BBDF: 3-point Block BDF method

3ESBBDF: 3-point Extended Superclass of Block BDF method

NFDIBBDF: A New Fixed Coefficient Diagonally Implicit Block BDF method

MAXE: Maximum Error

CPU TIME: Computation Time (in seconds).

Table 3: Numerical Result for Problem 1.

H	METHOD	NS	MAXE	CPU TIME
10^{-2}	3BBDF	33	1.75664e-001	2.00833e-004
	3ESBBDF	33	6.50071e-002	6.56100e-002
	NFDIBBDF	33	3.52244e-002	2.70300e-002
10^{-3}	3BBDF	333	2.63192e-002	1.36950e-003
	3ESBBDF	333	6.50122e-004	1.88100e-001
	NFDIBBDF	333	6.19415e-004	2.53900e-002
10^{-4}	3BBDF	3,333	2.69331e-003	1.29261e-002
	3ESBBDF	3,333	6.50122e-006	1.13100e-002
	NFDIBBDF	3,333	6.93783e-006	1.10700e-002
10^{-5}	3BBDF	33,333	2.69933e-004	1.28720e-001
	3ESBBDF	33,333	6.50123e-008	9.13000e+000
	NFDIBBDF	33,333	7.06579e-008	1.80500e-001
10^{-6}	3BBDF	333,333	2.69993e-005	1.30950e+000
	3ESBBDF	333,333	6.50123e-010	9.62100e+001
	NFDIBBDF	333,333	7.08360e-010	1.13500e+001

Table 4: Numerical Result for Problem 2.

H	METHOD	NS	MAXE	CPU TIME
10^{-2}	3BBDF	33	2.80735e-002	2.76330e-004
	3ESBBDF	33	4.83217e-003	6.23441e-005
	NFDIBBDF	33	4.16232e-003	6.20100e-005
10^{-3}	3BBDF	333	3.71852e-003	1.81850e-003
	3ESBBDF	333	5.95338e-005	1.88100e-001
	NFDIBBDF	333	7.01937e-005	1.79400e-001
10^{-4}	3BBDF	3,333	3.74700e-005	1.71443e-002
	3ESBBDF	3,333	5.95692e-007	6.48433e-003

	NFDIBBDF	3,333	7.86945e-007	5.55800e-003
10^{-5}	3BBDF	33,333	3.74970e-005	1.70042e-001
	3ESBBDF	33,333	5.95974e-009	6.58687e-002
	NFDIBBDF	33,333	8.02119e-009	6.55600e-002
10^{-6}	3BBDF	333,333	3.74997e-006	1.70308e+000
	3ESBBDF	333,333	6.18636e-011	9.62100e+001
	NFDIBBDF	333,333	8.04294e-011	1.49600e+001

Table 5: Numerical Result for Problem 3.

H	METHOD	NS	MAXE	CPU TIME
10^{-2}	3BBDF	666	9.15007e-002	5.69750e-004
	3ESBBDF	666	4.49329e-002	3.37200e-002
	NFDIBBDF	666	3.93265e-002	1.93500e-002
10^{-3}	3BBDF	6,666	2.08350e-002	4.54233e-003
	3ESBBDF	6,666	9.23116e-004	1.05100e-002
	NFDIBBDF	6,666	1.00816e-003	1.04300e-002
10^{-4}	3BBDF	66,666	2.19484e-003	4.34752e-002
	3ESBBDF	66,666	9.92032e-006	7.45800e-002
	NFDIBBDF	66,666	1.23565e-005	7.12400e-002
10^{-5}	3BBDF	666,666	2.20579e-004	4.34533e-002
	3ESBBDF	666,666	9.99200e-008	3.36900e+000
	NFDIBBDF	666,666	1.27985e-007	3.68200e-001
10^{-6}	3BBDF	6,666,666	2.20688e-005	4.33535e+000
	3ESBBDF	6,666,666	9.99920e-010	2.96100e+001
	NFDIBBDF	6,666,666	1.28640e-009	7.45500e+000

Table 6: Numerical Result for Problem 4.

H	METHOD	NS	MAXE	CPU TIME
10^{-2}	3BBDF	666	6.23032e-002	2.77590e-002
	3ESBBDF	666	8.83217e-004	7.68676e-002
	NFDIBBDF	666	7.44133e-002	2.81100e-002
10^{-3}	3BBDF	6,666	3.76165e-002	7.66636e-002
	3ESBBDF	6,666	6.05338e-005	7.64515e-001
	NFDIBBDF	6,666	3.30107e-003	3.24500e-002
10^{-4}	3BBDF	66,666	4.26516e-003	7.64385e-001
	3ESBBDF	66,666	6.26692e-006	7.68143e-001
	NFDIBBDF	66,666	4.56593e-005	5.44100e-001
10^{-5}	3BBDF	666,666	4.30707e-004	7.63788e+000
	3ESBBDF	666,666	6.32740e-008	7.59821e+000
	NFDIBBDF	666,666	4.84759e-007	6.97890e+000
10^{-6}	3BBDF	6,666,666	4.31123e-005	7.65356e+001
	3ESBBDF	6,666,666	6.33362e-010	7.53567e+001
	NFDIBBDF	6,666,666	4.89169e-009	7.20100e+001

Table 7: Numerical Result for Problem 5.

H	METHOD	NS	MAXE	CPU TIME
10^{-2}	3BBDF	333	1.07308e-002	1.37500e-002
	3ESBBDF	333	1.83217e-002	7.36289e-002
	NFDIBBDF	333	2.78034e-004	1.41100e-002
10^{-3}	3BBDF	3,333	1.10060e-002	2.72200e-002
	3ESBBDF	3,333	8.05338e-002	5.81512e-002
	NFDIBBDF	3,333	3.14469e-006	1.91300e-002
10^{-4}	3BBDF	33,333	1.10333e-004	2.02700e-001
	3ESBBDF	33,333	1.26692e-008	5.81491e-001

	NFDIBBDF	33,333	3.20816e-008	5.57300e-002
10^{-5}	3BBDF	333,333	1.10361e-005	1.92600e+000
	3ESBBDF	333,333	1.32740e-010	5.81122e+000
	NFDIBBDF	333,333	3.21710e-010	4.40900e+000
10^{-6}	3BBDF	3,333,333	1.10363e-006	1.91700e+001
	3ESBBDF	3,333,333	1.33362e-012	5.79987e+001
	NFDIBBDF	3,333,333	2.38012e-011	4.23200e+001

The visual impact on the performance of the method developed is presented blow. $Log_{10}(MAXE)$ against $Log_{10}(H)$each test problem is plotted with the aid of MATLAB (Figure 2-6).

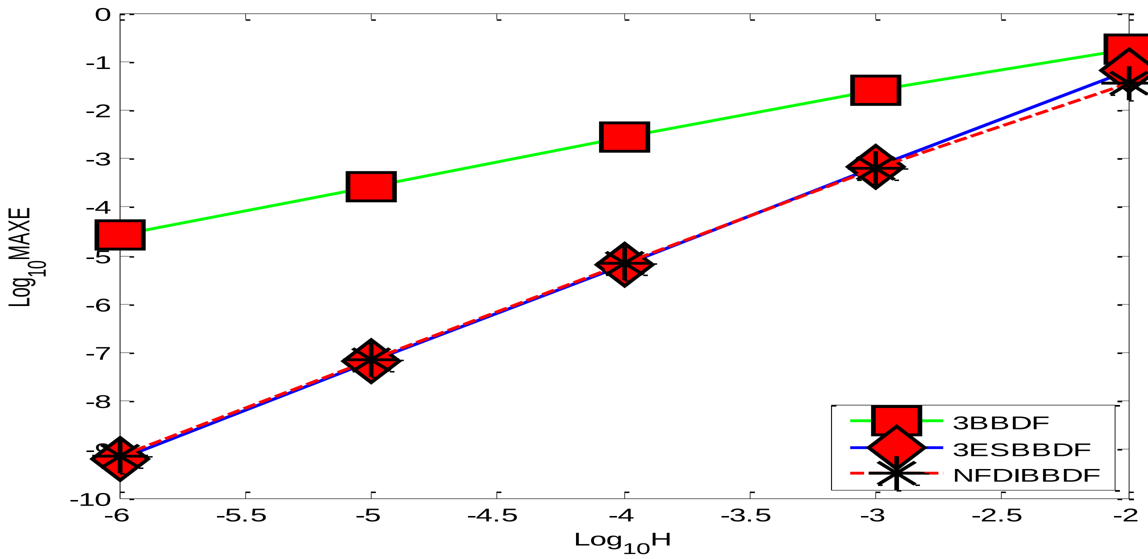

Figure 2: Efficiency Curve for Problem 1

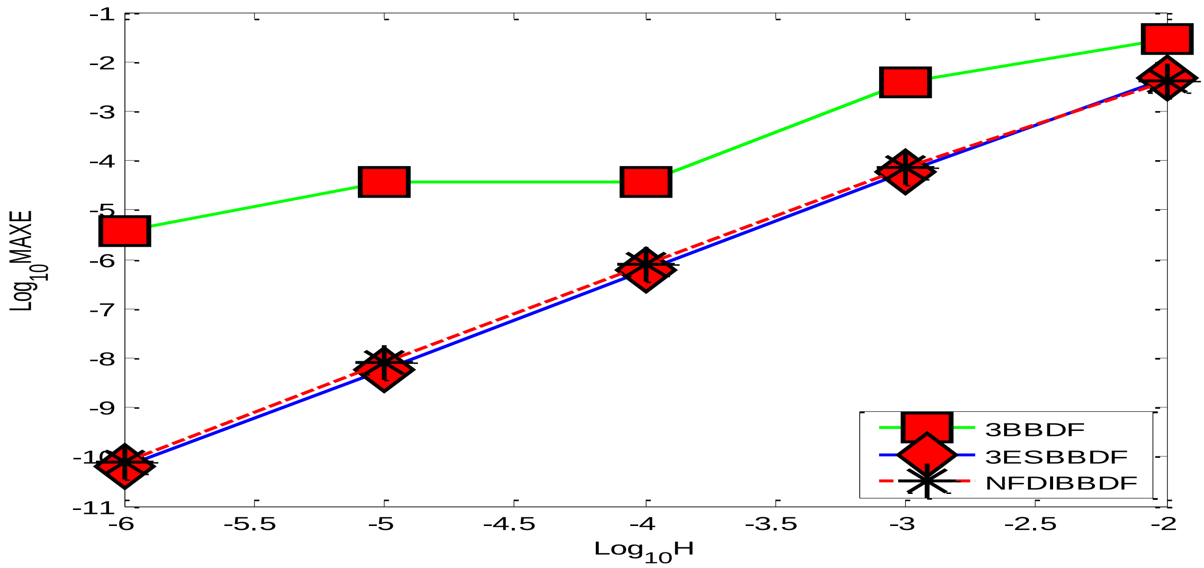

Figure 3: Efficiency Curve for problem 2

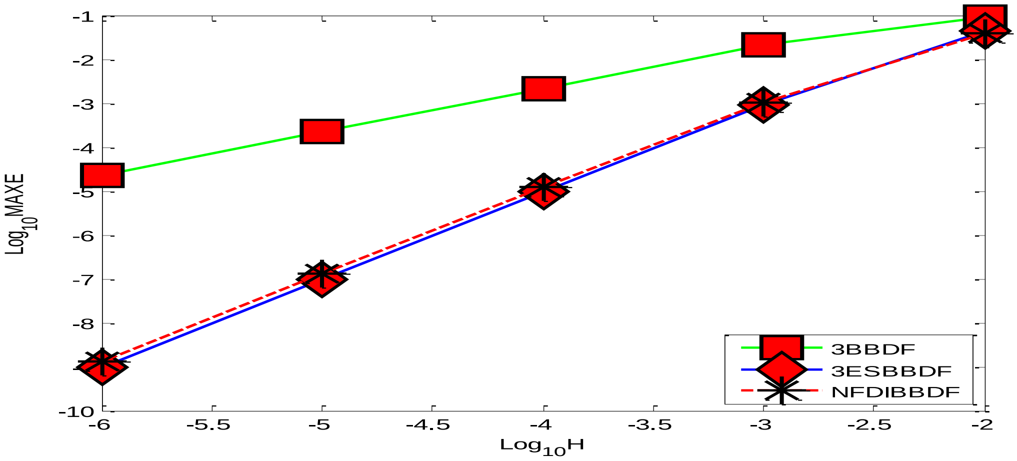

Figure 4: Efficiency Curve for problem 3

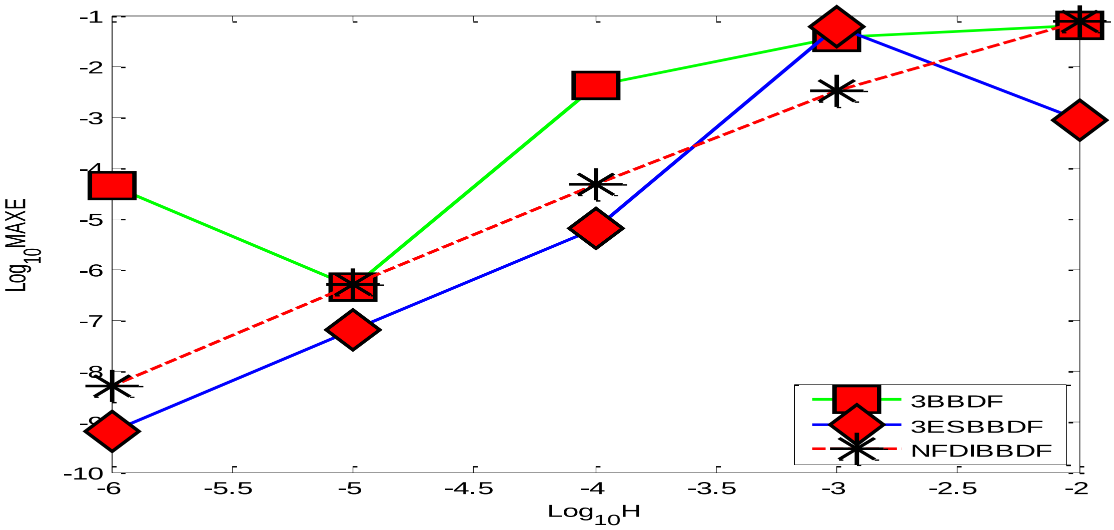

Figure 5: Efficiency Curve for problem 4

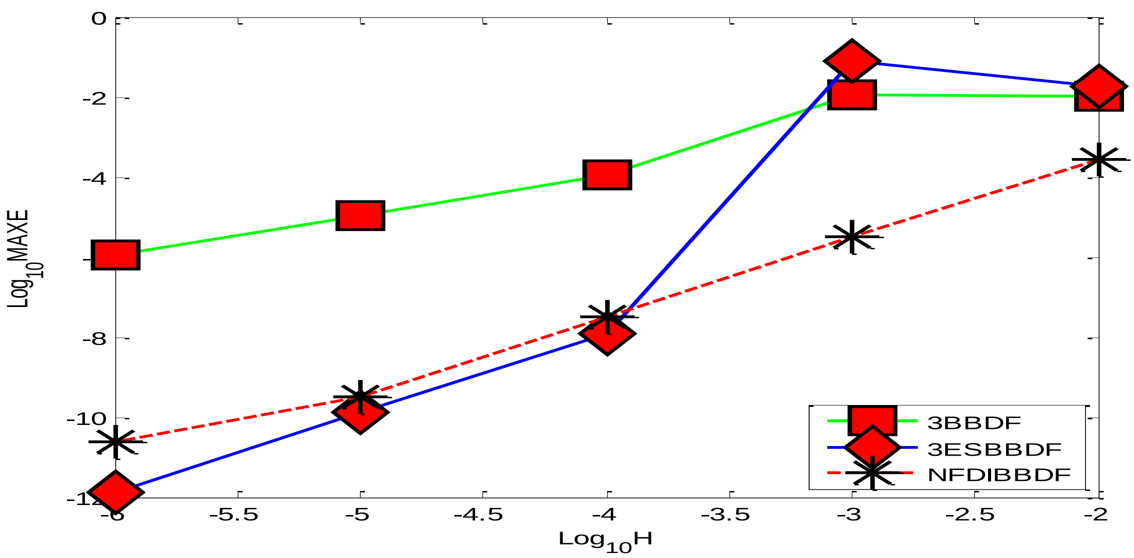

Figure 6: Efficiency Curve for problem 5

DISCUSSION:

From tables 3, 4, 5, 6, and 7 above, the comparative assessment of three numerical methods, 3-point BBDF, 3-point Extended SBBDF, and the newly introduced NFDIBBDF, is applied to five distinct test problems. The primary objective is to gauge the effectiveness and precision of these methods through an analysis of Maximum Errors (MAXE), the Number of Steps (NS), and Computational Time (CPU Time) for each problem. Key findings indicate that NFDIBBDF consistently outperforms 3BBDF and 3ESBBDF across various step sizes in terms of maximum error and computational time, showcasing its overall superiority in solving the specified problems.

The graphical portrayal of $\text{Log}_{10}\text{MAXE}$ against Log_{10}H further underscores the heightened performance of the NFDIBBDF method in contrast to 3BBDF and 3ESBBDF. The downward trends in NFDIBBDF's graphs signify enhanced scalability and accuracy as the step size varies.

CONCLUSION

The New Fixed coefficient Diagonally Implicit Block Backward Differentiation Formula (NFDIBBDF) is developed to solve stiff initial value problems. The NFDIBBDF method emerges as a resilient and effective approach for tackling stiff initial value problems, surpassing or competently matching existing methods in terms of accuracy and computational efficiency. The comprehensive analysis and visual representation reinforce the credibility of the NFDIBBDF method as a valuable addition to numerical techniques for stiff ODEs.

REFERENCES

Abasi N., Suleiman M.B., Ismail F., Ibrahim Z.B., & Musa H. (2014). A new formula of variable step 3-point block BDF method for solving stiff ODEs. Journal of Pure and Applied Mathematics: Advances and Application 12(1):49-76. **[Crossref]**

Alexander R. (1977). Diagonally implicit Runge-kutta for stiff ordinary differential equations. SIAM Journal on Numerical analysis, 14: 1006-1021. **[Crossref]**

Alhassan B., Yusuf H. Musa H. & Abasi N. (2023). A New Fifth Order Variable Step Size Block Backward Differentiation Formula with Off-Step Points for the Numerical of Stiff Initial Ordinary Differential Equations. Applied Mathematics and Computational Intelligence. 12: 94-121. **[Crossref]**

Bala N., Alhassan B., & Musa H. (2022). A New 3-point Implicit Block Numerical Method For Solving First Order Stiff Initial Value Problems. Timbou-Africa Academics Publications, International Journal of Science Research and Technology, 9(9): PP. 219-231.

Cash J.R. (1980). On the integration of stiff systems of ODEs using Extended Backward Differentiation Formula. Numerical Mathematics 34. 235-246. **[Crossref]**

Curtiss C.F., & Hirschfelder J. (1952). Integration of stiff equations. Proceedings of the National Academy of Sciences of the United States of America 38:235-243. **[Crossref]**

Dahlquist G. (1974). Problems related to the numerical treatment of stiff differential equations. International Computing symposium: 307-314.

Ibrahim Z.B., Noor N.M., Othman K.I. (2019). Fixed coefficients A() stable block backward differentiation formulas for stiff ordinary differential equations. Symmetry. 11, 864. **[Crossref]**

Ibrahim Z.B., Othman K.I., & Suleiman M.B. (2007a). Implicit r-point Block Backward Differentiation Formula for first order stiff ODEs. Applied Mathematics and Computational, 186:558-565. **[Crossref]**

Ibrahim Z.B., Suleiman M.B. & Othman K.I. (2007b). Fixed coefficients block backward differentiation formulas for the numerical solution of stiff ordinary differential equations. European Journal of Scientific Research. 21(3), 508-520.

Lambert J. D. (1991). Numerical Methods for Ordinary Differential Systems: The Initial Value Problem, John Wiley & Sons, New York, NY, USA.

Lambert J.D. (1973). Computational methods in ODEs. New York: John Wiley.

Lee H.C., Chan C.K., Hung C.I. (2002). A modified group-preserving scheme for solving the initial value problems of stiff ordinary differential equations. Applied Mathematics and Computation. 133(2-3):445-459. **[Crossref]**

Musa H. (2015). A new 3-point super class of BBDF for solving stiff ODEs. Katsina Journal of Natural sciences and Applied sciences. 4(1): ISNN2141-0755.

Musa H. and Muhammad U. (2019). Extended 3-point Super Class of Block Backward Differentiation Formula for solving Stiff IVP's. Abacus (Mathematical Sciences series), 44: 584-591.

Musa H., Alhassan B. & Abasi N. (2022). Diagonally Implicit 2-point Block Backward Differentiation Formula with Two Off-Step Points for Solving Stiff Initial Value Problems. Nigerian Journal of Mathematics and Applications. 32 (II):108-119. **[Crossref]**

Musa H., Suleiman M.B., & Senu N. (2012). Fully implicit 3-point block extended backward differentiation formulae for solving ODEs. Applied Mathematical Sciences, 6(85): 4211-4228.

Suleiman M.B., Musa H., Ismail F., Senu N., & Ibrahim Z.B. (2014). A new super class of Block Backward Differentiation Formulas for stiff ODEs. Asian–European journal of mathematics,7: 1350034. **[Crossref]**

Musa H., Suleiman M.B., Ismail F., Senu N., Majid Z.A., & Ibrahim Z.B. (2014). A new Fifth Order Implicit Block Method for Solving First Order Stiff Ordinary Differential Equations. Malaysian Journal of Mathematical Sciences. 8(8): 45-59.