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ORIGINAL RESEARCH ARTICLE

Development of Exponentiated Cosine Topp-Leone Generalized Family of Distributions and its Applications to Lifetime Data

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ABSTRACT
The Topp-Leone distribution is widely used for modeling lifetime data across many disciplines. This paper presents the Exponentiated Cosine Topp-Leone Generalized family (ExCTLG), which is an extension of the Topp-Leone family. The research examines the mathematical properties of the ExCTLG, including the survival and hazard functions, moments, moment-generating functions, and Renyi's entropy. Parameter estimation is carried out using the maximum likelihood estimation (MLE) techniques, and the estimated parameters consistency is validated using the Monte Carlo simulation method, thereby highlighting the superiority of MLE. The advantages and applicability of the proposed distribution are shown by analyzing two-lifetime datasets.

INTRODUCTION
Statistical distributions are utilized in various fields to represent real-world phenomena, such as life analysis, reliability, insurance, engineering, finance, economics, biology, medicine, and business materials (Al-Noor & Hilal, 2021; Al-saiary & Al-jadaani, 2022). However, classical distributions frequently found it difficult to effectively model diverse datasets due to skewness and multimodality. Consequently, interest in enhancing their performance is rising. (Al-shomrani, 2022; El-morshedy et al., 2020; Sangsanit & Bodhisuwan, 2016). This has led to the development of extensions and generalizations. Generators, also known as G families of distributions, are increasingly used to enhance the flexibility of existing distributions in modeling datasets by controlling their characteristics(Hassan et al., 2022; Nanga et al., 2022).

The Topp-Leone (T-L) distribution, proposed by (Topp & Leone, 1955), is a continuous unimodal distribution with bounded support. This distribution is particularly suitable for modeling finite support lifetimes and failure data. Numerous studies have been carried out to examine and explore various extensions of the Topp Leone-G distribution (TL-G) family. (Reyad et al., 2019) proposed the exponentiated generalized TL-G, further expanded by (Ibrahim et al., 2020) to include two additional shape parameters. In 2021,(Chamunorwa et al., 2021) presented the Exponentiated Odd Weibull-TL-G family, and (Chipepa & Oluyede, 2021) introduced the Topp-Leone Odd Exponential Half Logistic-G family. These studies have demonstrated the flexibility and potential of these extended families in modeling different types of data, offering a wide range of mathematical features and applications.

Most extensions of classical distributions are typically algebraic in nature. However, researchers have recently turned their attention towards statistical distributions based on trigonometric functions(Nanga et al., 2022, 2023; Souza et al., 2021, 2022; Souzay et al., 2019). These proposed statistical distributions using trigonometric functions could offer scholars additional options. For instance, the extended cosine Weibull, power, and half-logistic distributions (the extended cosine family) were initially introduced by (Muhammad et al., 2021). These distributions were found to have satisfactory performance in parameter estimation. Kumar et al. (2015) applied the sine function to a lifetime distribution, generating a new distribution that better fits bladder cancer patient data than existing distributions, as demonstrated by various tests. (Nanga et al., 2023) combined cosine-G with Topp-Leone to introduce the CTL-G family of distributions. Five special cases of the distributions were developed, with the CTL- Weibull and Cauchy distributions outperforming others. The log-CTL Weibull model

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showed better performance in fitting the Lung cancer dataset.

The study's objectives are to explore the theoretical properties, develop statistical inference methods, apply the Exponentiated Cosine Topp-Leone (ECTL) distribution to lifetime data, explore extensions, and contribute to methodological advancement in modeling lifetime data.

MATERIALS AND METHODS

Exponentiated Cosine Topp-Leone G family of distribution

Using the CTL family of distribution introduced by (Nanga et al., 2023) combined with the transformation proposed by (Gupta & Kundu, 1999). We proposed the exponentiated CTL family(ExCTL) of distribution with cumulative distribution function (cdf) defined below:

$$F_{ExCLT-G}(x;\alpha,\theta,\xi)=\left[1-\cos\left[\frac{\pi}{2}(1-(\bar{G}(x;\xi))^2)^{\alpha}\right]\ \right]^{\theta}\ ,\ x\in\Re,\theta,\alpha>0 \tag{1}$$

Where $\bar{G}(x;\xi)=1-G(x;\xi)$

The probability density function (pdf) can obtained by taking the first derivatives of the equation (1)

$$f_{ExCTL-G}(x;\alpha,\lambda,\xi)=\pi\alpha\theta g(x;\xi)\bar{G}(x;\varepsilon)(1-(\bar{G}(x;\xi))^2)^{\alpha-1}\sin\left[\frac{\pi}{2}(1-(\bar{G}(x;\xi))^2)^{\alpha}\right]\left[1-\cos\left[\frac{\pi}{2}(1-(\bar{G}(x;\xi))^2)^{\alpha}\right]\ \right]^{\theta-1} \tag{2}$$

$G(x;\xi)$ and $g(x;\xi)$ represent the baseline cumulative distribution function and probability distribution function respectively

The survival and hazard functions are defined, respectively, as

$$S_{ExCLT-G}(x;\alpha,\theta,\xi)=1-\left[1-\cos\left[\frac{\pi}{2}(1-(\bar{G}(x;\xi))^2)^{\alpha}\right]\ \right]^{\theta} \tag{3}$$

$$h_{ExCLT-G}(x;\alpha,\theta,\varepsilon)=\frac{\pi\alpha\theta g(x;\xi)\bar{G}(x;\xi)(1-(\bar{G}(x;\xi))^2)^{\alpha-1}\sin\left[\frac{\pi}{2}(1-(\bar{G}(x;\xi))^2)^{\alpha}\right]\left[1-\cos\left[\frac{\pi}{2}(1-(\bar{G}(x;\xi))^2)^{\alpha}\right]\ \right]^{\theta-1}}{1-\left[1-\cos\left[\frac{\pi}{2}(1-(\bar{G}(x;\xi))^2)^{\alpha}\right]\ \right]^{\theta}} \tag{4}$$

Useful expansion of the pdf and cdf

To determine the mathematical properties of the ExCTL-G family, we may now obtain the mixture representation for the ExCTL-G PDF and CDF. The cdf and pdf of the mixture representation of the ExCTL-G family can be computed with the help of the generalized binomial expansion, Taylor series, and the Table of integral series supplied by (Gradshteyn & Ryzhik, 2007).

The cdf is given by

$$F(x)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\varphi_{i,j,k,l}G_{c_i}^{l}(x;\xi) \tag{5}$$

Where $\varphi_{i,j,k,l}=(-1)^{i+j+k+l}\binom{\lambda}{j}\binom{2\alpha i}{k}\binom{2k}{l}$and $c_i[G(x;\xi)]^l=G_{c_i}^{l}(x;\xi)$

The pdf is given by

$$f(x)=\pi\alpha\lambda\sum_{p=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\phi_{i,j,k,l,m,n,p}H_{c_i}^{p+l}(x) \tag{6}$$

Where$H_{c_i}^{p+l}(x)=c_i g(x;\xi)(G(x;\xi))^{p+l}and\phi_{i,j,k,l,m,n,p}=$
$\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{l}^{\infty}\frac{(-1)^{n+m+p+j+k+l}\left(\frac{\pi}{2}\right)}{(2n+1)!}\binom{2\alpha(n+1)-1}{m}\binom{2m+1}{p}\binom{\lambda-1}{j}\binom{2\alpha i}{k}\binom{2k}{l}$

Moment and Moment Generating Function(mgf)

Preposition: The r^{th}non-central moment of the Exponentiated Topp Leone G family is defined as;

$$\mu_r'=\pi\alpha\lambda\sum_{p=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\phi_{i,j,k,l,m,n,p}\int_0^{\infty}x^r H_{c_i}^{p+l}(x)\,dx \tag{7}$$

Proof:

By definition, the r^{th}moment can be expressed as;

$\mu_r^{'} = E[x^r] = \int_{-\infty}^{\infty} x^r f(x)dx$ Where $f(x)$ is the pdf.

Substitute the pdf from eqn (18)

$$\mu_r^{'} = \pi\alpha\lambda \sum_{p=0}^{\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \phi_{i,j,k,l,m,n,p} \int_0^{\infty} x^r H_{c_i}^{p+l}(x)\, \mathrm{dx} \tag{8}$$

ExCTL-G's mgf is described as;

$$M_X(t) = \pi\alpha\lambda \sum_{p=0}^{\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \phi_{i,j,k,l,m,n,p} \int_0^{\infty} e^{tx} H_{c_i}^{p+l}(x)\, dx \tag{9}$$

Rényi's entropy

A random variable's entropy quantifies its degree of variation or uncertainty. It has diverse applications across numerous disciplines, including data processing, statistical physics, probability theory, engineering, communication theory, and quantum physics. Suppose X represents a random variable with a probability density function $f(x)$. The Rényi entropy, as defined by (Renyi, 1961), is defined as:

$$I_R(\gamma) = \frac{1}{1-\gamma} log\left[\int_{-\infty}^{\infty} f(x)^{\gamma} dx\right], \gamma \neq 1, \gamma>0 \tag{10}$$

The following is the expression for the ExCTL distributions' Rényi entropy.:

$$I_R(\gamma) = \frac{1}{1-\gamma} log\left[\int_{-\infty}^{\infty} f_{ExCTL-G}(x)^{\gamma} dx\right], \gamma \neq 1, \gamma>0 \tag{11}$$

By restructuring (2) algebraically, we can obtain an expression for $f_{ExCTL-G}(x)^{\gamma}$ as:

$$f_{ExCTL-G}(x)^{\gamma} = (\pi\theta\alpha)^{\gamma} g(x;\xi)^{\gamma}(1-G(x;\xi))[1-(1-G(x;\xi))^2]^{\gamma(\alpha-1)} sin^{\gamma}\left(\frac{\pi}{2}(1-(1-G(x;\xi))^2)^{\alpha}\right)\left[1-cos\left(\frac{\pi}{2}(1-(1-G(x;\xi))^2)^{\alpha}\right)\right]^{\gamma(\theta-1)} \tag{12}$$

Using the Taylor series and binomial expansion

$$f_{ExCTL-G}(x;\alpha,\theta,\xi)^{\gamma} = (\pi\theta\alpha)^{\gamma} g(x;\xi)^{\gamma} w_{krjlmnpq}(G(x;\xi))^p \tag{13}$$

$$w_{krjlmnpq} = \sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(-1)^{k+j+l+m+n+p+q}\left(\frac{\pi}{2}\right)^k \binom{k}{r}\binom{\alpha(r+\gamma)-\gamma}{j}\binom{\alpha j+\gamma}{l}\binom{\gamma(\theta-1)}{n}\binom{2\alpha n}{m}\binom{2m}{p} c_q$$

Substituting the term $f_{ExCTL-G}(x)^{\gamma}$into (11), we can derive the Rényi entropy for the ExCTL distributions as:

$$I_R(\gamma) = \frac{1}{1-\gamma} log\left[(\pi\alpha\theta)^{\gamma} \int_{-\infty}^{\infty} g(x;\xi)^{\gamma} w_{krjlmnpq}(G(x;\xi))^p dx\right], \gamma \neq 1, \gamma>0 \tag{14}$$

Parameter estimation

Here, we applied the MLE method to estimate the ExCTL-G parameters. Consider a random sample$x_1, x_2, \ldots, x_n$of size n from the ExCLT-G with parametersα, θ andε. Suppose $\vartheta = (\alpha, \theta, \varepsilon)^T$is a p × 1 vector of parameters. The following is an expression for the log-likelihood function:

$$\ell = n\, log(\pi) + n\, log(\theta) + n\, log(\alpha) + \sum_{i=1}^{n} log\, g\,(x_i;\xi) + \sum_{i=1}^{n} log(1-G(x_i;\xi)) + (\alpha-1)\sum_{i=1}^{n} log(1-(1-G(x_i;\xi))^2) + \sum_{i=1}^{n} log\left(sin\left(\frac{\pi}{2}(1-(1-G(x_i;\xi))^2)^{\alpha}\right)\right) + (1-\theta)\sum_{i=1}^{n} log\left(1-cos\left(\frac{\pi}{2}(1-(1-G(x_i;\xi))^2)^{\alpha}\right)\right) \tag{15}$$

By taking the partial derivative of the log-likelihood function, one can determine the score function components $U(\vartheta) = \left(\frac{\delta\ell}{\delta\alpha}, \frac{\delta\ell}{\delta\theta}, \frac{\delta\ell}{\delta\varepsilon}\right)^T$ as follows:

$$\frac{\delta\ell}{\delta\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}log(1-(1-G(x_i;\xi))^2)+\left(\frac{\pi}{2}\sum_{i=1}^{n}(1-(1-G(x_i;\xi)))\times log(1-(1-G(x_i;\xi))^2)\,cot\left[\frac{\pi}{2}(1-(1-G(x_i;\xi))^2)^{\alpha}\right]\right)+$$
$$\left(\frac{\pi}{2}\textstyle\sum_{i=1}^{n}(1-(1-G(x_i;\xi)))\times log(1-(1-G(x_i;\xi))^2)\,tan\left[\frac{\pi}{2}(1-(1-G(x_i;\xi))^2)^{\alpha}\right]\right) \quad (16)$$

$$\frac{\delta\ell}{\delta\theta}=\frac{n}{\theta}+\textstyle\sum_{i=1}^{n}log\left(1-cos\left[\frac{\pi}{2}(1-(1-G(x_i;\xi))^2)^{\alpha}\right]\right) \quad (17)$$

$$\frac{\delta\ell}{\delta\varepsilon}=\sum_{i=1}^{n}\frac{g'(x_i;\xi)}{g(x_i;\xi)}-\sum_{i=1}^{n}\frac{G'(x_i;\xi)}{1-G(x_i;\xi)}+2(\alpha-1)\sum_{i=1}^{n}\frac{G'(x_i;\xi)}{1-G(x_i;\xi)}+\alpha\pi\sum_{i=1}^{n}G'(x_i;\xi)(1-G(x_i;\xi))(1-(1-G(x_i;\xi))^2)^{\alpha-1}$$
$$cot\left(\frac{\pi}{2}(1-(1-G(x_i;\xi))^2)^{\alpha}\right)-(1-\theta)\alpha\pi\textstyle\sum_{i=1}^{n}G'(x_i;\xi)(1-G(x_i;\xi))(1-(1-G(x_i;\xi))^2)^{\alpha-1}\,tan\left(\frac{\pi}{2}(1-(1-G(x_i;\xi))^2)^{\alpha}\right) \quad (18)$$

Where $g'(x_i;\xi)=\frac{\delta g(x_i;\xi)}{\delta\varepsilon}$ and $G'(x_i;\xi)=\frac{\delta G(x_i;\xi)}{\delta\varepsilon}$

MLE estimates can be obtained by solving for unknown parameters in the score functions (16), (17), and (18) after setting them equal to zero. This can be accomplished using a variety of methods, such as the quasi-Newton-Raphson approach.

Some sub-distributions

The ExCTL-G family can be extended by incorporating various baseline distributions to enhance application performance and flexibility. Using specified parameter values, these distributions' density and hazard functions are plotted to investigate the shape flexibility of the selected distributions. The graphs were generated using R software.

Exponentiated Cosine Topp Leone Weibull distribution

Consider adopting the Weibull distribution (Weibull.1939) as the baseline distribution, with the baseline cumulative distribution function and probability density function given respectively as; $G(x;\xi)=1-e^{-\lambda x^{\beta}}$, $g(x;\xi)=\lambda\beta x^{\beta-1}e^{-\lambda x^{\beta}}$,$x>0;\ \lambda,\beta>0$respectively. The cdf and pdf of Exponentiated Cosine Topp Leone Weibull distribution (ExCTLW).

$$F_{ExCLT-W}(x;\alpha,\theta,\lambda,\beta)=\left[1-cos\left[\frac{\pi}{2}\left(1-e^{-2\lambda x^{\beta}}\right)^{\alpha}\right]\ \right]^{\theta}\ ,\ x>0;\ \alpha,\theta,\lambda.\beta>0 \quad (18)$$

$$f_{ExCTL-W}(x;\alpha,\theta,\lambda,\beta)=\pi\alpha\theta\lambda\beta x^{\beta-1}e^{-2\lambda x^{\beta}}\left(1-e^{-2\lambda x^{\beta}}\right)^{\alpha-1}sin\left[\frac{\pi}{2}\left(1-e^{-2\lambda x^{\beta}}\right)^{\alpha}\right]\left[1-cos\left[\frac{\pi}{2}\left(1-e^{-2\lambda x^{\beta}}\right)^{\alpha}\right]\ \right]^{\theta-1} \quad (19)$$

The corresponding hazard function is

$$h(x)=\frac{\pi\alpha\theta\lambda\beta x^{\beta-1}e^{-2\lambda x^{\beta}}\left(1-e^{-2\lambda x^{\beta}}\right)^{\alpha-1}sin\left[\frac{\pi}{2}\left(1-e^{-2\lambda x^{\beta}}\right)^{\alpha}\right]\left[1-cos\left[\frac{\pi}{2}\left(1-e^{-2\lambda x^{\beta}}\right)^{\alpha}\right]\ \right]^{\theta-1}}{\left\{1-\left[1-cos\left[\frac{\pi}{2}\left(1-e^{-2\lambda x^{\beta}}\right)^{\alpha}\right]\ \right]^{\theta}\right\}} \quad (20)$$

The density function plot and the hazard function graphs of the ExCTLW distribution for some arbitrary parameter values are shown in Figure 1. The density exhibits both left- and right-skewed shapes, as can be observed. However, the hazard function illustration shows bathtub increasing, and decreasing failure rates.

The Quantile function of ExCTLW is derived as

$$x_u=G^{-1}\left[-\frac{1}{2\lambda}log\left[1-\left(\frac{2}{\pi}arccos\left(1-u^{\frac{1}{\theta}}\right)\right)^{\frac{1}{\alpha}}\right]\right]^{\frac{1}{\beta}},\quad 0\le u\le 1 \quad (21)$$

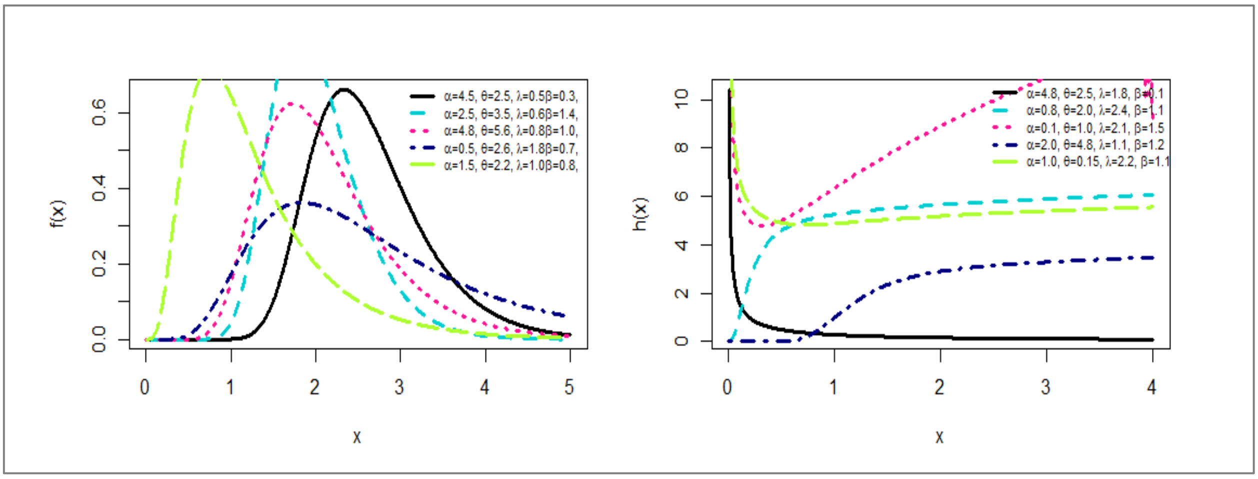

Figure. 1 ExCTLW density and hazard functions plots

Exponentiated Cosine Topp Leone Frechet distribution (ExCTLF)

using the Frechet distribution (Ramos et al., 2020) as the baseline distribution. The baseline cumulative distribution function and probability density function are:$G(x;\xi) = e^{-\lambda x^{-\beta}}$ $g(x;\xi) = \lambda\beta x^{-(\beta+1)}e^{-\lambda x^{-\beta}}$ $, x > 0;\ \lambda, \beta > 0$. The cdf and pdf of (ExCTLF) are respectively defined as

$$F_{ExCLT-W}(x;\alpha,\theta,\lambda,\beta) = \left[1 - cos\left[\frac{\pi}{2}\left(1-\left(1-e^{-\lambda x^{-\beta}}\right)^2\right)^{\alpha}\right]\ \right]^{\theta}\ ,\ x > 0;\ \alpha,\theta,\lambda.\beta > 0 \tag{22}$$

$$f_{ExCTL-W}(x;\alpha,\theta,\lambda,\beta) = \pi\alpha\theta\lambda\beta x^{-(\beta+1)}e^{-\lambda x^{-\beta}}\left(1-e^{-\lambda x^{-\beta}}\right)\left(1-\left(1-e^{-\lambda x^{-\beta}}\right)^2\right)^{\alpha-1}$$

$$sin\left[\frac{\pi}{2}\left(1-\left(1-e^{-\lambda x^{-\beta}}\right)^2\right)^{\alpha}\right]\left[1-cos\left[\frac{\pi}{2}\left(1-\left(1-e^{-\lambda x^{-\beta}}\right)^2\right)^{\alpha}\right]\ \right]^{\theta-1} \tag{23}$$

The corresponding hazard function is

$$h(x) = \frac{\pi\alpha\theta\lambda\beta x^{-(\beta+1)}e^{-\lambda x^{-\beta}}\left(1-e^{-\lambda x^{-\beta}}\right)\left(1-\left(1-e^{-\lambda x^{-\beta}}\right)^2\right)^{\alpha-1} sin\left[\frac{\pi}{2}\left(1-\left(1-e^{-\lambda x^{-\beta}}\right)^2\right)^{\alpha}\right]\left[1-cos\left[\frac{\pi}{2}\left(1-\left(1-e^{-\lambda x^{-\beta}}\right)^2\right)^{\alpha}\right]\ \right]^{\theta-1}}{\left\{1-\left[1-cos\left[\frac{\pi}{2}\left(1-\left(1-e^{-\lambda x^{-\beta}}\right)^2\right)^{\alpha}\right]\ \right]^{\theta}\right\}} \tag{24}$$

The density function and hazard function plots of the ExCTLF for various parameter values are displayed in Figure 2. While the hazard function shows both increasing and decreasing failure rates, the probability density function (pdf) exhibits a right skewed form.

ExCTLF's quantile function is provided as$x_u = G^{-1}\left[-\frac{1}{\lambda}log\left[\sqrt{1-\left(\frac{2}{\pi}arccos\left(1-u^{\frac{1}{\theta}}\right)\right)^{\frac{1}{\alpha}}}\right]\right]^{\frac{1}{\beta}}$, $\quad 0 \le u \le 1$ (25)

Exponentiated Cosine Topp Leone Lomax distribution (ExCTLLx).

The ExCTLLx is proposed using the Lomax distribution (Lomax.1954) as the baseline distribution with the cumulative distribution function and probability density function are; $G(x;\xi) = 1-\left(1+\frac{x}{\lambda}\right)^{-\beta}$, $g(x;\xi) = \frac{\beta}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-(\beta+1)}$ $, x > 0;\ \lambda, \beta > 0$. The cumulative distribution function and probability density function ExCTLLx distribution are defined as

$$F_{ExCLT-W}(x;\alpha,\theta,\lambda,\beta) = \left[1 - cos\left[\frac{\pi}{2}\left(1-\left(1+\frac{x}{\lambda}\right)^{-2\beta}\right)^{\alpha}\right]\ \right]^{\theta}\ ,\ x > 0;\ \alpha,\theta,\lambda.\beta > 0 \tag{26}$$

$$f_{ExCTL-W}(x;\alpha,\theta,\lambda,\beta) = \frac{\pi\alpha\theta\beta}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-(2\beta+1)}\left(1-\left(1+\frac{x}{\lambda}\right)^{-2\beta}\right)^{\alpha-1}$$

$$sin\left[\frac{\pi}{2}\left(1-\left(1+\frac{x}{\lambda}\right)^{-2\beta}\right)^{\alpha}\right]\left[1-cos\left[\frac{\pi}{2}\left(1-\left(1+\frac{x}{\lambda}\right)^{-2\beta}\right)^{\alpha}\right]\ \right]^{\theta-1} \tag{27}$$

The corresponding hazard function is

$$h(x)=\frac{\frac{\pi\alpha\theta\beta}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-(2\beta+1)}\left(1-\left(1+\frac{x}{\lambda}\right)^{-2\beta}\right)^{\alpha-1} sin\left[\frac{\pi}{2}\left(1-\left(1+\frac{x}{\lambda}\right)^{-2\beta}\right)^{\alpha}\right]\left[1-cos\left[\frac{\pi}{2}\left(1-\left(1+\frac{x}{\lambda}\right)^{-2\beta}\right)^{\alpha}\right]\ \right]^{\theta-1}}{\left\{1-\left[1-cos\left[\frac{\pi}{2}\left(1-\left(1+\frac{x}{\lambda}\right)^{-2\beta}\right)^{\alpha}\right]\ \right]^{\theta}\right\}} \tag{28}$$

For specified parameter values, the density function and hazard function of the ExCTLLx are plotted in Figure 3. It is evident that the density has a skewed shape to the right, and the hazard function indicates an increasing failure rate.

The quantile function of the ExCTLLx can be expressed as

$$x_u = G^{-1}\left[\lambda\left(\left(\sqrt{1-\left(\frac{2}{\pi}arccos\left(1-u^{\frac{1}{\theta}}\right)\right)^{\frac{1}{\alpha}}}\right)^{-\left(\frac{1}{\beta}\right)}-1\right)\right],\quad 0\le u\le 1 \tag{29}$$

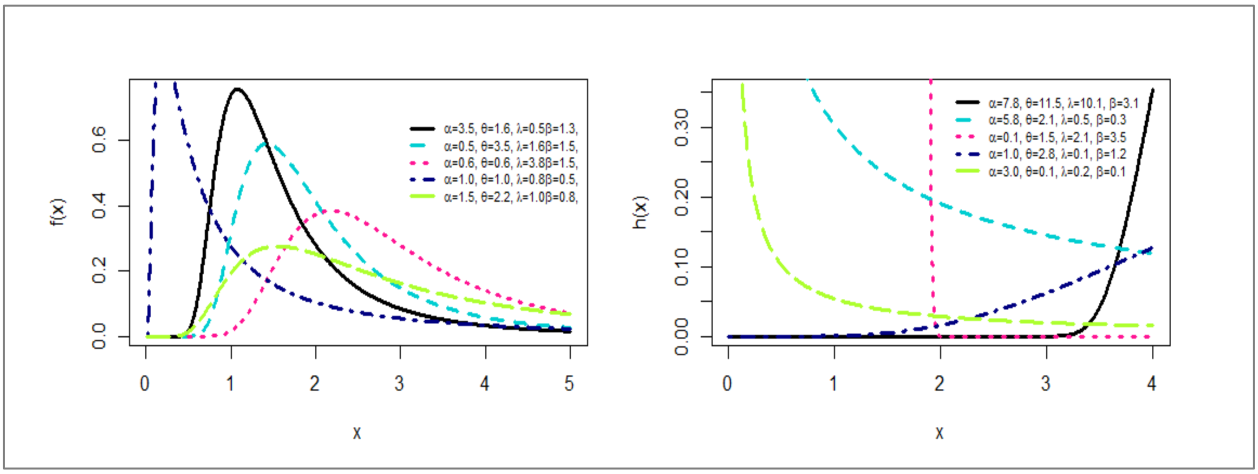

Figure 2 ExCTLF density and hazard functions plots

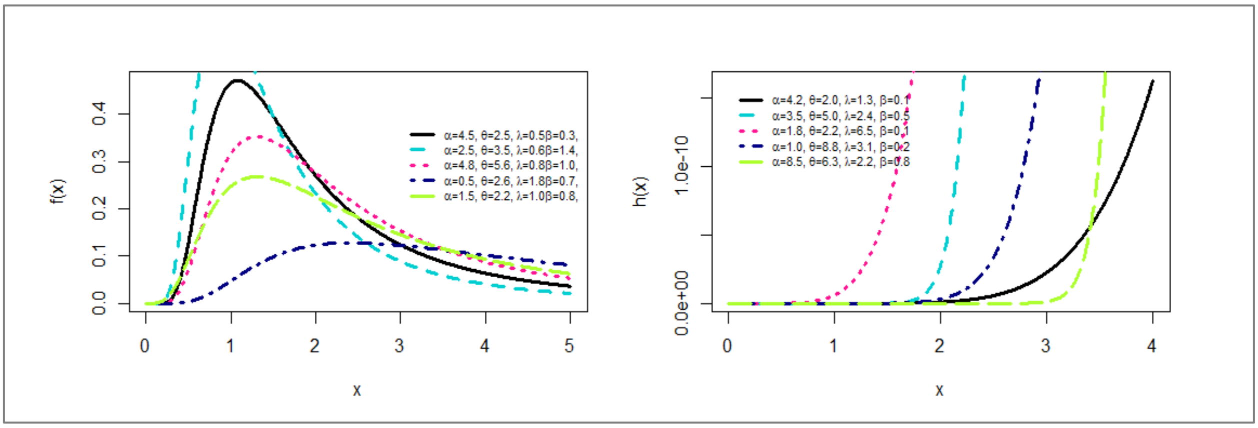

Figure. 3 ExCTLLx density and hazard functions plots

RESULTS AND DISCUSSION

Simulation results

We verified the performance of the maximum likelihood estimators and maximum product spacing (MPS) estimators using Monte Carlo simulations. We ran simulation experiments using the ExCTLW to illustrate this. Using the quantile function of the ExCTLW distribution, we generated random numbers. One thousand iterations of the experiment were conducted using different sample sizes $n = (20, 50, 100, 250, 500\ and\ 1000)$ and parameter values I $(\alpha = 1.4, \theta = 1.3, \lambda = 3.5, \beta = 1.5)$ and II $(\alpha = 5.4, \theta = 4.8, \lambda = 3.8, \beta = 2.5)$ for the ExCTLW distribution. We obtained estimates for the parameter estimators such as the average (mean), average bias (AB), and root mean square (RMSE). The outcomes are presented in Tables 1 and 2 for the MPS and MLE methods. The findings in both techniques consistently demonstrate that the AB and RMSE decrease as the sample size increases. This indicates that the methods successfully estimate the ExCTLW distribution's parameters. In contrast, the MLE approach is considered the best method for parameter estimation because of its approximately equal actual and iterative levels, but the MPS parameter estimates are less reliable. The MLE approach, however, demonstrated consistency in observing and estimating parameter values across various sample sizes and iteration levels.

Table 1: Results ExCTLW simulations using the MPS method

		I				II		
Parameter	n	Means	Bias	RMSE	Parameter	Means	Bias	RMSE
	20	1.6203	0.2203	0.6277		5.6021	0.2021	1.0365
	50	1.5532	0.1532	0.4791		5.6398	0.2398	0.9846
	100	1.5298	0.1298	03750		5.6291	0.2291	0.8062
α (1.4)	250	1.4858	0.0858	0.2656	α (5.4)	5.6349	0.2349	0.7171
	500	1.4578	0.0578	0.1972		5.5765	0.1765	0.5336
	1000	1.4398	0.0398	0.1411		5.5386	0.1386	0.4106
	20	1.4291	0.1291	0.6907		4.9723	0.1723	0.9405
	50	1.3472	0.0472	0.5274		4.8910	0.0910	0.8176
	100	1.3151	0.0151	0.3855		4.8795	0.0795	0.6933
θ (1.3)	250	1.2771	-0.0229	0.2624	θ (4.8)	4.7883	-0.0117	0.5624
	500	1.2867	-0.0133	0.1836		4.8056	0.0056	0.4191
	1000	1.2853	-0.0147	0.1253		4.8195	0.0195	0.3209
	20	3.4372	-0.0628	0.8967		3.7345	-0.0655	0.4915
	50	3.4379	-0.0621	0.6228		3.7473	-0.0527	0.3069
	100	3.4365	-0.0635	0.4023		3.7641	-0.0359	0.2140
λ (3.5)	250	3.4546	-0.0454	0.2556	λ (3.8)	3.7775	-0.0225	0.1527
	500	3.4682	-0.0318	0.1776		3.7897	-0.0103	0.1150
	1000	3.4826	-0.0174	0.1199		3.7944	-0.0016	0.0855
	20	1.4738	-0.0262	0.4934		2.3567	-0.1433	0.5265
	50	1.4879	-0.0121	0.3798		2.3923	-0.1077	0.3224
	100	1.4764	-0.0236	0.2788		2.4197	-0.0803	0.2282
β (1.5)	250	1.4913	-0.0087	0.1954	β (2.5)	2.4511	-0.0489	0.1587
	500	1.4877	-0.0123	0.1435		2.4638	-0.0362	0.1156
	1000	1.4915	-0.0085	0.1038		2.4724	-0.0276	0.0828

Table 2: Results ExCTLW simulations using the MLE method

Parameter	n	I Means	I Bias	I RMSE	Parameter	II Means	II Bias	II RMSE
α (1.4)	20	1.5943	0.1943	0.6943	α (5.4)	5.5538	0.1538	1.1528
	50	1.5326	0.1326	0.4969		5.5291	0.1291	0.9771
	100	1.5083	0.1083	0.3781		5.6194	0.2194	0.8274
	250	1.4740	0.0740	0.2503		5.5971	0.1971	0.6644
	500	1.4602	0.0602	0.1925		5.5970	0.1970	0.5345
	1000	1.4354	0.0354	0.1387		5.5442	0.1442	0.4130
θ (1.3)	20	1.5483	0.2483	0.7060	θ (4.8)	4.7953	-0.0047	0.9976
	50	1.4233	0.1233	0.5033		4.8989	0.0989	0.9141
	100	1.3825	0.0825	0.3942		4.8754	0.0754	0.6638
	250	1.3061	0.0061	0.2666		4.8508	0.0508	0.5394
	500	1.3006	0.0006	0.1824		4.8221	0.0221	0.4359
	1000	1.2935	-0.0065	0.1156		4.8434	0.0434	0.3101
λ (3.5)	20	4.1190	0.6190	1.3597	λ (3.8)	4.0747	0.2747	0.6823
	50	3.7698	0.2698	0.7463		3.9141	0.1941	0.3508
	100	3.6325	0.1325	0.4470		3.8635	0.0635	0.2308
	250	3.5520	0.0520	0.2673		3.8282	0.0282	0.1561
	500	3.5200	0.0200	0.1762		3.8189	0.0189	0.1180
	1000	3.5116	0.0116	0.1191		3.8169	0.0169	0.0886
β (1.5)	20	1.6150	0.1150	0.5007	β (2.5)	2.7419	0.2419	0.6600
	50	1.5478	0.0478	0.3608		2.5844	0.0844	0.3553
	100	1.5110	0.0110	0.2719		2.5236	0.0236	0.2264
	250	1.5095	0.0095	0.1898		2.4953	-0.0047	0.1492
	500	1.4955	-0.0045	0.1406		2.4869	-0.0131	0.1089
	1000	1.4973	-0.0027	0.0983		2.4850	-0.0150	0.0778

Application 1

In this section, we show that the ExCTLW distribution outperforms the Weibull distribution (WD) and the Cosine Topp Leone Weibull (CTLW) distribution when fitting real datasets. The dataset used in this analysis includes Egypt's actual monthly income taxes from January 2006 to November 2010 (measured in 1000 million Egyptian pounds). The source of the data was (Owoloko et al., 2015). The following is how the dataset appears:

5.9, 20.4, 14.9, 16.2, 17.2, 7.8, 6.1, 9.2, 10.2, 9.6, 13.3, 8.5, 21.6, 18.5, 5.1, 6.7, 17, 8.6, 9.7, 39.2, 35.7, 15.7, 9.7, 10, 4.1, 36, 8.5, 8, 9.2, 26.2, 21.9, 16.7, 21.3, 35.4, 14.3, 8.5, 10.6, 19.1, 20.5, 7.1, 7.7, 18.1, 16.5, 11.9, 7, 8.6, 12.5, 10.3, 11.2, 6.1, 8.4, 11, 11.6, 11.9, 5.2, 6.8, 8.9, 7.1, 10.8.

The results presented in Table 3 reveal that the ExCTLW distribution outperforms the other distributions. This is due to its minimum goodness-of-fit (GOF) test measurements and information criteria (IC) values. The empirical pdf and the fitted pdfs of the candidate distributions are plotted in Figure 4. It is clear that the fitted pdf of the ExCTLW and the empirical pdf of the dataset are very similar.

Table 3: Parameter estimates, IC, and GOF statistics for dataset one

Model		α	θ	λ	β	AIC	AICC	BIC	HQAIC
ExCTLW	191.77	0.7570	2.4050	0.0418	1.1780	391.5415	392.8319	397.8756	393.7523
CTLW	198.17	0.3900		0.0007	2.2483	402.3377	403.0877	407.0883	403.9953
WD	197.29			0.0066	1.8406	398.5811	398.9447	401.7481	399.6865

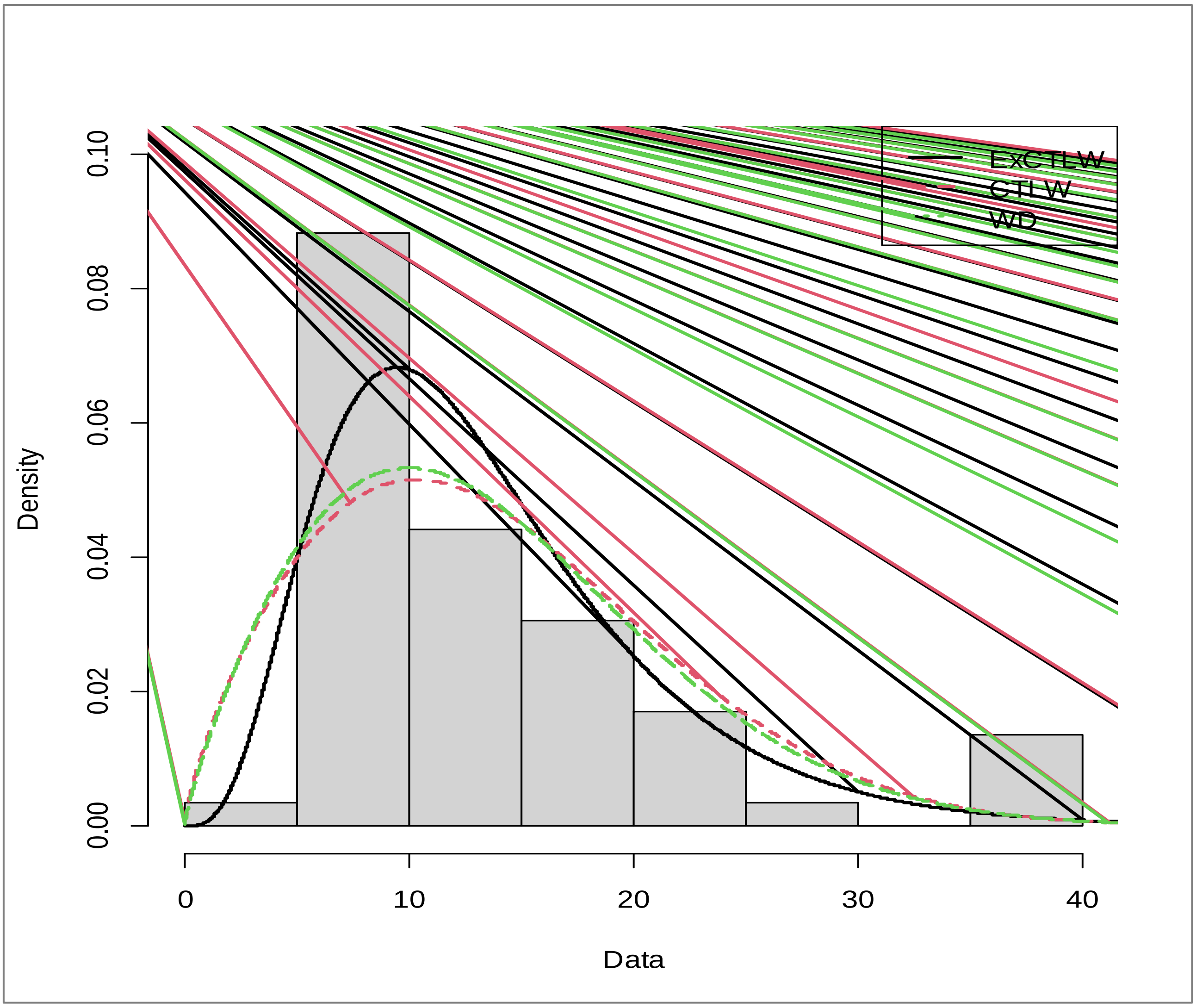

Figure 4 Estimated densities over histogram for dataset one.

Application 2

The second dataset consists of the red cell counts (RCC) of 202 Australian athletes. The red cell count data can be found in the "sn" package in the R software. This dataset is easily accessible and is provided below:

3.80, 3.90, 3.90, 3.91, 3.95, 3.95, 3.96, 3.96, 4.00, 4.02, 4.03, 4.06, 4.07, 4.08, 4.09, 4.09, 4.10, 4.11, 4.11, 4.12, 4.13, 4.13, 4.14, 4.15, 4.16, 4.16, 4.17, 4.17, 4.19, 4.20, 4.20, 4.21, 4.23, 4.23, 4.24, 4.24, 4.25, 4.26, 4.26, 4.27, 4.27, 4.30, 4.31, 4.31, 4.32, 4.32, 4.32, 4.35, 4.36, 4.36, 4.37, 4.38, 4.38, 4.39, 4.40, 4.40, 4.40, 4.41, 4.41, 4.41, 4.42, 4.42, 4.44, 4.44, 4.44, 4.45, 4.45, 4.46, 4.46, 4.46, 4.46, 4.46, 4.48, 4.49, 4.50, 4.50, 4.51, 4.51, 4.51, 4.51, 4.52, 4.53, 4.54, 4.55, 4.56, 4.57, 4.58, 4.62, 4.63, 4.63, 4.63, 4.64, 4.66, 4.68, 4.71, 4.71, 4.71, 4.71, 4.73, 4.75, 4.75, 4.76, 4.77, 4.77, 4.78, 4.81, 4.81, 4.82, 4.82, 4.83, 4.83, 4.83, 4.83, 4.84, 4.86, 4.86, 4.87, 4.87, 4.87, 4.87, 4.87, 4.87, 4.88, 4.89, 4.89, 4.90, 4.90, 4.91, 4.91, 4.92, 4.93, 4.93, 4.94, 4.95, 4.95, 4.96, 4.97, 4.97, 4.98, 4.99, 5.00, 5.00, 5.00, 5.01, 5.01, 5.01, 5.02, 5.02, 5.03, 5.03, 5.03, 5.03, 5.04, 5.04, 5.08, 5.09, 5.09, 5.09, 5.10, 5.11, 5.11, 5.11, 5.11, 5.11, 5.13, 5.13, 5.13, 5.13, 5.16, 5.16, 5.16, 5.16, 5.17, 5.17, 5.18, 5.21, 5.21, 5.22, 5.22, 5.24, 5.24, 5.25, 5.29, 5.31, 5.32, 5.33, 5.33, 5.34, 5.34, 5.34, 5.34, 5.38, 5.40, 5.48, 5.48, 5.49, 5.50, 5.59, 5.66, 5.69, 5.93, 6.72.

Table 4: Parameter estimates, IC, and GOF statistics for dataset two

Model		α	θ	λ	β	AIC	AICC	BIC	HQAIC
ExCTLW	126.22	5.2593	5.2629	0.0291	2.7710	260.4594	261.7497	266.7935	262.6702
CTLW	131.58	1.4721		0.00002	6.6759	269.1559	269.9059	273.9064	270.8140
WD	148.66			0.0000002	9.7931	301.3332	301.6968	304.5002	302.4386

The ExCTLW distribution was compared to the WD and CTLW distributions. The results of the goodness-of-fit test, information criterion values, and maximum likelihood estimates for dataset two are shown in Table 4. Since the ExCTLW distribution produces the minimum values for all IC and GOF metrics, it outperforms the competing models in fitting the RCC dataset. This is corroborated by the plot of the fitted pdfs of the candidate distributions and the empirical pdfs, as shown in Figure 5. It is evident that the fitted pdf of the ExCTLW distribution and the empirical pdf of the RCC sample are quite similar.

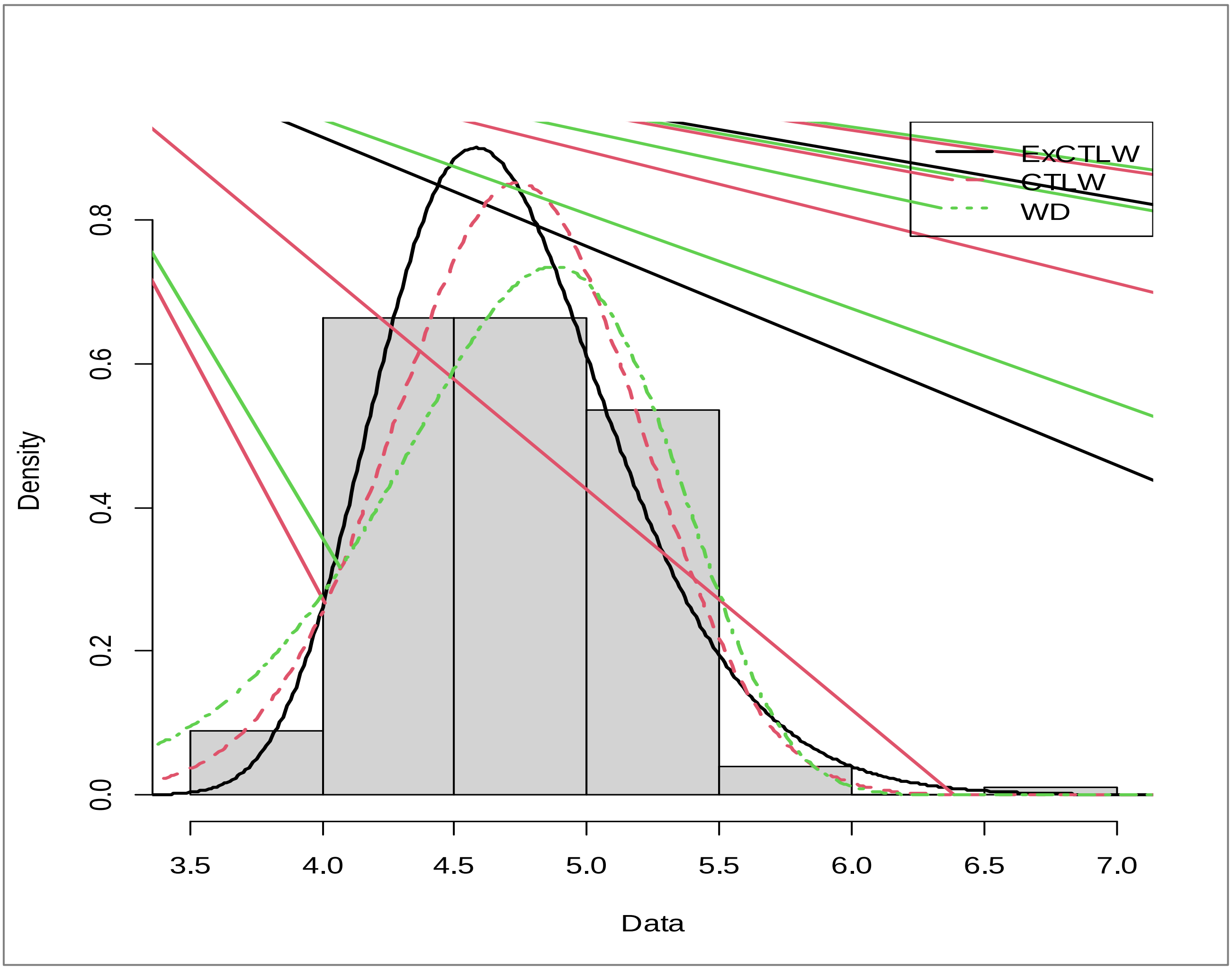

Figure 5 Estimated densities over histogram for dataset two.

CONCLUSION

In this work, we proposed the ExCTL family, which combines the trigonometric transformation and the exponentiation technique. We extract its mathematical properties, including entropies, moments, quantile, and moment-generating functions. We investigate the shapes of the density and hazard functions using three different sub-distributions. The MLE approach is applied to estimate the model parameters. Using Monte Carlo simulations, we validate the performance of the MPS and the MLE estimators. The results show that the MLE is more consistent. Using two real datasets, we illustrate an application of a sub-model generated from the family. Our results reveal that sub-distributions derived from the ExCTL-G family better fit these datasets than the other two distributions.

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