UMYU Scientifica, Vol. 3 NO. 2, June 2024, Pp 133 – 138
https://doi.org/10.56919/usci.2432.014
ISSN: 2955 – 1145 (print); 2955 – 1153 (online)
A periodical of the Faculty of Natural and Applied Sciences, UMYU, Katsina

ORIGINAL RESEARCH ARTICLE

Estimation of Extension of Topp-Leone Distribution using Two Different Methods: Maximum Product Spacing and Maximum Likelihood Estimate

Lawrence Habu[1], Abubakar Usman[1], Ibrahim Abubakar Sadiq[1]* and Usman Aliyu Abdullahi[2]
[1]Department of Statistics, Faculty of Physical Sciences, Ahmadu Bello University, Zaria, Kaduna, Nigeria.
[2]Department of Statistics Aliko Dangote University of Science and Technology, Wudil, Kano, Nigeria.

ARTICLE HISTORY
Received March 04, 2024.
Accepted June 09, 2024.
Published June 23, 2024.

KEYWORDS
Estimation, Topp-Leone, Maximum Likelihood, Maximum Product Spacing, Distribution

ABSTRACT
The parameters for Log-Topp-leone (L-T-L) distribution, Topp-leone-exponential (T-L-E) distribution, and Top-plane-Weibull (T-L-W) distribution were estimated via the methods of ML (maximum likelihood) and MPS (maximum product spacing). Simulations were carried out using R-Studio to estimate the parameters and check the efficiency of the estimators. As a consequence, if the sample size (SS) increases, the outcome of the estimations tends to the true PV (parameter values), which proves the consistency and adequacy (C & A) of all the estimators. Similarly, the MSE and biases using the two estimators approach zero, even though some MPS bias values fluctuated. This shows the suitability of each estimator, and MLE is more unbiased and adequate. The estimates using MLE and MPS were efficient. Thus, we recommend that MLE be employed in assessing and computing the parameters of extensions of the Topp-Leone (TL) distribution.

INTRODUCTION

The Topp-Leone (T-L) distribution was first developed by the well-known statisticians Topp and Leone (1955), equally to a probability distribution (PD) with limited realization, which is useful for describing and modelling lifetime occurrences (Pourdarvish *et al.,* 2015). The Topp-Leone (T-L) distribution is a simple hemmed-in J-shaped bathtub model that has enticed innumerable statisticians as a substitute for the beta model (Ali *et al.,* 2016). Estimating probability distributions is crucial in various fields of study, ranging from finance to economics and engineering. One such distribution that has gained considerable attention in recent years is the T-L distribution. The T-L model is a flexible probability dispersal that can model various data types with heavy tails and asymmetry. It is particularly useful for modelling insurance claim amounts, income distributions, and other phenomena exhibiting skewness and kurtosis (Pourdarvish *et al.,* 2015). El-Saeed *et al.* (2023) state that the determination of generating novel compound models from acknowledged continuous probability families of distributions remains towards extending the standard distributions and improving them by adding one or more contour parameters. These composite distributions have conversely been figured out to be improved than the parental distribution in connections of flexibility and describing and modelling competency. Some families of continuous probability models can be reproduced for this determination but in this research, the concern stands on estimating the parameters of the extensions of T-L distribution with a particular interest in Log-Top-Leone distribution developed by Okasha *et.al.* (2017), Top-Leone-Exponential Distribution proposed by Pourdarvish *et al.* (2015) and Top-Leone Weibull distribution developed according to Aryal *et al.* (2017). The parameter estimation of the T-L distribution has been the focus of numerous research studies. Precisely estimating these parameters is crucial for making dependable inferences and predictions based on the distribution. In this research work, we aim to evaluate and compare two commonly used estimation procedures for the extension of Topp-Leone distribution: the method of moments product spacing (MPS) and the MLE. The techniques of MPS is a classical estimation technique that equates sample statistics (such as mean and variance) to their theoretical counterparts (Sadiq *et al.,* 2023a). It provides initial estimates for the Topp-Leone parameters based on the technique of matching moments. Lastly, the MLE is a widely utilized statistical procedure that maximizes the likelihood function of the experimental data given a specific distribution (Sadiq *et al.,* 2023b). The MLE approach provides asymptotically efficient and consistent estimates (Sadiq *et al.,* 2023c). In this research, we will investigate the performance of each mentioned estimation process for the extensions of the T-L distribution and

Correspondence: Ibrahim Abubakar Sadiq. Department of Statistics, Faculty of Physical Sciences, Ahmadu Bello University, Zaria, Kaduna, Nigeria. ✉ isabubakar@abu.edu.ng. Phone Number: +234 813 752 6770.

How to cite: Habu, L., Usman, A., Abubakar Sadiq, I., & Abdullahi, U. A. (2024). Estimation of Extension of Topp-Leone Distribution using Two Different Methods: Maximum Product Spacing and Maximum Likelihood Estimate. *UMYU Scientifica*, 3(2), 113 – 138. https://doi.org/10.56919/usci.2432.014

compare the accuracy and robustness of the estimates obtained through simulation studies. By examining the performance of various estimation methods, this research seeks to provide valuable insights into estimating extensions of Topp-Leone distribution parameters and to guide researchers and practitioners in selecting an appropriate estimation method.

METHODOLOGY

THE LOG-TOP-LEONE (L-T-L) DISTRIBUTION

The "L-T-L distribution is a probability distribution presented by El-Saeed *et al.* (2023). It has been introduced as an extension of the Topp-Leone distribution. It shares many properties with the original Topp-Leone distribution, such as its usefulness for modelling lifetime phenomena and its simple, bounded J-shaped density function. The Log-Topp–Leone distribution has been studied in various contexts, including its application in reliability studies and its comparison with other distributions, such as the exponential T-L distribution". The CDF (cumulative distribution function) and PDF (probability density function) of the L-T-L distribution are given correspondingly as;

$$F(x/\alpha) = (1 - exp\{-2x\})^{\alpha}, x > 0, \alpha > 0 \quad (1)$$

$$f(x/\alpha) = 2\alpha\, exp\{-2x\}\,(1 - exp\{-2x\})^{\alpha}, x > 0, \alpha > 0 \quad (2)$$

The Quantile Function (QF) of the L-T-L Distribution

The uth quantile function (QF) of the L-T-L distribution using equation (1) can be consequentially derived as;

$$(1 - exp\{-2x\})^{\alpha} = u$$

$$1 - exp\{-2x\} = u^{1/\alpha}$$

$$exp\{-2x\} = 1 - u^{1/\alpha}$$

Taking the $\log_e$ we have,

$$-2x = log(1 - u^{1/\alpha})$$

$$x = \frac{1}{2} log(1 - u^{1/\alpha}), 0 < u > 1 \quad (3)$$

The L-T-L Distribution Parameter Estimation Using MLE

From equation (2), the $\log_e$ of the likelihood-function for Log-Topp-Leone distribution is:

$$l = n\,log(2) + n\,log(\alpha) - 2\sum_{i=1}^{n} x_i + (\alpha - 1)\sum_{i=1}^{n} log(1 - exp\{-2x\}) \quad (4)$$

Differentiating the log-likelihood function given in equation (4) partially, w.r.t the parameter (α), we obtain;

$$\frac{\delta l}{\delta \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} log(1 - exp\{-2x\}) \quad (5)$$

The L-T-L Distribution Parameter Estimation Using MPS

From equation (1), the MPS function for Log-Topp-Leone distribution is;

$$\omega(x; \alpha, \lambda) = \frac{1}{n+1}\sum_{i=1}^{n} log\,\psi_i \quad (6)$$

where;

$$\psi_i = F(x_i) - F(x_{i-1})$$

$$F(x_i) = (1 - exp\{-2x\})^{\alpha}$$

$$F(x_{i-1}) = (1 - exp\{-2x_{i-1}\})^{\alpha}$$

$$\psi_i = (1 - exp\{-2x\})^{\alpha} - (1 - exp\{-2x_{i-1}\})^{\alpha}$$

Hence:

$$\omega(x; \alpha, \lambda) = \frac{1}{n+1}\sum_{i=1}^{n} log((1 - exp\{-2x\})^{\alpha} - (1 - exp\{-2x_{i-1}\})^{\alpha}) \quad (7)$$

TOPP-LEONE EXPONENTIAL (T-L-E) DISTRIBUTION

Pourdarvish *et al.* (2015) developed a generality of the T-L distribution declared to as the T-L-E distribution. They derived and studied various characteristics of the novel distribution, like the hazard rate (hr) function, the statistical moments, the order statistics, etc. The study considered the MLE of the distribution parameters. The wide range of this model is called the T-L-E distribution. The CDF is carved as follows;

$$F_{TL-E}(x) = (1 - exp\{-2\lambda x\})^{\alpha} \quad (8)$$

The density function (PDF) of the T-L-E model is;

$$f_{TL-E}(x) = 2\alpha\lambda\, exp\{-2\lambda x\}\,(1 - exp\{-2\lambda x\})^{\alpha-1}; x, \alpha, \lambda > 0 \quad (9)$$

The Quantile Function (QF) of the T-L-E Distribution

The quantile function (qf) of an RV (random variable) X is accomplished by unravelling the inverse CDF of $F(x)^{-1} = u$. Thus, given CDF defined in equation (8), the uth quantile function of the T-L-E distribution can be obtained as";

$$F_{TL-E}(x) = (1 - exp\{-2\lambda x\})^{\alpha}$$

$$(1 - exp\{-2\lambda x\})^{\alpha} = u$$

$$1 - exp\{-2\lambda x\} = u^{1/\alpha}$$

$$exp\{-2\lambda x\} = 1 - u^{1/\alpha}$$

Multiply the Log to both sides.

$$-2\lambda x = log(1 - u^{1/\alpha})$$

$$x = -\frac{1}{2\lambda x} log(1 - u^{1/\alpha}) \quad (10)$$

The T-L-E Distribution Parameter Estimation Using MLE

From equation (9), the log_e of the likelihood function (LF) for the T-L-E model is:

$$l = nIn(2) + nIn(\alpha) + nIn(\lambda) - 2\lambda \sum_{i=1}^{n} X_i + (\alpha - 1) \sum_{i=1}^{n} In(1 - exp\{-2\lambda x_i\}) \quad (11)$$

Differentiating the log-likelihood function given in equation (11) partially w.r.t each parameter, we get;

$$\frac{\delta l}{\delta \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} In(1 - exp\{-2\lambda x_i\}) \quad (12)$$

$$\frac{\delta l}{\delta \lambda} = \frac{n}{\lambda} + 2\sum_{i=1}^{n} X_i + (\alpha - 1) \sum_{i=1}^{n} \frac{2x_i\, exp\{-2\lambda x_i\}}{1 - exp\{-2\lambda x_i\}} \quad (13)$$

Equations (12) and (13) should be solved arithmetically to obtain MLEs of α and λ.

Parameter Estimation of T-L-E Distribution using MPS

From equation (8), the MPS mathematical function for T-L-E distribution is;

$$\omega(x; \alpha, \lambda) = \frac{1}{n+1} \sum_{i=1}^{n} log\, \psi_i \quad (14)$$

where;

$$\psi_i = F(x_i) - F(x_{i-1})$$

$$F(x_i) = (1 - exp\{-2\lambda x_i\})^{\alpha}$$

$$F(x_{i-1}) = (1 - exp\{-2\lambda x_{i-1}\})^{\alpha}$$

$$\psi_i = (1 - exp\{-2\lambda x_i\})^{\alpha} - (1 - exp\{-2\lambda x_{i-1}\})^{\alpha}$$

Hence:

$$\omega(x; \alpha, \lambda) = \frac{1}{n+1} \sum_{i=1}^{n} log((1 - exp\{-2\lambda x_i\})^{\alpha} - (1 - exp\{-2\lambda x_{i-1}\})^{\alpha}) \quad (15)$$

THE TOPP-LEONE WEIBULL (T-L-W) DISTRIBUTION

Aryal *et al.* (2017) introduced a novel 4-parameter (4-P) continuous probability model named the T-L-W distribution. This continuous distribution is an overview of the 2-parameter (2-P) Weibull distribution (WD) using the foundation of the T-L distribution. They derived various of its essential properties containing the ordinary moments (OM), the incomplete moments (IM), the quantile function (QF), moment generating functions (MGF), order statistics (OS) and so on. We awaited the MLE parameter estimation and simulation results to be a reliable measure of the model's efficiency will be justified and discussed in this study. The CDF and PDF of the T-L-W distribution are correspondingly clear;

$$F(x) = (1 - exp\{-2\theta x^{\alpha}\})^{\lambda}; \quad x, \lambda, \alpha, \theta > 0 \quad (16)$$

and

$$f(x) = 2\lambda\alpha\theta x^{\alpha-1} exp\{-2\theta x^{\alpha}\}(1 - exp\{-2\theta x^{\alpha}\})^{\lambda-1}; \quad x, \lambda, \alpha, \theta > 0 \quad (17)$$

Quantile Function (QF) of the T-L-Weibull Distribution

The quantile function (qf) of an RV (random variable) X may achieved by manipulating the iverse CDF of $F(x)^{-1} = u.$ Thus, setting the CDF presented in equation (16), the u^{th} quantile function of the T-L-W model can be obtained as;

$$(1 - exp\{-2\theta x^{\alpha}\})^{\lambda} = u$$
$$1 - exp\{-2\theta x^{\alpha}\} = u^{1/\lambda}$$
$$exp\{-2\theta x^{\alpha}\} = 1 - u^{1/\lambda}$$

Taking the log_e we have,

$$exp\{-2\theta x^{\alpha}\} = 1 - u^{1/\lambda}$$
$$x^{\alpha} = -\frac{1}{2\theta} log(1 - u^{1/\lambda})$$

$$x = (-\frac{1}{2\theta} log(1 - u^{1/\lambda}))^{1/\alpha} \quad (18)$$

Parameter Estimation of the T-L-W Distribution Using MLE

From equation (17), supposed that $(x_1, x_2, \dots, x_n)$ be an (RS) random sample arises from the T-L-W model, then the Log-likelihood (LL) function for the T-L-W distribution is well-defined by;

$$l = n\, log(2\lambda\alpha\theta) + (\alpha - 1) \sum_{i=1}^{n} log\, x\, i - 2\theta \sum_{i=1}^{n} x_i^{\alpha} log(x_i) + 2\theta(\lambda - 1) \sum_{i=1}^{n} \frac{x_i^{\alpha} log(x_i)\, exp\{-2\theta x^{\alpha}\}}{(1 - exp\{-2\theta x^{\alpha}\})} \quad (19)$$

Differentiating the log-likelihood function given in equation (19) partially w.r.t each parameter, we get;

$$\frac{\delta l}{\delta \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} log\, x_i - 2\theta \sum_{i=1}^{n} {x_i}^{\alpha} log(x_i) + 2\theta(\lambda - 1) \sum_{i=1}^{n} \frac{{x_i}^{\alpha} log(x_i)\, exp\{-2\theta x^{\alpha}\}}{(1-exp\{-2\theta x^{\alpha}\})} \quad (20)$$

$$\frac{\delta l}{\delta \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} log(1 - exp\{-2\theta x^{\alpha}\}) \quad (21)$$

$$\frac{\delta l}{\delta \theta} = \frac{n}{\theta} - 2\sum_{i=1}^{n} {x_i}^{\alpha} + 2(\lambda - 1) \sum_{i=1}^{n} \frac{{x_i}^{\alpha} log(x_i)\, exp\{-2\theta x^{\alpha}\}}{(1-exp\{-2\theta x^{\alpha}\})} \quad (22)$$

The above equations (20), (21), and (22) can be subjected and solved arithmetically to get MLEs of α, ƛ and θ.

Parameter Estimation Topp-Leone Weibull Distribution Using MPS

From equation (16), the MPS function for TLWD Distribution is;

$$F(x) = (1 - exp\{-2\theta x^{\alpha}\})^{\lambda}$$

$$\omega(x; \alpha, \lambda) = \frac{1}{n+1} \sum_{i=1}^{n} log\, \psi_i \quad (23)$$

where;

$$\psi_i = F(x_i) - F(x_{i-1})$$
$$F(x_i) = (1 - exp\{-2\theta x^{\alpha}\})^{\lambda}$$
$$F(x_{i-1}) = (1 - exp\{-2\theta x_{i-1}^{\alpha}\})^{\lambda}$$

$$\psi_i = (1 - exp\{-2\theta x^{\alpha}\})^{\lambda} - (1 - exp\{-2\theta x_{i-1}^{\alpha}\})^{\lambda}$$

Hence:

$$\omega(x; \alpha, \lambda) = \frac{1}{n+1} \sum_{i=1}^{n} log((1 - exp\{-2\theta x^{\alpha}\})^{\lambda} - (1 - exp\{-2\theta x_{i-1}^{\alpha}\})^{\lambda}) \quad (24)$$

RESULT AND DISCUSSIONS

Monte Carlo Simulation

This segment covers an iterative numerical procedure termed a "simulation study" to evaluate the consistency and unbiasedness demonstration of the estimators for the model parameters using two distinct methods. These include maximum product spacing (MPS) and Maximum likelihood estimates (MLE). These simulations were carried out using quantile functions given in equations (3), (10), and (18), respectively, for the three different distributions by generating different random samples of 20, 50, 100, 250, 500, and 1000, setting α = 1, ƛ = 1 and θ = 1 to determine the mean estimate (mean), bias and mean square error (MSE) as shown in the Tables 1, 2, and 3 respectively.

Table 1 is the result of the Log-Topp-leone distribution simulation, which presented the parameter estimates using MPS and MLE methods by setting alpha (α = 1). As observed from the table for an increase in sample sizes, the result using different methods of estimations is approaching the true parameter values. This attests to the consistency and uniformity of all the distributional estimators. Similarly, the RMSE (root mean square error) of both estimators approaches zero, this showed the suitability of each estimator and can be equally employed to estimate the parameter of the L-T-L distributions. The Bias for MLE tends to zero as the sample increases, while that of MPS increases as the sample size increases, this shows that MLE is more efficient and unbiased as compared to MPS.

Table 2 is the result of the T-L-E distribution simulation, which presented the parameter estimates using MLE and MPS methods by setting alpha (α = 1) and lambda (ƛ = 1). Equally observed from Table 2 for any small or large increase in sample sizes, the result of different estimation methods is approaching the actual parameter values. This verifies the consistency with unbiasedness of all the estimators. Similarly, the RMSE using the two estimators approaches zero. This showed the suitability of each estimator and can equally be employed to assess the parameter of the T-L-E model. The Bias for MLE tends to be null as the continuously increases the size of the samples while that of MPS increases as the sample size increases, this shows that MLE is more efficient and unbiased as related to MPS.

Table 3 is the result of the T-L-Weibull distribution simulation analysis, which presented parameter estimates using MPS and MLE methods by setting alpha (α = 1), lambda (ƛ = 1), and theta (θ = 1). As observed from the table for any small or large increase in the sample sizes, the result using different methods of estimations is approaching the actual parameter values, this substantiates the consistency with unbiasedness of all the estimators. Similarly, the MSE and Bias using the two estimators approach zero showed the suitability and unbiasedness of each estimator and can equally be employed to estimate the parameter of the Topp-Leone Weibull distribution.

Table 1: Log-Toppleone Simulation (Alpha = 1)

n	MPS			M.L.E		
	Means	Bias	RMSE	Means	Bias	RMSE
20	0.9703	-0.0297	0.2314	1.046	0.046	0.2505
50	0.9749	-0.0251	0.142	1.0141	0.0141	0.1461
100	0.9875	-0.0125	0.1011	1.0106	0.0106	0.1034
250	0.9938	-0.0062	0.063	1.005	0.005	0.0636
500	0.9951	-0.0049	0.0427	1.0014	0.0014	0.0427
1000	0.9957	-0.0043	0.0318	0.9992	-8e-04	0.0316

Table 2: Topp-Leone Exponential Distribution Simulation (Alpha = 1, Lambda = 1)

n	MPS			M.L.E		
	Means	Bias	RMSE	Means	Bias	RMSE
20	0.9384	-0.0616	0.3416	1.1583	0.1583	0.4043
	0.9285	-0.0715	0.3195	1.1823	0.1823	0.4937
50	0.9466	-0.0534	0.1986	1.0612	0.0612	0.217
	0.9478	-0.0522	0.1973	1.0608	0.0608	0.2296
100	0.9659	-0.0341	0.136	1.0285	0.0285	0.1383
	0.9626	-0.0374	0.1336	1.0319	0.0319	0.1468
250	0.9805	-0.0195	0.0859	1.009	0.009	0.0843
	0.9778	-0.0222	0.0846	1.0118	0.0118	0.088
500	0.9862	-0.0138	0.0579	1.0031	0.0031	0.0585
	0.9855	-0.0145	0.0593	1.0038	0.0038	0.0578
1000	0.991	-0.009	0.042	1.0021	0.0021	0.04
	0.9923	-0.0077	0.0405	1.0007	0.0007	0.0414

Table 3: Topp-Leone Weibull Distribution Simulation (Alpha = 1, Lambda = 1 and Theta = 1)

n	MPS			M.L.E		
	Means	Bias	RMSE	Means	Bias	RMSE
20	1.1430	0.1430	0.7189	1.1528	0.1528	0.8059
	1.0941	0.0941	0.5735	1.3174	0.3174	0.7660
	0.9986	-0.0014	0.3590	1.1385	0.1385	0.4830
50	1.1102	0.1102	0.5807	1.1153	0.1153	0.6189
	1.0328	0.0328	0.3622	1.1268	0.1268	0.4224
	0.9900	-0.0100	0.2648	1.0476	0.0476	0.3035
100	1.0883	0.0883	0.4248	1.0767	0.0767	0.4252
	1.0012	0.0012	0.2459	1.0549	0.0549	0.2696
	0.9973	-0.0027	0.2009	1.0265	0.0265	0.2125
250	1.0444	0.0444	0.2889	1.0767	0.0302	0.2806
	0.9992	-0.0008	0.1624	1.0549	0.0278	0.1694
	0.9958	-0.0042	0.1425	1.0265	0.0054	0.1460
500	1.0283	0.0283	0.1945	1.0190	0.0190	0.1975
	0.9929	-0.0071	0.1123	1.0109	0.0109	0.1187
	0.9993	-0.0007	0.1016	1.0040	0.0040	0.1050
1000	1.0167	0.0167	0.1366	1.0102	0.0102	0.1371
	0.9938	-0.0062	0.0794	1.0041	0.0041	0.0816
	1.0015	0.0015	0.0708	1.0036	0.0036	0.0714

CONCLUSION

Generally, there have been so many developed extensions of the T-L distributions. Hence there is an additional nec essity to evaluate the parameters of the novel developed models using different estimators. The model parameters for the L-T-L distribution, T-L-E distribution, and T-L-W distribution were estimated using the practice of MLE an d that of MPS. As the size of samples

continually increases, the outcome and justification of the estimations approach the true assumed PV (parameter val ues). This ascertains the consistency, adequacy, and applic ability of all the estimators. Similarly, the RMSE and Bias using the two estimators approach zero even though

some bias values for the MPS were fluctuating; this show s the suitability of each estimator, and MLE is more unbi ased. The estimates using MLE and MPS were efficient.

RECOMMENDATIONS

i. The estimates using MLE and MPS were efficient. Thus, I recommend that one be a substitute for the other.
ii. MLE is considered a more unbiased estimator for extensions of the T-L distributions.
iii. Any other parameter estimation apart from MPS and MLE methods should be considered for further study.

REFERENCES

Ali Al-Shomrani, Osama Arif, A Shawky, Saman Hanif, Muhammad Qaiser Shahbaz (2016). Topp–Leone Family of Distributions: Some Properties and Application. Pak.j.stat.oper.res. Vol.XII No.3 2016 pp443-451. **[Crossref]**

Aryal, G. R., Ortega, E. M., Hamedani, G. G., & Yousof, H. M. (2017). The Topp-Leone generated Weibull distribution: regression model, characterizations and applications. *International Journal of Statistics and Probability,* 6(1), 126-141. **[Crossref]**

El-Saeed, A. R., Hassan, A. S., Elharoun, N. M., Al Mutairi, A., Khashab, R. H., & Nassr, S. G. (2023). A class of power inverted Topp-Leone distribution: Properties, different estimation methods & applications. *Journal of Radiation Research and Applied Sciences*, *16*(4), 100643. **[Crossref]**

Okasha, H. M. (2017). A New Family of Topp and Leone Geometric Distribution with Reliability Applications. *Journal of Failure Analysis and Prevention*, *17*, 477-489. **[Crossref]**

Pourdarvish, A., Mirmostafaee, S. M. T. K., & Naderi, K. (2015). The exponentiated Topp-Leone distribution: Properties and application. *Journal of Applied Environmental and Biological Sciences,* 5(7), 251-256.

Sadiq, I. A., Doguwa, S. I. S., Yahaya, A., & Garba, J. (2023c). New Generalized Odd Fréchet-Odd Exponential-G Family of Distribution with Statistical Properties and Applications. *FUDMA Journal of Sciences*, *7*(6), 41-51. **[Crossref]**

Sadiq, I. A., Doguwa, S. I. S., Yahaya, A., & Usman, A. (2023b). Development of New Generalized Odd Fréchet-Exponentiated-G Family of Distribution. *UMYU Scientifica*, *2*(4), 169-178. **[Crossref]**

Sadiq, I. A., Doguwa, S. I., Yahaya, A., & Garba, J. (2023a). New Generalized Odd Frechet-G (NGOF-G) Family of Distribution with Statistical Properties and Applications. *UMYU Scientifica*, *2*(3), 100-107. **[Crossref]**

Topp, C. W., & Leone, F. C. (1955). A family of J-shaped frequency functions. *Journal of the American Statistical Association*, *50*(269), 209-219. **[Crossref]**