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ORIGINAL RESEARCH ARTICLE

Truncated Exponential Log-Topp-Leone Rayleigh Distributions: Properties with Application to Bladder Cancer Data

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ABSTRACT
In this article, we introduce a new truncated exponential log-topp-leone Rayleigh distribution on the basis of the truncated exponential log-topp-leone family of distributions. We discussed some properties, including survival function, hazard function, entropy, moment, moment generating function, quantile, and order statistics. We also estimate the parameters of the distribution using maximum likelihood, least squares, and Cramer von-mises. We demonstrated how suitable the proposed distribution is for modeling right-skewed data, as shown from the pdf plot in Figure 1. Finally, we apply the right-skewed (see Figure 3) bladder cancer data sets and compare the performance of the new model using information criteria (see Table 3), and we conclude that the new model outperforms the other standard models with smaller values of AIC, CAIC, and BIC.

INTRODUCTION

Currently, many statistical distributions are flexible and reliable in modeling and fitting real-life data sets in many fields, including engineering, biological sciences, economics, and environmental sciences, among others. As a result, there is a further need to develop more probability distributions that can handle these challenges (Yahaya and Abba, 2017). The existing truncated exponential top-leone family, introduced by (Al-Noor and Hilal, 2021), is used for modeling data with a bounded interval, which limits its ability to model different forms of data. Since the topp-leone generalized distribution provides a more flexible generalized family distribution, there is more interest in making some transformations on it and combining it with the truncated exponential as a baseline distribution.

The unimodel distribution was introduced by (Topp and Leone, 1955) as the Topp-Leone distribution, but the distribution was not popularly known until the work of (Nadarajah and Kozt, 2003), where they provided the density function of the model. The distribution gained the attention of many researchers, including (Al-shomrani et al., 2016), who provided a generalization of top-leone, and (Usman et al., 2023), where they extended the interval of the topp-leone distribution to be unbounded, leading to the introduction of a log top-leone distribution.

Continuously, several studies were introduced by incorporating top-leone with other existing distributions, including Topp Leone extended exponential distribution by (Aidi et al., 2022), Topp Leone odd log-logistic exponential distribution by (Afify et al., 2021), Topp Leone modified Weibull distribution by (Alyami et al., 2022), Topp Leone-inverted Kumaraswamy distribution by (Behairy et al., 2020), Kumaraswamy inverted Topp-Leone distribution by (Hassan et al., 2021).

Truncated exponential (TE) distribution is exclusively specified over a specific area and is exponential in nature. (Akahira, 2017) considers the estimation of a truncation parameter with a natural parameter as a nuisance one for a one sided TE. The pdf of TE distribution is provided as

$$f(x) = \frac{\delta e^{-\delta x}}{1-e^{-\delta}} \quad 0 < x < a \tag{1}$$

Where a is the point of truncation

Rayleigh distribution (Rayleigh, 1980), as a special case of weibull distribution when 2 is the shape parameter, was used by many researchers due to its uniformity and capability of modeling continuous data with cdf and pdf given as,

$$F(y) = 1 - e^{\frac{-y^2}{2\xi^2}} \tag{2}$$

And

$$f(y) = \frac{y}{\xi^2} e^{\frac{-y^2}{2\xi^2}} \quad y>0 \tag{3}$$

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The generalized family of distribution called the truncated exponential log topp-leone generalized family of distributions (TELTL-G) introduced by (Abubakar et al., 2024), which was derived using the cdf of topp-leone generalized family (Al-shomrani et al., 2016), log top-leone family by (Usman et at., 2023) and the pdf of truncated exponential distribution by (Akaira, 2017) using the integral link function as follows;

$$F_{TELTL-G}(z,\delta,\theta,\psi)=\int_0^{\left(1-e^{-2G(z,\psi)}\right)^{\theta}}\frac{\delta e^{-\delta z}}{1-e^{-\delta}}dz \qquad (4)$$

Where $F(z)=\left(1-e^{-2G(z,\psi)}\right)^{\theta}$ is the cumulative distribution function (cdf) of log top-leone generalized family derived from $F(z)=(1-\bar{G}(z)^2)^{\theta}$, which is the cdf of topp-leone Generalized family, and $f(z)=\frac{\delta e^{-\delta z}}{1-e^{-\delta}}$ is the probability density function (pdf) of truncated exponential distribution. Therefore, the pdf and cdf of TELTL-G are respectively.

$$F_{TELTL-G}(z,\delta,\theta,\psi)=\frac{1-e^{-\delta(1-e^{-2G(z,\psi)})^{\theta}}}{1-e^{-\delta}} \qquad (5)$$

$$f_{TELTL-G}(z,\delta,\theta,\psi)=\frac{2\delta\theta g(z,\psi)e^{-2G(z,\psi)}(1-e^{-2G(z,\psi)})^{\theta-1}e^{-\delta(1-e^{-2G(z,\psi)})^{\theta}}}{1-e^{-\delta}} \quad z,\theta,\delta>0 \qquad (6)$$

Where $g(y)$ and $G(y)$ are the probability density function and cumulative density function of the baseline distribution, δ is a shape parameter, θ is a second shape parameter, and ψ is the parameter vector of the baseline distribution.

This paper aims to introduce a new family member of TELTL-G called the truncated exponential log top-leone rayleigh distribution, and the objectives are to:

- Derive the statistical properties of the proposed distribution.
- Estimate the parameter of the distribution using maximum likelihood, least squares, and cramer von-mises.
- Examine the performance of the distribution using bladder cancer patients data.

METHODOLOGY

In this section, we focus on the exploration of the new distribution, understanding its properties, and estimating the parameters, which enable us to accurately describe the characteristics of the proposed distribution.

Truncated exponential log top-leone rayleigh distribution.

The pdf and the cdf of truncated exponential log top-leone rayleigh distribution would be introduced with three parameters by substituting equations (2) and (3) into equations (5) and (6). We have the cdf and pdf of truncated exponential log topp-leone rayleigh distribution as;

$$F(y,\delta,\theta,\xi)=\frac{1-e^{-\delta(1-e^{-2(1-e^{\frac{-y^2}{2\xi^2}})})^{\theta}}}{1-e^{-\delta}} \quad y,\theta,\delta,\xi>0 \qquad (7)$$

And the corresponding probability function is;

$$f(y,\delta,\theta,\xi)=\frac{2\delta\theta y e^{\frac{-y^2}{2\xi^2}}e^{-2(1-e^{\frac{-y^2}{2\xi^2}})}(1-e^{-2(1-e^{\frac{-y^2}{2\xi^2}})})^{\theta-1}e^{-\delta(1-e^{-2(1-e^{\frac{-y^2}{2\xi^2}})})^{\theta}}}{\xi^2(1-e^{-\delta})}$$

$y,\theta,\delta,\xi>0$ (8)

The pdf plot of the TELTL-R distribution is illustrated in Figure 1, which shows that the distribution has a positive skewed distribution with a monotonic increasing cdf in Figure 2, showing an increasing upward.

Properties of TELTL-R Distribution

Some summarized properties of TELTL-R Distribution, which comprises the survival function, Hazard function, and quantile are given in the following equations

$$S(y)=\frac{e^{-\delta(1-e^{-2(1-e^{\frac{-y^2}{2\xi^2}})})^{\theta}}-e^{-\delta}}{1-e^{-\delta}} \qquad (9)$$

$$H(y)=\frac{2\delta\theta y e^{\frac{-y^2}{2\xi^2}}e^{-2(1-e^{\frac{-y^2}{2\xi^2}})}(1-e^{-2(1-e^{\frac{-y^2}{2\xi^2}})})^{\theta-1}e^{-\delta(1-e^{-2(1-e^{\frac{-y^2}{2\xi^2}})})^{\theta}}}{e^{-\delta(1-e^{-2(1-e^{\frac{-y^2}{2\xi^2}})})^{\theta}}-e^{-\delta}} \qquad (10)$$

$$y_{\lambda}=\sqrt{-2\xi^2\ln\left\{1+\frac{1}{2}\ln\left\{1-\left\{-\frac{\ln(1-\lambda(1-e^{-\delta}))}{\delta}\right\}^{\frac{1}{\theta}}\right\}\right\}} \qquad (11)$$

Likewise, the Renyi's entropy, moment, moment generating function, and Order Statistics of TELTL-R Distribution were drived as;

$$RE=I_x(y)=\frac{1}{1-z}log\int_{-\infty}^{\infty}f(y)^z\,dy \qquad (12)$$

For the TELTL-R Distribution, the entropy is given by:

$$f(y)^z=(\omega\alpha)^z \qquad (13)$$

Where

$$\omega=ye^{\frac{-y^2}{2\xi^2}}e^{-2(1-e^{\frac{-y^2}{2\xi^2}})}(1-e^{-2(1-e^{\frac{-y^2}{2\xi^2}})})^{\theta-1}e^{-\delta(1-e^{-2(1-e^{\frac{-y^2}{2\xi^2}})})^{\theta}}$$

and $\alpha=\frac{2\delta\theta}{\xi^2(1-e^{-\delta})}$

$$I_x(y)=\frac{1}{1-z}\left[log\alpha^z+log\int_0^{\infty}\delta^z dy\right] \qquad (14)$$

Implies that

$$I_x(y) = \frac{1}{1-z}\left[z\log\alpha + \log\int_0^\infty \delta^z dy\right] \quad (15)$$

The rth moment of a continuous distribution is given by;

$$E(y^r) = \mu^r = \int_{-\infty}^{\infty} y^r f(y,\delta,\theta,\xi)dy \quad (16)$$

$$\mu^r = \alpha\int_0^\infty y^r \omega dy \quad (17)$$

Where ω and α are given above

The moment generating function $M_y(t)$ is;

$$M_y(t) = E(e^{ty}) = \int_{-\infty}^{\infty} e^{ty} f(y,\delta,\theta,\xi)dy \quad (18)$$

$$M_y(t) = \int_0^\infty e^{ty}\delta\alpha dy \quad (19)$$

But $\quad e^{ty} = \sum_{i=0}^{\infty}\frac{t^i y^i}{i!}$

Implies that

$$M_y(t) = \sum_{i=0}^{\infty} gk \quad (20)$$

Where

$$g = \int_0^\infty y^{i+1} e^{\frac{-y^2}{2\xi^2}} e^{-2\left(1-e^{\frac{-y^2}{2\xi^2}}\right)}\left(1-e^{-2\left(1-e^{\frac{-y^2}{2\xi^2}}\right)}\right)^{\theta-1} e^{-\delta\left(1-e^{-2\left(1-e^{\frac{-y^2}{2\xi^2}}\right)}\right)^{\theta}} dy$$

and $\quad k = \frac{2\delta\theta t^i}{\xi^2(1-e^{-\delta})i!}$

For order statistics, Let $y_1, y_2, y_3, y_4, \ldots\ldots, y_n$ be a random sample from the TELTL-R distribution and $y_{(1)}, y_{(2)}, y_{(3)}, y_{(4)}, \ldots\ldots, y_{(n)}$ be the corresponding order statistics. The nth order statistic's pdf written as;

$$f_{(i,n)}(y) = \frac{n!}{(i-1)(n-i)!} f(y)[F(y)]^{i-1}[1-F(y)]^{n-i} \quad (21)$$

Now using power series expansion;

$$[1-F(y)]^{n-i} = \sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}[F(y)]^j \quad (22)$$

Implies that equation (21) becomes;

$$f_{(i,n)}(y) = \frac{n!(-1)^j}{(i-1)(n-i-j)!j!}\sum_{j=0}^{n-i} f(y)[F(y)]^{i+j-1} \quad (23)$$

$$f_{(i,n)}(y) = \sum_{j=0}^{n-i}\frac{2\delta\theta n!(-1)^j y e^{\frac{-y^2}{2\xi^2}} e^{-2\left(1-e^{\frac{-y^2}{2\xi^2}}\right)}\left(1-e^{-2\left(1-e^{\frac{-y^2}{2\xi^2}}\right)}\right)^{\theta-1}}{(i-1)(n-i-j)!j!\xi^2(1-e^{-\delta})} \text{x}$$

$$e^{-\delta\left(1-e^{-2\left(1-e^{\frac{-y^2}{2\xi^2}}\right)}\right)^{\theta}}\left(\frac{1-e^{-\delta\left(1-e^{-2\left(1-e^{\frac{-y^2}{2\xi^2}}\right)}\right)^{\theta}}}{1-e^{-\delta}}\right)^{j+i-1} \quad (24)$$

Likewise, When i=1,

$$f_{(1,n)}(y) = \sum_{j=0}^{n-i}\binom{n-1}{j}\frac{2\delta\theta(-1)^j y e^{\frac{-y^2}{2\xi^2}} e^{-2\left(1-e^{\frac{-y^2}{2\xi^2}}\right)}\left(1-e^{-2\left(1-e^{\frac{-y^2}{2\xi^2}}\right)}\right)^{\theta-1}}{\xi^2(1-e^{-\delta})^{j+1}} \text{x}$$

$$e^{-\delta\left(1-e^{-2\left(1-e^{\frac{-y^2}{2\xi^2}}\right)}\right)^{\theta}}\left(1-e^{-\delta\left(1-e^{-2\left(1-e^{\frac{-y^2}{2\xi^2}}\right)}\right)^{\theta}}\right)^{j} \quad (25)$$

And when i=n,

$$f_{(1,n)}(y) = \frac{2n\delta\theta(-1)^j y e^{\frac{-y^2}{2\xi^2}} e^{-2\left(1-e^{\frac{-y^2}{2\xi^2}}\right)}\left(1-e^{-2\left(1-e^{\frac{-y^2}{2\xi^2}}\right)}\right)^{\theta-1}}{\xi^2(1-e^{-\delta})^n} \text{x}$$

$$e^{-\delta\left(1-e^{-2\left(1-e^{\frac{-y^2}{2\xi^2}}\right)}\right)^{\theta}}\left(1-e^{-\delta\left(1-e^{-2\left(1-e^{\frac{-y^2}{2\xi^2}}\right)}\right)^{\theta}}\right)^{n-1} \quad (26)$$

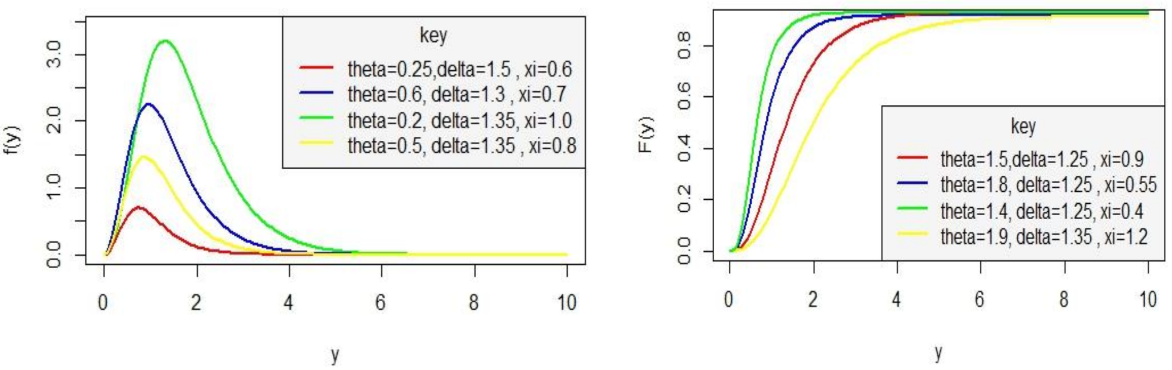

Figure 1: The plot of the pdf and cdf of TELTL-R Distribution for some selected parameters.

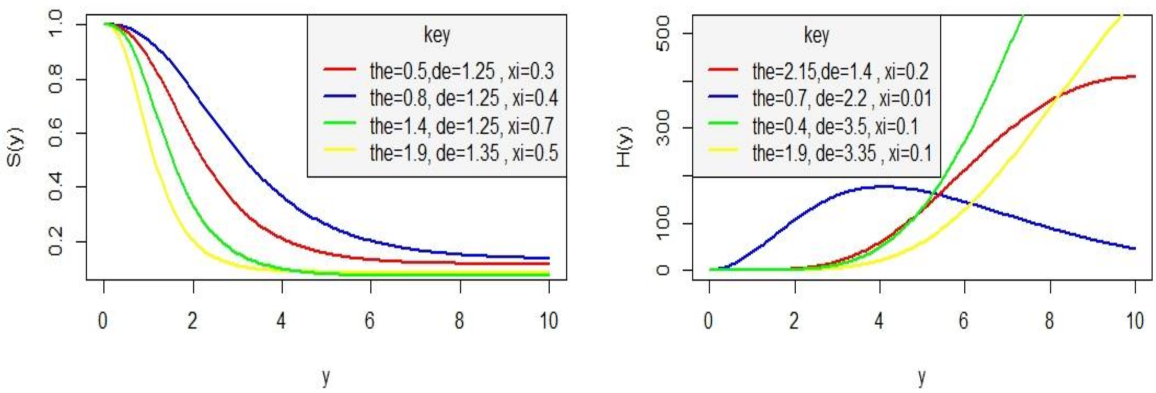

Figure 2: The Survival and Hazard rate function of TELTL-R distribution.

Estimation

Method of Maximum Likelihood Estimation

Maximum likelihood estimation is one of the important methods of finding the estimate of the unknown parameter. To estimate the parameters of the TELTL-R distribution using a complete sample, let $y_1, y_2, y_3, y_4, \ldots\ldots, y_n$ be a random sample from the TELTL-R distribution. The log-likelihood of the parameter vector is written as.

$$l = nlog2 + nlog\delta + nlog\theta + \sum_{i=1}^{n} logy - \frac{1}{2\xi^2}\sum_{i=1}^{n} y^2 - 2\sum_{i=1}^{n}\left(1 - e^{\frac{-y^2}{2\xi^2}}\right) + (n\theta - n)\log\left(1 - e^{-2(1-e^{\frac{-y^2}{2\xi^2}})}\right) - \delta\sum_{i=1}^{n}\left(1 - e^{-2(1-e^{\frac{-y^2}{2\xi^2}})}\right)^{\theta} - 2nlog\xi - nlog(1 - e^{-\delta}) \quad (27)$$

Implies that

$$\frac{dl}{d\delta} = \frac{n}{\delta} - \sum_{i=1}^{n}\left(1 - e^{-2(1-e^{\frac{-y^2}{2\xi^2}})}\right)^{\theta} - \frac{ne^{-\delta}}{(1-e^{-\delta})} \quad (28)$$

$$\frac{dl}{d\theta} = \frac{n}{\theta} + \text{nlog}\left(1 - e^{-2(1-e^{\frac{-y^2}{2\xi^2}})}\right) - \delta\sum_{i=1}^{n}\left(1 - e^{-2(1-e^{\frac{-y^2}{2\xi^2}})}\right)^{\theta} ln\left(1 - e^{-2(1-e^{\frac{-y^2}{2\xi^2}})}\right) \quad (29)$$

$$\frac{dl}{d\xi} = \frac{1}{\xi^3}\sum_{i=1}^{n} y^2 + \frac{2}{\xi^3}\sum_{i=1}^{n} y^2 e^{\frac{-y^2}{2\xi^2}} - \frac{2(n\theta-n)y^2 e^{\frac{-y^2}{2\xi^2}} e^{-2(1-e^{\frac{-y^2}{2\xi^2}})}}{\xi^3\left(1-e^{-2(1-e^{\frac{-y^2}{2\xi^2}})}\right)} + \frac{2\delta\theta\sum_{i=1}^{n} y^2 e^{\frac{-y^2}{2\xi^2}} e^{-2(1-e^{\frac{-y^2}{2\xi^2}})}\left(1-e^{-2(1-e^{\frac{-y^2}{2\xi^2}})}\right)^{\theta-1}}{\xi^3} - \frac{2n}{\xi} \quad (30)$$

Method of Least Square Estimation

Least square is another technique for calculating the parameters of the probability model (Swain et al., 1988). Alternative methods are developed to deal with the situation where obtaining the explicit forms of the maximum likelihood estimators is not always feasible. Let $y_1, y_2, y_3, y_4, \ldots\ldots, y_n$ represent the ordered samples from the TELTL-R distribution that were taken from a sample of size n.

$$R(z) = \sum_{i=0}^{n}\left\{F(y,\delta,\theta,\xi) - \frac{i}{n+1}\right\}^2 \quad (31)$$

$$R(z) = \sum_{i=0}^{n}\left\{\frac{1-e^{-\delta(1-e^{-2(1-e^{\frac{-y^2}{2\xi^2}})})^{\theta}}}{1-e^{-\delta}} - \frac{i}{n+1}\right\}^2 \quad (32)$$

The estimate of $\bar{G}_{LSE} = (\bar{\delta}, \bar{\theta}, \bar{\xi})^T$were obtained by differentiating equation (32).

$$\frac{dR(z)}{d\theta} = -2\sum_{i=1}^{n}\left\{\frac{1-e^{-\delta(1-e^{-2(1-e^{\frac{-y^2}{2\xi^2}})})^{\theta}}}{1-e^{-\delta}} - \frac{i}{n+1}\right\}w_{\theta} \quad (33)$$

$$\frac{dR(z)}{d\delta} = -2\sum_{i=1}^{n}\left\{\frac{1-e^{-\delta(1-e^{-2(1-e^{\frac{-y^2}{2\xi^2}})})^{\theta}}}{1-e^{-\delta}} - \frac{i}{n+1}\right\}w_{\delta} \quad (34)$$

$$\frac{dR(z)}{d\xi} = -2\sum_{i=1}^{n}\left\{\frac{1-e^{-\delta(1-e^{-2(1-e^{\frac{-y^2}{2\xi^2}})})^{\theta}}}{1-e^{-\delta}} - \frac{i}{n+1}\right\}w_{\xi} \quad (35)$$

Where w_{θ}, w_{δ}, w_{ξ} are the derivatives of $F(y,\delta,\theta,\xi)$ and were derived as follows.

$$w_\theta = \frac{1}{1-e^{-\delta}}\frac{d\left(1-e^{-\delta\left(1-e^{-2\left(1-e^{\frac{-y^2}{2\xi^2}}\right)}\right)^\theta}\right)}{d\theta} =$$

$$\frac{\delta e^{-\delta\left(1-e^{-2\left(1-e^{\frac{-y^2}{2\xi^2}}\right)}\right)^\theta}\left(1-e^{-2\left(1-e^{\frac{-y^2}{2\xi^2}}\right)}\right)^\theta \log\left(1-e^{-2\left(1-e^{\frac{-y^2}{2\xi^2}}\right)}\right)}{1-e^{-\delta}} \quad (36)$$

$$w_\delta = \frac{d}{d\delta}\left(\frac{1-e^{-\delta\left(1-e^{-2\left(1-e^{\frac{-y^2}{2\xi^2}}\right)}\right)^\theta}}{1-e^{-\delta}}\right) =$$

$$\frac{(1-e^{-\delta})\left(1-e^{-2\left(1-e^{\frac{-y^2}{2\xi^2}}\right)}\right)^\theta e^{-\delta\left(1-e^{-2\left(1-e^{\frac{-y^2}{2\xi^2}}\right)}\right)^\theta} - \delta e^{-\delta}\left(1-e^{-\delta\left(1-e^{-2\left(1-e^{\frac{-y^2}{2\xi^2}}\right)}\right)^\theta}\right)}{(1-e^{-\delta})^2} \quad (37)$$

$$w_\xi = \frac{1}{1-e^{-\delta}}\frac{d\left(1-e^{-\delta\left(1-e^{-2\left(1-e^{\frac{-y^2}{2\xi^2}}\right)}\right)^\theta}\right)}{d\xi} =$$

$$\frac{2\delta\theta y^2\left(1-e^{-2\left(1-e^{\frac{-y^2}{2\xi^2}}\right)}\right)^{\theta-1} e^{\frac{-y^2}{2\xi^2}} e^{-\delta\left(1-e^{-2\left(1-e^{\frac{-y^2}{2\xi^2}}\right)}\right)^\theta}}{\xi^3} \quad (38)$$

Method of Cramer-Von-Mises Estimation

The Cramer-von-Mises estimates are the least biased when compared to the other goodness-of-fit statistical estimators. (Boos, 1982) reveals the formula CVM(z), for which the estimators provide a minimum about the unknown parameters.

$$CVM(z) = \frac{1}{12n} + \sum_{i=0}^{n}\left\{F(y,\delta,\theta,\xi) - \frac{2i-1}{2n}\right\}^2 \quad (39)$$

$$CVM(z) = \frac{1}{12n} + \sum_{i=0}^{n}\left\{\frac{1-e^{-\delta\left(1-e^{-2\left(1-e^{\frac{-y^2}{2\xi^2}}\right)}\right)^\theta}}{1-e^{-\delta}} - \frac{2i-1}{2n}\right\}^2 \quad (40)$$

By solving equation (40) above, the estimate were obtained as;

$$\frac{dCVM(z)}{d\theta} = -2\sum_{i=1}^{n}\left\{\frac{1-e^{-\delta\left(1-e^{-2\left(1-e^{\frac{-y^2}{2\xi^2}}\right)}\right)^\theta}}{1-e^{-\delta}} - \frac{2i-1}{2n}\right\}\mathrm{x}$$

$$\frac{\delta e^{-\delta\left(1-e^{-2\left(1-e^{\frac{-y^2}{2\xi^2}}\right)}\right)^\theta}\left(1-e^{-2\left(1-e^{\frac{-y^2}{2\xi^2}}\right)}\right)^\theta \log\left(1-e^{-2\left(1-e^{\frac{-y^2}{2\xi^2}}\right)}\right)}{1-e^{-\delta}} \quad (41)$$

$$\frac{dCVM(z)}{d\delta} = -2\sum_{i=1}^{n}\left\{\frac{1-e^{-\delta\left(1-e^{-2\left(1-e^{\frac{-y^2}{2\xi^2}}\right)}\right)^\theta}}{1-e^{-\delta}} - \frac{2i-1}{2n}\right\}\mathrm{x}$$

$$\frac{(1-e^{-\delta})\left(1-e^{-2\left(1-e^{\frac{-y^2}{2\xi^2}}\right)}\right)^\theta e^{-\delta\left(1-e^{-2\left(1-e^{\frac{-y^2}{2\xi^2}}\right)}\right)^\theta} - \delta e^{-\delta}\left(1-e^{-\delta\left(1-e^{-2\left(1-e^{\frac{-y^2}{2\xi^2}}\right)}\right)^\theta}\right)}{(1-e^{-\delta})^2} \quad (42)$$

$$\frac{dCVM(z)}{d\xi} = -2\sum_{i=1}^{n}\left\{\frac{1-e^{-\delta\left(1-e^{-2\left(1-e^{\frac{-y^2}{2\xi^2}}\right)}\right)^\theta}}{1-e^{-\delta}} - \frac{2i-1}{2n}\right\}\mathrm{x}$$

$$\frac{2\delta\theta y^2\left(1-e^{-2\left(1-e^{\frac{-y^2}{2\xi^2}}\right)}\right)^{\theta-1} e^{\frac{-y^2}{2\xi^2}} e^{-\delta\left(1-e^{-2\left(1-e^{\frac{-y^2}{2\xi^2}}\right)}\right)^\theta}}{\xi^3} \quad (43)$$

RESULTS

Application to bladder cancer patients Data sets

The data below is the remission times (in months) of sample of 128 bladder cancer patients, and was used by (Tahir et al., 2016), (Zea et al, 2012) and (Rady et al., 2016). The actual data are:

0.08, 0.20, 0.40, 0.50, 0.51, 0.81, 0.90, 1.05, 1.19, 1.26, 1.35, 1.40, 1.46, 1.76, 2.02, 2.02, 2.07, 2.09, 2.23, 2.26, 2.46, 2.54, 2.62, 2.64, 2.69, 2.69, 2.75, 2.83, 2.87, 3.02, 3.25, 3.31, 3.36, 3.36, 3.48, 3.52, 3.57, 3.64, 3.70, 3.82, 3.88, 4.18, 4.23, 4.26, 4.33, 4.34, 4.40, 4.50, 4.51, 4.87, 4.98, 5.06, 5.09, 5.17, 5.32, 5.32, 5.34, 5.41, 5.41, 5.49, 5.62, 5.71, 5.85, 6.25, 6.54, 6.76, 6.93,6.94, 6.97, 7.09, 7.26, 7.28, 7.32, 7.39, 7.59, 7.62, 7.63, 7.66, 7.87, 7.93, 8.26, 8.37, 8.53, 8.65, 8.66, 9.02, 9.22, 9.47, 9.74, 10.06, 10.34, 10.66, 10.75, 11.25, 11.64, 11.79, 11.98, 12.02, 12.03, 12.07, 12.63, 13.11, 13.29, 13.80, 14.24, 14.76, 14.77, 14.83, 15.96, 16.62, 17.12, 17.14, 17.36, 18.10, 19.13, 20.28, 21.73, 22.69, 23.63, 25.74, 25.82, 26.31, 32.15, 34.26, 36.66, 43.01, 46.12, 79.05.

Table 1: Summary of the Bladder cancer data.

Min	Q(1)	Median	Mean	Q(3)	Max
0.08	3.348	6.395	9.366	11.838	79.05

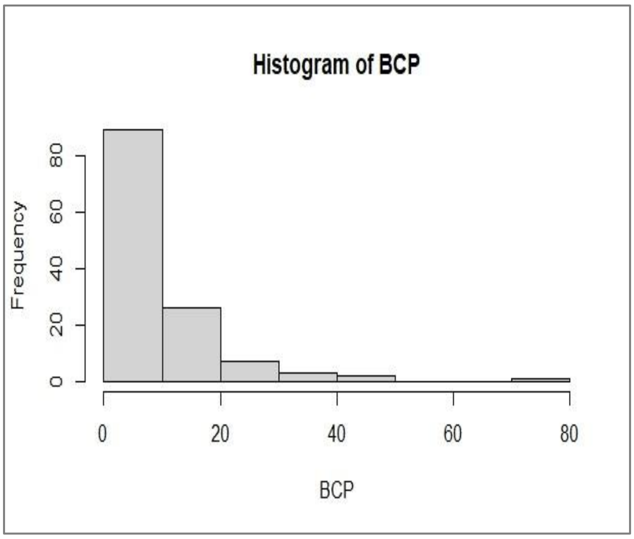

Figure 3: Histogram illustration of right skewed bladder cancer patient's data sets

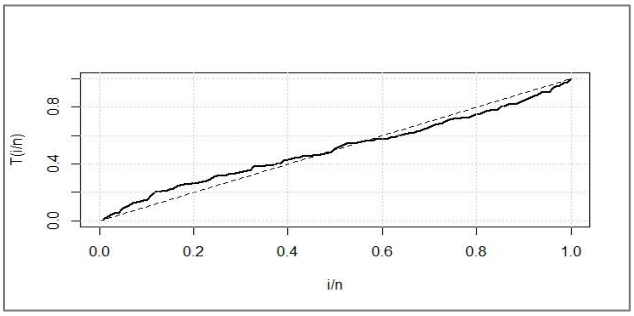

Figure 4: TTT plot illustration for model adequacy

In order to assess the fit of the fitted model, we employed information criteria, including the log-likelihood function evaluated at the maximum likelihood estimate (MLEs), the Akaike information criterion (AIC), the Bayesian information criterion (BIC), the Akaike information corrected criterion (CAIC), and goodness of fit techniques including Anderson-Darling (A*), Cramer-von Mises (W*), and Kolmogorov-Smirnov (K-S*). Typically, a better fit is indicated by smaller values for these statistics (Aryal et al., 2017).

$$AIC = -2L+2k \quad (44)$$

$$BIC = -2L+k\log(n) \quad (45)$$

And

$$CAIC = -2L + \frac{2kn}{n-k-1} \quad (46)$$

Table 2: Existing models

Distributions.	Author(s)
Truncated exponential top-leone exponential	(Al-Noor & Hilal, 2021)
Exponentiated Pareto	(Nadarajah, 2005)
Additive Weibull	(Lemonte et al.,2014)
Exponentiated inverted Weibull	(Flaih et al., 2012)

Table 3: Information criteria and estimate of the parameters

Distr.	$\hat{a}$	$\hat{b}$	$\hat{c}$	$\hat{d}$	LL	AIC	CAIC	BIC
TELTLR	1.693	-1.3×10^{-16}	2.003	-	2044.66	-4083.32	-4083.13	-4074.77
TETLE	-35.001	0.0617	0.0361	-	-413.3	832.723	832.917	841.279
ExP	3.9×10^{-7}	0.3800	-	-	-36.969	77.9386	78.3023	81.10565
ADW	8.9×10^{-5}	2.4730	8.3×10^{-2}	0.06	-412.39	832.7834	833.1086	844.1915
ExIW	4.6×10^{-7}	0.01356	-	-	-378.17	760.3364	760.4324	766.0405

Table 4: Goodness of fit test for the fitted models

Distribution	KS*	AD*	W*	P-value
TELTLR	0.9145	0.07734	0.1667	<0.00
TETLE	0.7447	0.71382	0.1958	0.4767
ExP	1.00	0.97476	0.15983	<0.00
ADW	0.06298	0.5467	0.9158	0.6901
ExIW	1.00	0.85646	0.1273	<0.00

DISCUSSION

The plot of cdf and pdf in Figure 1 shows the nature of the proposed distribution, which converges at one for the cdf plot and the right-skewed pdf. The hazard rate plot with an upward-increasing shape shows that the risk of system failure increases as time increases. The analysis conducted in Section 3 demonstrates the flexibility of the proposed TELTL-R distribution compared to the existing truncated exponential top-leone distribution, exponentiated pareto, additive weibull, and exponentiated inverse weibull distributions based on likelihood, information criteria, and goodness of fit measures. Considering the likelihood and information criteria results of the TELTL-R given in Table 3 with an AIC value of -4083.32, a BIC of -4083.13, and a CAIC of -4074.77, which is greater than that of the TETL-E with an AIC of 832.723, a CAIC of 832.917, and a BIC of 841.279, ExP with an AIC of 77.9386, a CAIC of 78.3023, and a BIC of 81.10565, ADW with an AIC of 832.7834, a CAIC of 833.1086, and a BIC of 844.1915, ExIW with an AIC of 832.7834, a CAIC of 832.7834, and a BIC of 844.1915, The larger the likelihood value, the stronger the fit and distribution supported by the data. This shows that the TELTL-R outperforms better with smaller values of information criteria. Likewise, in Table 3, the Anderson darling (AD*) result of TELTL-R performs best with a value of 0.07734 compared to the other existing distribution.

CONCLUSION

In this paper, we introduce a new distribution called the truncated exponential log topp-leone rayleigh distribution, which is a family member of the truncated exponential log topp-leone generalized family of distributions. We examined some of its statistical and mathematical properties, including survival function, hazard rate function, moment, moment generating function, entropy, and order statistics, as well as the estimation of parameters using the maximum likelihood method, least squares, and cramer von- Mises. Lastly, we applied the data of bladder cancer patients and suggested that the new model is a better one for modeling right-skewed data sets, which also perform better when compared with other existing models.

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ETHICAL CONSIDERATION

No Ethical clearance obtained before commencement of this work as it is not applicable.

CONTRIBUTION TO KNOWLEDGE

It has been recommended that a novel generalized family of distributions is superior to some J-shape probability distributions already in use for modeling right-skewed data sets.

AREA OF FURTHER RESEARCH

Researchers with an interest in this field of study can conduct a simulation study and also examine the estimate of confidence intervals for the suggested distribution parameters. Additionally, for the sake of theoretical comparison and methodology validation, researchers might estimate the parameters of the new distribution through the use of regression and Bayesian methodologies.

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