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ORIGINAL RESEARCH ARTICLE

Exploring Partially Ordered Multisets: Definitions, Applications, and Combinatorial Parameters

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ABSTRACT
A partially ordered set $(S, \preccurlyeq)$ is a nonempty set S equipped with a partial order $\preccurlyeq$. Ordered structures are useful for representing application problems that involve comparable and incomparable parameters or inputs. Ordered sets are studied based on classical set theory, where objects in a collection are assumed to be distinct. However, mathematical objects are not always distinguishable, especially in applications. Multisets are mathematical models of entities with repetition. Multiset theory is distinguished from a set by carrying information on the number of times an element appears in a given collection, making it suitable for modeling real-life situations. A multiset M over S can be formally defined as a cardinal-valued function, $M: S \rightarrow N$ such that $x \in Dom\ (M)$ implies $M(x)$ is a cardinal number and $M(x) = m_M(x) > 0$, where $m_M(x)$ denotes the number of times x occurs in M. Multisets generalize the classical sets and are apt for representing partial orders. There has been a growing interest in extending results on ordered sets to multisets. Unlike in classical set theory, there is still no unanimous way of studying foundational concepts in multisets. For instance, different approaches have been proposed for studying orderings on multisets due to the multiplicity parameter that is peculiar to multisets. This paper surveys studies on ordered multisets and compares existing orderings proposed on multiset structures. In particular, cases where the definition results in a partial order are of great interest in this exposition; their properties and theoretical implications make them suitable for application. We focus on definitions consistent with the Dershowitz-Manna multiset ordering, usually referred to as the standard multiset ordering, due to their relevance in applications, e.g., in computer programming. The strengths and possible limitations of the multiset orderings presented in this work are highlighted. This would aid in identifying potentially suitable definitions when dealing with studies that involve ordered multisets. Combinatorial parameters that have been studied on ordered multiset structures are also presented in this paper. The generalized notions of these parameters are investigated with some recommendations.

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INTRODUCTION

Ordered structures are present in all aspects of mathematics. They occur in fields like graph theory and are useful for studying various algebraic structures (Jokela, 2023; Gosh, 2019; Caspard et al., 2012). Investigations on partially ordered sets remain a growing sphere of study, largely due to the many useful applications of ordered sets. For instance, ordered sets are employed in problems involving scheduling, ranking, and prioritization (Yap et al., 2021; Garmendia et al., 2017; Fattore, 2014; Blanchet-Sadri, 2012). They also have practical applications in decision-making (Muren et al., 2023; Muren et al., 2022; Qiao & Hu, 2020; Chen et al., 2013). Ordered set theory assumes the axioms of the Cantorian set, and as such, the repetition of elements is insignificant.

A multiset extends the idea of a set by allowing multiple instances of its elements. This notion is more compatible with practical experiences. The study of multisets has both theoretical and practical implications (Gisin & Volkova, 2024; Alhazor et al., 2024; Liu et al., 2021; Felisiak et al., 2020; Jurgense, 2020; Singh et al., 2007). The range of a multiset M varies based on the theory under consideration. For most application problems, the image of any element in the set S is assumed to be in $\mathbb{N}$. However, in theoretical studies, the range can take values

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from any of ℤ, ℝ, etc. (Felisiak et al., 2020). Multiset-like structures have been frequently used in applications - long before the concept was formalized. These structures have appeared under different names in the literature (e.g., heap, bag, bunch, finitely generated set). They have been rediscovered several times due to the practical need for models of structures that admit repetition. For instance, in combinatorial analysis, combinations with repetitions are common (Knuth, 1981; Kuba, 2022), also in membrane computing (Alhazor et al., 2024; Liu et al., 2021). Although classical sets and multisets are, in essence, at different abstraction levels, it is natural to explore a set theory-like framework to study the theory of multisets using set-theoretical foundations; Blizard's well-known results (Blizard, 1988) is a good foundational work on multisets (see also, Felisiak et al., 2020; Dang, 2014; Blizard, 1991). Since the introduction of multisets, there has been a growing interest in extending results from set theory to multisets. A partially ordered multiset (or pomset) generalizes an ordered set. As such, concepts and results from ordered set theory can be generalized to ordered multisets under suitable constructions and with the appropriate rules (Kuba, 2022; Balogun et al., 2020, 2021, 2022; Anusuya & Vimala, 2018; Balogun & Singh, 2017; Wilson, 2016; Girish and Sunil, 2009; Pratt, 1986). Ordered multisets have applications in fields such as computer science, economics, linguistics, and biology (Liu et al., 2021; Paun, 2010; Conder et al., 2007; Dershowitz & Ellerman, 2007). In computer science, for instance, multiset orderings are exploited in proving termination of programs and term rewriting systems, including production systems where programs are written in terms of rewrite rules (Martin, 1989; Dershowitz & Manna, 1979). Relative to ordered sets, the theory of ordered multisets is still under-explored. Different definitions have been proposed for comparing multisets or objects in a multiset using different approaches, with each approach embodying certain advantages and constraints. Some of these definitions are modifications or extensions of the standard multiset ordering (Dershowitz & Manna, 1979), and they are consistent with it. However, some other approaches differ completely from the Dershowitz-Manna definition. In this paper, we survey different proposals on multiset ordering with particular interest in definitions that result in a partial order. We also focus on definitions that are consistent with the standard multiset ordering due to their practical implications, these orderings are presented in section 2. The advantages and possible drawbacks of the orderings investigated are presented. By highlighting the strengths and limitations of these orderings via comparative analysis, this study aims to aid researchers working in related fields in identifying suitable orderings that can be adopted or modified for modeling problems that admit repeated elements. Multisets are valuable in combinatorics (Kuba, 2022; Balogun et al., 2020, 2021). In particular, partially ordered multisets are apt for modeling concurrent behaviors. Existing theories on partially ordered multisets are presented in section 3. Essential combinatorial objects, such as the height and width of a partially ordered multiset, are presented in the next section. Several characterizations exist for these parameters. Linear extensions, realizers, and dimensions of ordered sets are significant combinatorial concepts with practical applications. Analogous forms of these combinatorial parameters have been proposed for ordered multisets. For instance, the notion of linear extension has been studied on partially ordered multiset structures (Balogun et al., 2022; Singh et al., 2012). These generalized combinatorial parameters need further vindication as they could efficiently model problems involving scheduling, ranking, and multi-criteria decision-making where repetition is significant.

ORDERINGS ON MULTISETS

In this section, we review orderings proposed on multisets with some examples. The similarities and peculiarities, strengths and limitations of these multiset orderings are also discussed.

Dershowitz-Manna Multiset Ordering

Dershowitz & Manna (1979) presented a multiset ordering by defining multisets over well-founded sets. In their definition, the well-founded set S is assumed to induce an ordering on $\boldsymbol{M}(S)$ i.e., the class of finite multisets defined on S. This ordering is defined as follows:

For $A, B \in \boldsymbol{M}(S)$, $A \prec\prec B$ if $\exists X, Y \in \boldsymbol{M}(S)$ such that,

i. $\emptyset \neq X \subseteq B$
ii. $A = (B \backslash X) + Y$, and
iii. X dominates Y, that is, for all $y \in Y$, there is some $x \in X$ such that $y \prec x$.

This ordering is usually referred to as the standard multiset ordering and often serves as a reference for other multiset orderings - especially for orderings introduced to prove termination of programs and term rewriting systems. The ordering is well-founded, provided the ordering on the generic set is well-founded.

Example:

Let $A, B \in \boldsymbol{M}(S)$ where

$A = \{2/a_1, 4/a_2, 2/a_3, 4/a_4\}$

$B = \{2/a_1, 2/a_3, 6/a_4, 2/a_5\}$

$X = \{2/a_4, 2/a_5\}$

$Y = \{4/a_2\}$ (n/a_i implies the object a_i occurs in the multiset under consideration n times).

Suppose $a_i \prec a_j \Leftrightarrow i < j$.

Since conditions i - iii above are satisfied, $A \prec\prec B$

The Dershowitz-Manna ordering is monotonic, a desirable property, especially in application. Jouannaud and Lescanne (1982) observed that it is, so far, the strongest monotonic ordering that has been defined for these structures. A major drawback of this ordering is its limitation in describing the inclusion relation between two multisets; a different definition would have to be adopted to achieve this.

An important application of the Dershowitz-Manna multiset ordering is proving the termination of program and term rewriting systems.

Huet-Oppen Multiset Ordering

Building on the Dershowitz-Manna definition, a more tractable multiset ordering was proposed by Huet & Oppen (1980) as follows:

For any $A, B \in M(S)$, $A \prec\!\prec B \Leftrightarrow A \neq B$ and $[A(y) > B(y) \Longrightarrow (\exists x \in S) x \succ y \,\&\, A(x) < B(x)]$

Example:

Let $A = \{2/a_1, 3/a_2, a_3, 2/a_4\}$ and $B = \{a_1, a_2, 2/a_3, 6/a_4\}$

Suppose $a_i \prec a_j$ whenever $i < j$. Clearly $A \prec\!\prec B$.

This ordering is also induced by the ordering $\prec$ on the ground set S. It is well-founded if and only if $(S, \prec)$ is well-founded. The ordering is linear and becomes a lexicographic ordering if $\prec$ on the set S is total. This multiset ordering also has applications in establishing program termination and in term rewriting systems. This ordering is equivalent to the Dershowitz-Manna multiset ordering.

Jouannaud-Lescanne Multiset Ordering

Another efficient multiset ordering was proposed by Jouannaud & Lescanne (1982). These authors extended Dershowitz and Manna's ordering via two definitions. In the first definition, $A \ll_1 B \Leftrightarrow \bar{A} \ll_1^{lex} \bar{B}$. Where the following holds for the partition $\bar{A} = \{A_k | k = 1 \dots p\}$:

i. $t \in A_k \Longrightarrow A_k(t) = A(t)$.
ii. $t \in A_k$ and $s \in A_k \Rightarrow t$ *and* s are incomparable.
iii. $\forall k \in [2, \dots, p] t \in A_k \Rightarrow (\exists s \in A_{k-1}) s \succ t$.

Example:

If $a_1 \prec a_2$ and $A = \{a_1, a_1, a_2, 3\}$ then $A_1 = \{3, a_2\}, A_2 = \{a_1, a_1\}$. If $B = \{a_1, a_2, a_2, a_1, 3,3,3\}$ then $B_1 = \{3,3,3, a_2, a_2\}$, $B_2 = \{a_1, a_1\}$ and $A \ll_1 B$.

Clearly, $\ll_1$ is an ordering because lexicographic extension preserves it. These authors proved that the ordering $\ll_1$ contains the Dershowitz and Manna ordering, whereas the converse is false.

On the other hand, the ordering $\ll_1$ has a strong limitation; it is not monotonic. Hence it cannot be employed when the initial ordering is increased.

The second definition uses another approach to building the partition. Here, the construction puts elements that are the same in different multisets of the partition. $A \ll_2 B \Leftrightarrow \bar{A} \ll_2^{lex} \bar{B}$ and the following hold for $\bar{A} = \{A_k | k = 1 \dots p\}$:

i. A_k is a set that is $A_k(t) \leq 1$.
ii. $t \in A_k$ and $s \in A_k \Rightarrow t$ *and* s are incomparable.
iii. $\forall k \in [2, \dots, p] t \in A_k \Rightarrow (\exists s \in A_{k-1}) s \succcurlyeq t$

The two definitions differ only in the first condition. This ordering contains the Dershowitz and Manna multiset ordering, but the converse is false. Using their definitions, the authors presented an efficient implementation of Dershowitz and Manna's ordering. Also, the Jouannaud-Lescanne definition produces a well-founded ordering provided $\succ$ on the ground set has this property. An important application of this multiset ordering is defining combinatorial parameters in the multiset setting (Balogun et al., 2021, 2022).

Krom Multiset Ordering

Also building on the Dershowitz-Manna multiset ordering, Krom (1985) defined two multiset orderings together with their respective transitive closures as follows:

For multisets A and B defined over a set X, $A \ll^n B$ if and only if

i. $\exists!\, y \in X$ with $A(y) < B(y)$
ii. $A(y) + 1 = B(y)$ for the one element in i above
iii. if $A(x) > B(x)$ *then* $x \prec y$ *in* X
iv. $\sum(A(x) - B(x)) \leq n$, the sum is over all $x \in X$ *such that* $A(x) > B(x)$

For $\ll^\infty$, if condition iv is omitted, then $A \ll^\infty B$.

An extension of the partial order $\prec$ on X is the ordering. $\prec_\in$ Defined thus:

$A \prec_\in B$ if and only if there exists $x \in X$ such that $B(x) > 0$ and for any $y \in X$, if $A(y) > 0$, then there exists $z \in X$ such that $y \prec z$ and $B(z) > 0$ i.e.,

$B \neq \emptyset \wedge \forall y \in A \rightarrow \exists z \in B \wedge y \prec z$.

The defined ordering inherits the well-founded property if it exists on the underlying set $(X, \prec)$, making it useful for application in computer programs.

Martin Multiset Ordering

Martin (1989) presented a definition via a geometric approach. Martin's definition is a generalization of Dershowitz-Manna ordering using matrix action. Martin exploited the notion of cone in $\mathbb{R}^n$. The author showed that tame orderings (i.e., an ordering on a multiset that preserves both multiset union and an ordering on the ground set) can be classified geometrically. He also showed that this method can be employed in presenting a unification of different orderings on multiset structures.

The following definition was given:

For two multisets $A\ and\ B$ defined on S, if $\prec$ is a strict order on S and K an $n\ by\ n$ matrix over $\mathbb{N}$ indexed by the elements of S, then $A \ll B \Leftrightarrow f_x(A) \leq f_x(B)\ \forall x \in S$ where $f_x(B) = \sum_{y \geq x} B(y)$

Martins' multiset ordering is well-founded and monotonic. The proposed ordering embodies properties that arise naturally in termination proofs. Hence, it is useful in proofs of program termination and in equational reasoning algorithms based on term rewriting systems. In their classification, the standard multiset ordering (or an ordering got from it by the action of an invertible matrix) is shown to be the strongest tame ordering defined on the class $\boldsymbol{M}(S)$ of finite multisets on S.

Savaglio-Vannucci Multiset Ordering

The following multiset ordering was proposed in Savaglio & Vannucci (2007):

Let $A, B \in \boldsymbol{M}(S)$, then $A \gg B \Leftrightarrow A(x) > B(x)\ \forall x \in S$, i.e., the multiset A dominates the multiset B if and only if B is a proper submultiset of A.

In this case, the ordering $\gg$ does not depend on the underlying ordering on S. This ordering cannot be total as the inclusion relation on $\boldsymbol{M}(S)$ is not total. Also, the different properties of the underlying base set, such as well-foundedness, do not naturally extend to the elements of $\boldsymbol{M}(S)$ via this definition.

PARTIALLY ORDERED MULTISETS (POMSETS)

Partially ordered multisets are useful for representing concurrent behaviors. Through different notions of comparability/incomparability, various partially ordered multiset structures have been proposed for modeling concurrency (Balogun et al., 2021, 2022; Girish & Sunil, 2009; Pratt, 1986; Gischer, 1984; Mazurkiewicz, 1984). An extensive study of equational theories of ordered multisets, sets of ordered multisets, ideals of ordered multisets, and how they relate to theories of languages is presented in Gischer (1984). Gischer's work appears to be, so far, the most comprehensive work on the algebraic theory of ordered multisets. He examined some theories and established whether they are finitely axiomatizable for each equational theory of ordered multisets and languages under concatenation, concurrence, and other operations that can be defined on an ordered multiset structure.

A natural generalization of language as a set of strings is the notion of a set of ordered multisets as a process. A *string* is an ordered set of symbols, or preferably multiset; considering the fact that repetition of symbols is permitted. Quite a number of formal language theory intuitions generalize easily when the total order is relaxed to a partial one. The model of a partial string, called a *partial word*, as a generalization of formal language theory was initiated by Grabowski (1981). This generalization of a string is called a pomset (the term partially ordered multiset or pomset together with an in-depth characterization was proposed in Pratt, 1986). Via a formulation that exploits the notion of a partial string (obtained by relaxing the linearity constraint on a string), Pratt (1986) presented a single hybrid approach for expressing concurrency. A combination of formal languages, partial orders, and temporal logic was used to propose a rich language that blends logic and algebra, giving rise to a natural class of models of concurrent processes. The method employed is direct and involves fewer artificial constructs- an advantage over some of the existing concurrency models- and still applies to various systems. Various effective operations for formally defining concurrent processes were introduced and categorized under combinatorial, Boolean, and homomorphisms. Some of the operations proposed by Pratt correspond to familiar programming language constructs, while others require further vindication.

Some concepts and definitions proposed for pomsets are presented below.

Definition: A structure $(V, \preccurlyeq, \sigma, \sum)$ is called a *labeled partial order* (lpo) over a set $\sum$ where $\preccurlyeq$ is a partial order on V and $\sigma: V \to \sum$ maps each $v \in V$ to an element of $\sum$.

Note that $\sum$ is an alphabet of actions and V denotes instances of that alphabet, or events that form a word, where $\preccurlyeq$ represents the order of occurrences of letters in the word. The usual formal language theoretical notion of a word obtains. If $\preccurlyeq$ is a total order, we get the regular formal language theoretical concept of a word. In an atomic *labeled partial order* the cardinality of V is 1.

An isomorphism of labeled partial orders is a map (f, t) such that f is an isomorphism of posets and the identity map on $\sum$ is t. A map (f, t) where t is the identity and f is the identity function on the elements of the ordered set (but not necessarily an isomorphism of posets) is called an *augmentation* of labeled partial orders. An augmentation yields an augment of its argument.

Definition: The isomorphism class of a labeled partial order is called a *pomset*.

A pomset can be informally viewed as a labeled partial order where attention is paid to the selection of the set V, instead of its size, but other information are maintained. Hence, replacing $V = \{0,1,2\}$ by $V = \{5,6,7\}$ while retaining σ or $\preccurlyeq$ does not alter the pomset.

Rensik (1996) describes a pomset as node-labelled directed graphs where the edge-relation is irreflexive and transitive, interprets up to label and edge-preserving isomorphism such that the identity of the nodes is abstracted from them, while their ordering is preserved. His investigation is around a particular class of pomset called *order-deterministic*. This class of pomsets has as a proper subset all ordered sets. Order-deterministic pomsets form a distributive lattice under pomset prefix. Among other attributes, order-deterministic pomsets give rise to a reflexive subcategory of the category of all pomsets.

In Girish & Sunil (2009), multiset relations are introduced as follows:

Definition: Let A and B be two multisets drawn from a set S; then $A \times B$ (*Cartesian product*) is given by

$$A \times B = \{(m/x, n/y)/mn; x \in^m A, y \in^n B\}$$

A submultiset R of $A \times A$ is a *multiset relation* on A if every member $(m/x, n/y)$ of R has count $C_1(x, y) \cdot C_2(x, y)$. We write $m/xRn/y$ whenever m/x is related to n/y.

The domain $(Dom\ R)$ and range $(Ran\ R)$ of the relation R are given by the following sets, respectively:

$\{x \in^r A; \exists y \in^s A\ such\ that\ r/xRs/y\}$,
$\{y \in^s A; \exists\ x \in^r A\ such\ that\ r/xRs/y\}$

Based on these relations, a pomset is defined as a multiset with a reflexive, antisymmetric and transitive relation. The authors proposed different ways of ordering a multiset with respect to the defined relations.

Balogun & Tella (2018) proposed a partial ordering on a multiset structure with a non-trivial involvement of the multiplicity. The ordering on the set of multiplicities is assumed to be a partial order instead of the usual natural ordering. The authors defined a pomset as follows:

Let $(S, \preccurlyeq)$ and $(\mathbb{N}, \preccurlyeq)$ be ordered sets such that $\mathbb{N}$ is also partially ordered. For any two points m_i/s_i and m_j/s_j (where m_i represents the multiplicity of the object s_i) in $M \in \boldsymbol{M}(S)$, we have $m_i/s_i \preccurlyeq\leq m_j/s_j \Leftrightarrow ((s_i \preccurlyeq s_j) \wedge (m_i \leq m_j))$. The two points coincide if and only if $s_i = s_j$ ($m_i = m_j$ follows from the exact multiplicity axiom of MST in Blizard, 1988). The two points are comparable if and only if $(m_i/s_i \preccurlyeq\leq m_j/s_j) \vee (m_j/s_j \preccurlyeq\leq m_i/s_i)$ otherwise they are incomparable.

It is easily deducible from the above definition that, given any of the conditions below, a point m_i/s_i cannot precede another point m_j/s_j

i. $m_i/s_i \not\preccurlyeq\leq m_j/s_j$
ii. $m_i/s_i \preccurlyeq\nleq m_j/s_j$
iii. $m_i/s_i \not\preccurlyeq\nleq m_j/s_j$

The definition gives rise to a new concept called *semimset chains*. A semimset chain is obtained whenever any two objects of the underlying ordered set or their multiplicities are comparable. The proposed definition needs further vindication as it promises to be useful for modeling complex systems.

SPECIAL INSTANCES AND COMBINATORIAL PARAMETERS OF A PARTIALLY ORDERED MULTISET

Some special cases of pomsets are described in this section (see Rensink, 1996). We also discuss combinatorial parameters that have been generalized to ordered multisets.

Mazurkiewicz traces: Also known as partially commutative monoids, Mazurkiewicz traces consist of strings and multisets. Here, the set S under consideration (also known as a concurrent alphabet) has the structure $(S, \prec)$ where $\prec \subseteq S \times S$ is an irreflexive and symmetric independency relation. Mazurkiewicz traces and the pomset model studied by Pratt (1986) are somewhat closely related. The independency relation determines the degree to which the concatenation operator $\odot$ is commutative. The traces are freely generated by $\sum_{maz} = \langle \varepsilon, \odot, \prec \rangle$ where ε is the identity element and $\odot$ an associative operation, satisfying the following:

$$\vdash \quad \varepsilon \prec a \tag{1}$$
$$a \prec b \vdash b \prec a \tag{2}$$
$$a \prec b, a \prec c \vdash a \prec (b \odot c) \tag{3}$$
$$a \prec b \vdash a \odot b = b \odot a \tag{4}$$

The above system is a model for strings if $\prec$ is the empty order. The system is a model for multisets whenever $\prec = (S \times S) \backslash \{(a, a) | a \in S\}$.

Series-parallel pomsets: Consists of a direct combination of the algebras of strings and multisets, where the neutral element of a string is the same as that of a multiset. Series-parallel pomsets are also known as N-free pomsets (Gischer, 1988; Pratt, 1986). This is probably the most intensively studied approach for obtaining an extensive theory of pomsets. The signature freely generates series-parallel pomsets $\sum_{sp} = \langle \varepsilon, \cdot, \uplus \rangle$ with these operations:

$$\varepsilon \cdot a = a \tag{1}$$
$$a \cdot \varepsilon = a \tag{2}$$
$$(a \cdot b) \cdot c = a \cdot (b \cdot c) \tag{3}$$
$$\varepsilon \uplus a = a \tag{4}$$
$$(a \uplus b) \uplus c = a \uplus (b \uplus c) \tag{5}$$
$$a \uplus b = b \uplus a \tag{6}$$

Only $N-free$ models (probably not all $N-free$ models) can be generated via this signature e.g. the empty and singleton pomsets.

Forests: These pomsets (also known as multisets of trees) possess a special attribute that all predecessors of a given element are linearly ordered. They are viewed as an algebraic extension of multisets with an associative concatenation operator (denoted by ;) for which the empty forest is left cancellative, instead of left and right neutral obtainable for strings, and distributes over addition from the right. They are freely generated by the signature $\sum_{tr} = \langle \varepsilon, ;, \uplus \rangle$ using operations 4, 5, and 6 combined with the axioms below:

$$\varepsilon; a = a \tag{7}$$
$$(a; b); c = a; (b; c) \tag{8}$$
$$(a \uplus b); c = a; c \uplus b; c \tag{9}$$

The above algebra is presented with the additional axiom $a \uplus a = a$ in Beaten and Weijland (1990). This system is a model for pomsets that are hierarchical orders with termination. Forest addition coincides with multiset addition.

Order-deterministic pomsets: These pomsets are constructed by replacing pomset union with a new operator $\sqcup$ called join. Order-deterministic pomsets are generated by the signature $\sum_{det} = \langle \varepsilon, \cdot, \sqcup \rangle$ with operations 1, 2 and 3 of series parallel pomsets together with:

$$\varepsilon \sqcup a = a \tag{10}$$
$$(a \sqcup b) \sqcup c = a \sqcup (b \sqcup c) \tag{11}$$
$$a \sqcup b = b \sqcup a \tag{12}$$
$$a \cdot (b \sqcup c) = (a \cdot b) \sqcup (a \cdot c) \tag{13}$$

Mazurkiewicz traces are properly contained in order-deterministic pomsets.

Combinatorial Parameters

Combinatorial parameters that have been studied on ordered multiset structures are presented in this section.

The notions of homomorphism and substitution are well-established in formal language theory. Both notions have been generalized from strings to pomsets. Homomorphism in formal language theory is defined on sets of pomsets, whereas the same notion as a mapping defined on pomsets goes between single pomsets.

As noted in section 2, the multiset ordering presented in Jouannaud & Lescanne (1982) is apt for defining combinatorial parameters in the multiset setting. The multiset ordering that employs set-based partitioning was adopted in Balogun et al. (2021) to construct sets of incomparable elements of a pomset. With this construction, the authors generalized results on antichains from ordered set theory to multisets. The method of set-based partitioning guarantees that in any antichain, no two elements are comparable or equal (since elements are allowed to repeat) under the defined order.

Linear extension, realizer and dimension of ordered sets have been extended to multisets. These combinatorial parameters are useful in application problems such as the scheduling problem also known as the jump number problem.

Linear extensions

A linear argument of an ordered multiset p is a *linearization* of p. The set containing linearizations of an ordered multiset is usually denoted by $\lambda(p)$. This extends to $\lambda(P)$ where P is a set of pomsets and $\lambda(P) = \bigcup_{p \in P} \lambda(P)$.

The method of constructing a linearization, also known as linear extension, of an ordered structure is key in determining an optimal realizer for the structure.

In Balogun et al. (2022), linear extensions were studied on a partially ordered multiset structure. The extension theorem was generalized under suitable constraints. Also, a heuristic algorithm that constructs multiset linear extensions of a finite ordered multiset was presented. This study can be extended to multisets in the class of finite multisets **M**(S). Singh et al. (2012) presented an application of topological sorting to obtain linear extensions of a partially ordered class of finite multisets. The multisets used are built from a partially ordered base set. The authors established that the partially ordered class of finite multisets is an intersection of its linear extensions (i.e., the realizer of the ordered multisets)

Dimension

The dimension of an ordered structure is an important combinatorial parameter and has several characterizations (Joret et al., 2016). This notion has been generalized using ordered multiset structures. For instance, the dimension of an ordered multiset p can be described as the dimension of the base poset of a representative labeled partial order of p.

In Balogun and Singh (2017), the dimension of a pomset was studied relative to the dimension of the underlying poset. Among other findings, the authors established that the dimension of a pomset is bounded by the dimension of the generating poset.

The generalized combinatorial parameters presented in this study are valuable in applications, especially in modeling problems where repetition is significant. Studies on decision-making, for instance, that are based on set theory can be extended via these generalized parameters. These combinatorial parameters need further exploration using multiset structures for more efficient outcomes.

CONCLUSION

Different orderings proposed on multiset structures were studied in this paper, with a focus on definitions that are consistent with the standard multiset ordering. These orderings have applications in areas such as computer programming. They are particularly useful in proving program termination. The strengths and limitations of the proposed orderings were investigated via a comparative analysis. The findings presented will aid in identifying suitable orderings that can be adopted for modeling problems involving repeated elements. The study of combinatorial parameters of an ordered multiset is a growing field with possible applications in areas such as scheduling, ranking, and decision-making. Extensions, realizer and dimension of an ordered multiset have been defined via existing multiset orderings. These combinatorial parameters can be further generalized by modifying or extending existing theories of partially ordered multisets, especially based on the multiplicity parameter.

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