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ORIGINAL RESEARCH ARTICLE

Analysis of Convergence and Stability Properties of Diagonally Implicit 3-Point Block Backward Differentiation Formula for First Order Stiff Initial Value Problems

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ABSTRACT

This paper presents a comprehensive analysis of the diagonally implicit 3-point block backward differentiation formula (BDF) for solving first-order stiff initial value problems. We establish the necessary and sufficient conditions for convergence, including consistency and zero stability, and derive the method's order of accuracy, which is found to be 5. Stability analysis reveals that the method is almost A-stable, with an absolute stability region that is plotted. A C programming language code is developed using Newton's Iteration for numerical implementation and compiled in the Microsoft Dev C++ compiler environment. Comparative numerical results demonstrate the competitive performance of the proposed method over existing fully implicit 3-point block backward differentiation formula (3BBDF), in terms of maximum error and CPU time. Therefore, this method offers a new and efficient numerical solution for integrating stiff initial value problems.

INTRODUCTION

Stiff initial value problems (IVPs) are a class of differential equation problems that pose significant challenges in numerical computation. These problems arise in various fields, including physics, engineering, chemistry, and biology, and are characterized by their rapid changes in solution behavior, high sensitivity to initial conditions, and large differences in timescales (Dalquist, 1974). The stiffness of these problems makes them difficult to solve using traditional numerical methods, leading to issues such as numerical instability, accuracy loss, and slow convergence.

The development of efficient and reliable numerical schemes for integrating stiff systems of ordinary differential equations (ODEs) has been a significant challenge in modern numerical analysis (Ibrahim et al., 2003). Stiff equations are those where implicit numerical methods, particularly the backward differentiation formula (BDF), outperform explicit schemes (Curtiss and Hirsfielder, 1952). An ordinary differential equation is considered stiff if the eigenvalues of the Jacobian matrix have negative real parts and the ratio of the real parts of the largest and smallest eigenvalues is extremely large (Lambert, 1991).

The development of efficient standard fully and diagonally implicit block numerical methods for solving stiff ordinary differential equations (ODEs) remains an active area of research, as evidenced by numerous studies such as those found in (Ibrahim, *et al.*, 2007; Musa, *et al.*, 2014; Musa and Bala, 2019; Zawawi, *et al.*, 2012; Haziza, *et al.*, 2019; Abasi, *et al.*, 2014; Nasir, *et al.*, 2011; Ibrahim, *et al.*, 2020; Noor, *et al.*, 2024; Suleiman, *et al.*, 2015;Alhassan, *et al.*, 2022; Alhassan and Musa, 2023b) and so on. These methods aim to balance accuracy, stability, and computational efficiency in solving stiff ODEs.

The existing literature lacks a comprehensive analysis of the convergence and stability properties of the diagonally implicit block backward differentiation formula for solving stiff initial value problems. Previous studies have not thoroughly addressed the limitations of the scheme,

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particularly regarding its stability and convergence characteristics under various conditions.

This paper aims to fill this gap by providing a detailed investigation into the mathematical formulation of the method using interpolation polynomials. Furthermore, it will rigorously analyze the stability and convergence properties, focusing on key aspects such as the order and error constant, consistency, zero stability, and absolute stability region of the diagonally implicit 3-point block backward differentiation formula developed by Bala et al. (2022) using Lagrange interpolating polynomial. Through this analysis, the paper seeks to enhance our understanding of the method's behavior and performance in solving first-order stiff initial value problems.

Mathematical Formulation of the Method

In this section, we will derive the 3-point diagonally implicit block backward differentiation formula of constant step size that compute three solution values y_{n+1}, y_{n+2} and y_{n+3}simultaneously at each integration step in block. Contrary to the fully implicit almost A-stable 3-point block backward differentiation formula that has been developed by Ibrahim et al. (2007), the first point of the diagonally implicit has one less interpolating point.

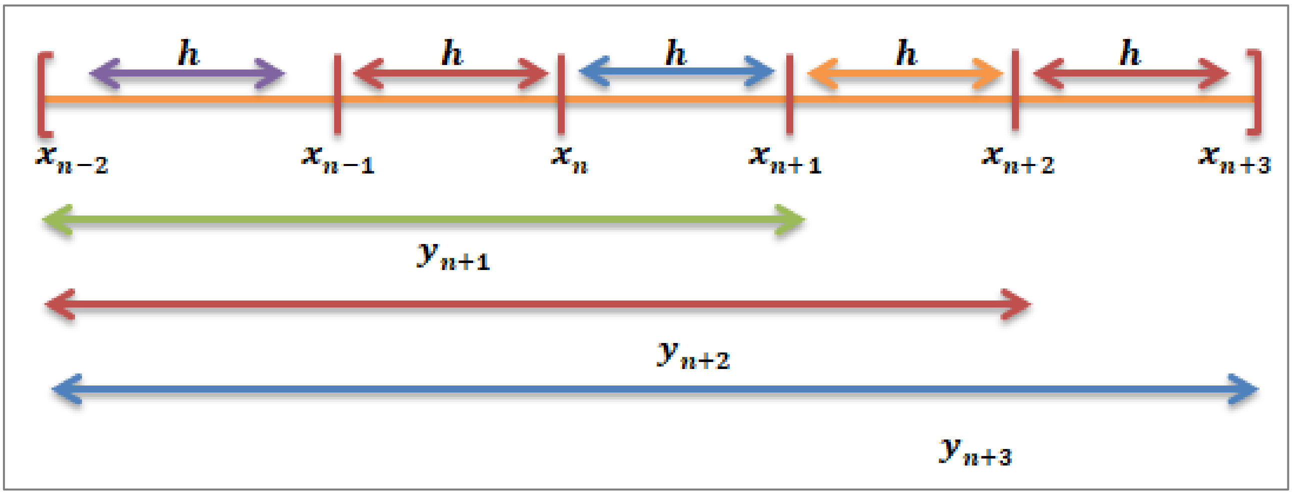

Figure 1: Interpolation points involved in the 3-point diagonally implicit BBDF method.

The derivation of the method using Lagrange polynomial $P_k(x)$ of degree $k$is defined as follows:

$$P_k(x) = \sum_{j=0}^{k} L_{k,j}(x) y(x_{n+2-j}), \tag{1}$$

Where

$$L_{k,j}(x) = \prod_{\substack{i=0\\ i\neq j}}^{k} \frac{(x - x_{n+2-i})}{(x_{n+2-j} - x_{n+2-i})} \text{for each} j = 0,1,\ldots,k$$

To obtain the formula for the first point y_{n+1}, we use the associated polynomial (1) with interpolating points x_{n-2} , x_{n-1} , $x_n$as follows:

$$P(x) = \frac{(x - x_{n-1})(x - x_n)(x - x_{n+1})}{(x_{n-2} - x_{n-1})(x_{n-2} - x_n)(x_{n-2} - x_{n+1})} y_{n-2} + \frac{(x - x_{n-2})(x - x_n)(x - x_{n+1})}{(x_{n-1} - x_{n-2})(x_{n-1} - x_n)(x_{n-1} - x_{n+1})} y_{n-1}$$
$$+ \frac{(x - x_{n-2})(x - x_{n-1})(x - x_{n+1})}{(x_n - x_{n-2})(x_n - x_{n-1})(x_n - x_{n+1})} y_n + \frac{(x - x_{n-2})(x - x_{n-1})(x - x_n)}{(x_{n+1} - x_{n-3})(x_{n+1} - x_{n-2})(x_{n+1} - x_{n-1})} y_{n+1} \tag{2}$$

Replacing $x = sh + x_{n+1}$ into (2) gives

$$P(x_{n+1} + sh) = \frac{(sh + 2h)(sh + h)(sh)}{(-h)(-2h)(-3h)} y_{n-2} + \frac{(sh + 3h)(sh + h)(sh)}{(h)(-h)(-2h)} y_{n-1}$$
$$+ \frac{(sh + 3h)(sh + 2h)(sh)}{(2h)(h)(-h)} y_n + \frac{(sh + 3h)(sh + 2h)(sh + h)}{(3h)(2h)(h)} y_{n+1} \tag{3}$$

Differentiating (3) with respect to s at the point $x = x_{n+1}$ and then substituting $s = 0$ gives

$$P'(x_{n+1}) = -\frac{1}{3} y_{n-2} + \frac{3}{2} y_{n-1} - 3y_n + \frac{11}{6} y_{n+1} \tag{4}$$

Substituting$P'(x_{n+1}) = hf_{n+1}$, the formula for the first point $y_{n+1}$is obtained as follows:

$$hf_{n+1} = -\frac{1}{3} y_{n-2} + \frac{3}{2} y_{n-1} - 3y_n + \frac{11}{6} y_{n+1} \tag{5}$$

To obtain the formula for the second point y_{n+2} we use the associated polynomial for (1) with interpolating pointsx_{n-2}, x_{n-1},x_n, x_{n+1},x_{n+2}as follows:

$$P(x)=\frac{(x-x_{n-1})(x-x_n)(x-x_{n+1})(x-x_{n+2})}{(x_{n-2}-x_{n-1})(x_{n-2}-x_n)(x_{n-2}-x_{n+1})(x_{n-2}-x_{n+2})}y_{n-2}+\frac{(x-x_{n-2})(x-x_n)(x-x_{n+1})(x-x_{n+2})}{(x_{n-1}-x_{n-2})(x_{n-1}-x_n)(x_{n-1}-x_{n+1})(x_{n-1}-x_{n+2})}y_{n-1}$$
$$+\frac{(x-x_{n-2})(x-x_{n-1})(x-x_{n+1})(x-x_{n+2})}{(x_n-x_{n-2})(x_n-x_{n-1})(x_n-x_{n+1})(x_n-x_{n+2})}y_n+\frac{(x-x_{n-2})(x-x_{n-1})(x-x_n)(x-x_{n+2})}{(x_{n+1}-x_{n-2})(x_{n+1}-x_{n-1})(x_{n+1}-x_n)(x_{n+1}-x_{n+2})}y_{n+1}$$
$$+\frac{(x-x_{n-2})(x-x_{n-1})(x-x_n)(x-x_{n+1})}{(x_{n+2}-x_{n-2})(x_{n+2}-x_{n-1})(x_{n+2}-x_n)(x_{n+2}-x_{n+1})}y_{n+1} \quad (6)$$

Substituting $x=sh+x_{n+2}$ into (6) gives

$$P(x_{n+2}+sh)=\frac{(sh+3h)(sh+2h)(sh+h)(sh)}{(-h)(-2h)(-3h)(-4h)}y_{n-2}+\frac{(sh+4h)(sh+3h)(sh+h)(sh)}{(h)(-h)(-2h)(-3h)}y_{n-1}$$
$$+\frac{(sh+4h)(sh+2h)(sh+h)(sh)}{(2h)(h)(-h)(-2h)}y_n+\frac{(sh+4h)(sh+3h)(sh+2h)(sh)}{(3h)(2h)(h)(-h)}y_{n+1}+\frac{(sh+4h)(sh+3h)(sh+2h)(sh+h)}{(4h)(3h)(2h)(h)}y_{n+2} \quad (7)$$

Differentiating (7) with respect to s at the point $x=x_{n+2}$ and then substituting $s=0$ gives

$$P'(x_{n+2})=\frac{1}{4}y_{n-2}-\frac{4}{3}y_{n-1}+3y_n-4y_{n+1}+\frac{25}{12}y_{n-2} \quad (8)$$

Replacing $P'(x_{n+2})=hf_{n+2}$, the formula for the second point y_{n+2} is obtained as follows:

$$hf_{n+2}=\frac{1}{4}y_{n-2}-\frac{4}{3}y_{n-1}+3y_n-4y_{n+1}+\frac{25}{12}y_{n-2} \quad (9)$$

To obtain the formula for the third point y_{n+3}, we use the associated polynomial for (1) with interpolating points x_{n-2}, x_{n-1}, x_n, x_{n+1}, x_{n+2}, x_{n+3} as follows:

$$\left.\begin{aligned}P(x)=&\frac{(x-x_{n-2})(x-x_{n-1})(x-x_n)(x-x_{n+1})(x-x_{n+2})}{(x_{n+3}-x_{n-2})(x_{n+3}-x_{n-1})(x_{n+3}-x_n)(x_{n+3}-x_{n+1})(x_{n+3}-x_{n+2})}y_{n+3}\\&+\frac{(x-x_{n-2})(x-x_{n-1})(x-x_n)(x-x_{n+1})(x-x_{n+3})}{(x_{n+2}-x_{n-2})(x_{n+2}-x_{n-1})(x_{n+2}-x_n)(x_{n+2}-x_{n+1})(x_{n+2}-x_{n+3})}y_{n+2}\\&+\frac{(x-x_{n-2})(x-x_{n-1})(x-x_n)(x-x_{n+2})(x-x_{n+3})}{(x_{n+1}-x_{n-2})(x_{n+1}-x_{n-1})(x_{n+1}-x_n)(x_{n+1}-x_{n+2})(x_{n+1}-x_{n+3})}y_{n+1}\\&+\frac{(x-x_{n-2})(x-x_{n-1})(x-x_{n+1})(x-x_{n+2})(x-x_{n+3})}{(x_n-x_{n-2})(x_n-x_{n-1})(x_n-x_{n+1})(x_n-x_{n+2})(x_n-x_{n+3})}y_n\\&+\frac{(x-x_{n-2})(x-x_n)(x-x_{n+1})(x-x_{n+2})(x-x_{n+3})}{(x_{n-1}-x_{n-2})(x_{n-1}-x_n)(x_{n-1}-x_{n+1})(x_{n-1}-x_{n+2})(x_{n-1}-x_{n+3})}y_{n-1}\\&+\frac{(x-x_{n-1})(x-x_n)(x-x_{n+1})(x-x_{n+2})(x-x_{n+3})}{(x_{n-2}-x_{n-1})(x_{n-2}-x_n)(x_{n-2}-x_{n+1})(x_{n-2}-x_{n+2})(x_{n-2}-x_{n+3})}y_{n-2}\end{aligned}\right\} \quad (10)$$

Substituting $x=sh+x_{n+3}$ into (10) gives

$$\left.\begin{aligned}&P(x_{n+3}+sh)\\&=\frac{(sh+5h)(sh+4h)(sh+3h)(sh+2h)(sh)}{(5h)(4h)(3h)(2h)(h)}y_{n+3}+\frac{(sh+5h)(sh+4h)(sh+3h)(sh+2h)(sh)}{(4h)(3h)(2h)(h)(-h)}y_{n+2}\\&+\frac{(sh+5h)(sh+4h)(sh+3h)(sh+h)(sh)}{(3h)(2h)(h)(-hh)(-2h)}y_{n+1}+\frac{(sh+5h)(sh+4h)(sh+2h)(sh+h)(sh)}{(2h)(h)(-h)(-2h)(-3h)}y_n\\&+\frac{(sh+5h)(sh+3h)(sh+2h)(sh+h)(sh)}{(h)(-h)(-2h)(-3h)(-4h)}y_{n-1}+\frac{(sh+4h)(sh+3h)(sh+2h)(sh+h)(sh)}{(-h)(-2h)(-3h)(-4h)(-5h)}y_{n-2}\end{aligned}\right\} \quad (11)$$

Differentiating (11) with respect to s at the point $x=x_{n+3}$ and then substituting $s=0$ gives

$$P'(x_{n+3})=-\frac{1}{5}y_{n-2}+\frac{5}{4}y_{n-1}-\frac{10}{3}y_n+5y_{n+1}-5y_{n+2}+\frac{137}{60}y_{n+3} \quad (12)$$

Replacing $P'(x_{n+3})=hf_{n+3}$, the formula for the third point y_{n+3} is obtained as follows:

$$hf_{n+3}=-\frac{1}{5}y_{n-2}+\frac{5}{4}y_{n-1}-\frac{10}{3}y_n+5y_{n+1}-5y_{n+2}+\frac{137}{60}y_{n+3} \quad (13)$$

By merging equations (5), (9), and (13), we have developed a diagonally implicit, nearly A-stable block numerical scheme for addressing stiff initial value problems. This scheme computes three solution values simultaneously using a step size of h as detailed below:

$$\left.\begin{aligned}hf_{n+1}&=-\frac{1}{3}y_{n-2}+\frac{3}{2}y_{n-1}-3y_n+\frac{11}{6}y_{n+1}\\hf_{n+2}&=\frac{1}{4}y_{n-2}-\frac{4}{3}y_{n-1}+3y_n-4y_{n+1}+\frac{25}{12}y_{n-2}\\hf_{n+3}&=-\frac{1}{5}y_{n-2}+\frac{5}{4}y_{n-1}-\frac{10}{3}y_n+5y_{n+1}-5y_{n+2}+\frac{137}{60}y_{n+3}\end{aligned}\right\} \quad (14)$$

After rearranging and collecting the like terms, we equivalently obtain the following formulae:

$$\left.\begin{aligned}y_{n+1}&=\frac{2}{11}y_{n-2}-\frac{9}{11}y_{n-1}+\frac{18}{11}y_n+\frac{6}{11}hf_{n+1},\\y_{n+2}&=-\frac{3}{25}y_{n-2}+\frac{16}{25}y_{n-1}-\frac{36}{25}y_n+\frac{48}{25}y_{n+1}+\frac{12}{25}hf_{n+2},\\y_{n+3}&=\frac{12}{137}y_{n-2}-\frac{75}{137}y_{n-1}+\frac{200}{137}y_n-\frac{300}{137}y_{n+1}+\frac{300}{137}y_{n+2}+\frac{60}{137}hf_{n+3}.\end{aligned}\right\} \quad (15)$$

The formula (15) is referred to diagonally implicit 3-point block backward differentiation formula(3DBBDF) for solving stiff initial value problems. The detailed derivation of order, error constant, stability and consistency of the method will be discussed in the subsequent sections.

Order, Error Constant and Consistency of the Method

The order and error constant of a numerical method are important properties that determine its accuracy and reliability. The order of a method determines the rate at which the error decreases as the step size is reduced, while the error constant determines the magnitude of the error (Alhassan et al, 2022). This section derives the order of the method (15). To derive the order of the method, equation (15) can be rearranged and rewritten as

$$\left.\begin{aligned} -\frac{2}{11}y_{n-2}+\frac{9}{11}y_{n-1}-\frac{18}{11}y_n+y_{n+1}&=\frac{6}{11}hf_{n+1}\\ \frac{3}{25}y_{n-2}-\frac{16}{25}y_{n-1}+\frac{36}{25}y_n-\frac{48}{25}y_{n+1}+y_{n+2}&=\frac{12}{25}hf_{n+2}\\ -\frac{12}{137}y_{n-2}+\frac{75}{137}y_{n-1}-\frac{200}{137}y_n+\frac{300}{137}y_{n+1}-\frac{300}{137}y_{n+2}+y_{n+3}&=\frac{60}{137}hf_{n+3}\end{aligned}\right\} \quad (16)$$

The matrix formulation of equation (16) is

$$\begin{bmatrix} -\frac{2}{11} & \frac{9}{11} & -\frac{18}{11}\\ \frac{3}{25} & -\frac{16}{25} & \frac{36}{25}\\ -\frac{12}{137} & \frac{75}{137} & -\frac{200}{137}\end{bmatrix}\begin{bmatrix} y_{n-2}\\ y_{n-1}\\ y_n\end{bmatrix}+\begin{bmatrix} 1 & 0 & 0\\ -\frac{48}{25} & 1 & 0\\ \frac{300}{137} & -\frac{300}{137} & 1\end{bmatrix}\begin{bmatrix} y_{n+1}\\ y_{n+2}\\ y_{n+3}\end{bmatrix}=h\begin{bmatrix} 0&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix} f_{n-2}\\ f_{n-1}\\ f_n\end{bmatrix}+h\begin{bmatrix} \frac{6}{11} & 0 & 0\\ 0 & \frac{12}{25} & 0\\ 0 & 0 & \frac{60}{137}\end{bmatrix}\begin{bmatrix} f_{n+1}\\ f_{n+2}\\ f_{n+3}\end{bmatrix} \quad (17)$$

Let α_0^*, α_1^*, β_0^* and β_1^* be block matrices defined by:

$$\alpha_0^*=(\alpha_0\alpha_1\alpha_2),\ \alpha_1^*=(\alpha_3\ \alpha_4\alpha_5),$$
$$\beta_0^*=(\beta_0\beta_1\beta_2),\ \beta_1^*=(\beta_3\ \beta_4\beta_5),$$

Where, $\alpha_0=\begin{bmatrix} -\frac{2}{11}\\ \frac{3}{25}\\ -\frac{12}{137}\end{bmatrix}$, $\alpha_1=\begin{bmatrix} \frac{9}{11}\\ -\frac{16}{25}\\ \frac{75}{137}\end{bmatrix}$, $\alpha_2=\begin{bmatrix} -\frac{18}{11}\\ \frac{36}{25}\\ -\frac{200}{137}\end{bmatrix}$, $\alpha_3=\begin{bmatrix} 1\\ -\frac{48}{25}\\ \frac{300}{137}\end{bmatrix}$, $\alpha_4=\begin{bmatrix} 0\\ 1\\ -\frac{300}{137}\end{bmatrix}$, $\alpha_5=\begin{bmatrix} 0\\ 0\\ 1\end{bmatrix}$

$$\beta_0=\begin{bmatrix}0\\0\\0\end{bmatrix}, \beta_1=\begin{bmatrix}0\\0\\0\end{bmatrix}, \beta_2=\begin{bmatrix}0\\0\\0\end{bmatrix}, \beta_3=\begin{bmatrix}\frac{6}{11}\\0\\0\end{bmatrix}, \beta_4=\begin{bmatrix}0\\ \frac{12}{25}\\0\end{bmatrix}, \beta_5=\begin{bmatrix}0\\0\\ \frac{60}{137}\end{bmatrix}$$

Definition 1 (Order): The order of the block method (15) and its associated linear operator Lgiven by

$$L[y(x),h]=\sum_{j=0}^{5}\left[\alpha_j y(x+jh)-h\beta_j y'(x+jh)\right] \quad (18)$$

is considered of order p if $C_0=C_1=C_2=\cdots=C_p=0$ and $C_{p+1}\neq 0$. It follows that

$$C_0=\sum_{j=0}^{5}(\alpha_j)=(\alpha_0+\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5)$$

$$=\begin{bmatrix} -\frac{2}{11}\\ \frac{3}{25}\\ -\frac{12}{137}\end{bmatrix}+\begin{bmatrix} \frac{9}{11}\\ -\frac{16}{25}\\ \frac{75}{137}\end{bmatrix}+\begin{bmatrix} -\frac{18}{11}\\ \frac{36}{25}\\ -\frac{200}{137}\end{bmatrix}+\begin{bmatrix} 1\\ -\frac{48}{25}\\ \frac{300}{137}\end{bmatrix}+\begin{bmatrix} 0\\ 1\\ -\frac{300}{137}\end{bmatrix}+\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$C_1=\sum_{j=0}^{5}(j\alpha_j)-\sum_{j=0}^{5}\beta_j=\big((0)\alpha_0+(1)\alpha_1+(2)\alpha_2+(3)\alpha_3+(4)\alpha_4+(5)\alpha_5\big)$$
$$-(\beta_0+\beta_1+\beta_2+\beta_3+\beta_4+\beta_5)$$

$$= \left[(0)\begin{bmatrix} -\frac{2}{11} \\ \frac{3}{25} \\ -\frac{12}{137} \end{bmatrix} + (1)\begin{bmatrix} \frac{9}{11} \\ -\frac{16}{25} \\ \frac{75}{137} \end{bmatrix} + (2)\begin{bmatrix} -\frac{18}{11} \\ \frac{36}{25} \\ -\frac{200}{137} \end{bmatrix} + (3)\begin{bmatrix} 1 \\ -\frac{48}{25} \\ \frac{300}{137} \end{bmatrix} + (4)\begin{bmatrix} 0 \\ 1 \\ -\frac{300}{137} \end{bmatrix} + (5)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right]$$

$$- \left[\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{6}{11} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{12}{25} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{60}{137} \end{bmatrix} \right] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_2 = \sum_{j=0}^{5} \frac{(j^2\alpha_j)}{2!} - \sum_{j=0}^{5} (j\beta_j) = \frac{1}{2!}((0)^2\alpha_0 + (1)^2\alpha_1 + (2)^2\alpha_2 + (3)^2\alpha_3 + (4)^2\alpha_4 + (5)^2\alpha_5)$$
$$-((0)\beta_0 + (1)\beta_1 + (2)\beta_2 + (3)\beta_3 + (4)\beta_4 + (5)\beta_5)$$

$$= \frac{1}{2!}\left[(0)^2\begin{bmatrix} -\frac{2}{11} \\ \frac{3}{25} \\ -\frac{12}{137} \end{bmatrix} + (1)^2\begin{bmatrix} \frac{9}{11} \\ -\frac{16}{25} \\ \frac{75}{137} \end{bmatrix} + (2)^2\begin{bmatrix} -\frac{18}{11} \\ \frac{36}{25} \\ -\frac{200}{137} \end{bmatrix} + (3)^2\begin{bmatrix} 1 \\ -\frac{48}{25} \\ \frac{300}{137} \end{bmatrix} + (4)^2\begin{bmatrix} 0 \\ 1 \\ -\frac{300}{137} \end{bmatrix} + (5)^2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right]$$

$$- \left[(0)\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + (1)\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + (2)\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + (3)\begin{bmatrix} \frac{6}{11} \\ 0 \\ 0 \end{bmatrix} + (4)\begin{bmatrix} 0 \\ \frac{12}{25} \\ 0 \end{bmatrix} + (5)\begin{bmatrix} 0 \\ 0 \\ \frac{60}{137} \end{bmatrix} \right] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_3 = \sum_{j=0}^{5} \frac{(j^3\alpha_j)}{3!} - \sum_{j=0}^{5} \frac{(j^2\beta_j)}{2!} = \frac{1}{3!}((0)^3\alpha_0 + (1)^3\alpha_1 + (2)^3\alpha_2 + (3)^3\alpha_3 + (4)^3\alpha_4 + (5)^3\alpha_5)$$
$$-\frac{1}{2!}((0)^2\beta_0 + (1)^2\beta_1 + (2)^2\beta_2 + (3)^2\beta_3 + (4)^2\beta_4 + (5)^2\beta_5)$$

$$= \frac{1}{3!}\left[(0)^3\begin{bmatrix} -\frac{2}{11} \\ \frac{3}{25} \\ -\frac{12}{137} \end{bmatrix} + (1)^3\begin{bmatrix} \frac{9}{11} \\ -\frac{16}{25} \\ \frac{75}{137} \end{bmatrix} + (2)^3\begin{bmatrix} -\frac{18}{11} \\ \frac{36}{25} \\ -\frac{200}{137} \end{bmatrix} + (3)^3\begin{bmatrix} 1 \\ -\frac{48}{25} \\ \frac{300}{137} \end{bmatrix} + (4)^3\begin{bmatrix} 0 \\ 1 \\ -\frac{300}{137} \end{bmatrix} + (5)^3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right]$$

$$- \frac{1}{2!}\left[(0)^2\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + (1)^2\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + (2)^2\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + (3)^2\begin{bmatrix} \frac{6}{11} \\ 0 \\ 0 \end{bmatrix} + (4)^2\begin{bmatrix} 0 \\ \frac{12}{25} \\ 0 \end{bmatrix} + (5)^2\begin{bmatrix} 0 \\ 0 \\ \frac{60}{137} \end{bmatrix} \right] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_4 = \sum_{j=0}^{5} \frac{(j^4\alpha_j)}{4!} - \sum_{j=0}^{5} \frac{(j^3\beta_j)}{3!} = \frac{1}{4!}((0)^4\alpha_0 + (1)^4\alpha_1 + (2)^4\alpha_2 + (3)^4\alpha_3 + (4)^4\alpha_4 + (5)^4\alpha_5)$$
$$-\frac{1}{3!}((0)^3\beta_0 + (1)^3\beta_1 + (2)^3\beta_2 + (3)^3\beta_3 + (4)^3\beta_4 + (5)^3\beta_5)$$

$$= \frac{1}{4!}\left[(0)^4\begin{bmatrix} -\frac{2}{11} \\ \frac{3}{25} \\ -\frac{12}{137} \end{bmatrix} + (1)^4\begin{bmatrix} \frac{9}{11} \\ -\frac{16}{25} \\ \frac{75}{137} \end{bmatrix} + (2)^4\begin{bmatrix} -\frac{18}{11} \\ \frac{36}{25} \\ -\frac{200}{137} \end{bmatrix} + (3)^4\begin{bmatrix} 1 \\ -\frac{48}{25} \\ \frac{300}{137} \end{bmatrix} + (4)^4\begin{bmatrix} 0 \\ 1 \\ -\frac{300}{137} \end{bmatrix} + (5)^4\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right]$$

$$- \frac{1}{3!}\left[(0)^3\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + (1)^3\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + (2)^3\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + (3)^3\begin{bmatrix} \frac{6}{11} \\ 0 \\ 0 \end{bmatrix} + (4)^3\begin{bmatrix} 0 \\ \frac{12}{25} \\ 0 \end{bmatrix} + (5)^3\begin{bmatrix} 0 \\ 0 \\ \frac{60}{137} \end{bmatrix} \right] = \begin{bmatrix} -\frac{3}{22} \\ 0 \\ 0 \end{bmatrix}$$

$$C_5 = \sum_{j=0}^{5}\frac{(j^5\alpha_j)}{5!} - \sum_{j=0}^{5}\frac{(j^4\beta_j)}{4!} = \frac{1}{5!}((0)^5\alpha_0 + (1)^5\alpha_1 + (2)^5\alpha_2 + (3)^5\alpha_3 + (4)^5\alpha_4 + (5)^5\alpha_5) - \frac{1}{4!}((0)^4\beta_0 + (1)^4\beta_1 + (2)^4\beta_2 + (3)^4\beta_3 + (4)^4\beta_4 + (5)^4\beta_5)$$

$$= \frac{1}{5!}\left[(0)^5\begin{bmatrix}-\frac{2}{11}\\ \frac{3}{25}\\ -\frac{12}{137}\end{bmatrix} + (1)^5\begin{bmatrix}\frac{9}{11}\\ -\frac{16}{25}\\ \frac{75}{137}\end{bmatrix} + (2)^5\begin{bmatrix}-\frac{18}{11}\\ \frac{36}{25}\\ -\frac{200}{137}\end{bmatrix} + (3)^5\begin{bmatrix}1\\ -\frac{48}{25}\\ \frac{300}{137}\end{bmatrix} + (4)^5\begin{bmatrix}0\\ 1\\ -\frac{300}{137}\end{bmatrix} + (5)^5\begin{bmatrix}0\\0\\1\end{bmatrix}\right]$$

$$-\frac{1}{4!}\left[(0)^4\begin{bmatrix}0\\0\\0\end{bmatrix} + (1)^4\begin{bmatrix}0\\0\\0\end{bmatrix} + (2)^4\begin{bmatrix}0\\0\\0\end{bmatrix} + (3)^4\begin{bmatrix}\frac{6}{11}\\0\\0\end{bmatrix} + (4)^4\begin{bmatrix}0\\ \frac{12}{25}\\0\end{bmatrix} + (5)^4\begin{bmatrix}0\\0\\ \frac{60}{137}\end{bmatrix}\right] = \begin{bmatrix}-\frac{27}{110}\\ -\frac{12}{125}\\ 0\end{bmatrix}$$

$$C_6 = \sum_{j=0}^{5}\frac{(j^6\alpha_j)}{6!} - \sum_{j=0}^{5}\frac{(j^5\beta_j)}{5!} = \frac{1}{6!}((0)^6\alpha_0 + (1)^6\alpha_1 + (2)^6\alpha_2 + (3)^6\alpha_3 + (4)^6\alpha_4 + (5)^6\alpha_5) - \frac{1}{5!}((0)^5\beta_0 + (1)^5\beta_1 + (2)^5\beta_2 + (3)^5\beta_3 + (4)^5\beta_4 + (5)^5\beta_5)$$

$$= \frac{1}{6!}\left[(0)^6\begin{bmatrix}-\frac{2}{11}\\ \frac{3}{25}\\ -\frac{12}{137}\end{bmatrix} + (1)^6\begin{bmatrix}\frac{9}{11}\\ -\frac{16}{25}\\ \frac{75}{137}\end{bmatrix} + (2)^6\begin{bmatrix}-\frac{18}{11}\\ \frac{36}{25}\\ -\frac{200}{137}\end{bmatrix} + (3)^6\begin{bmatrix}1\\ -\frac{48}{25}\\ \frac{300}{137}\end{bmatrix} + (4)^6\begin{bmatrix}0\\ 1\\ -\frac{300}{137}\end{bmatrix} + (5)^6\begin{bmatrix}0\\0\\1\end{bmatrix}\right]$$

$$-\frac{1}{5!}\left[(0)^5\begin{bmatrix}0\\0\\0\end{bmatrix} + (1)^5\begin{bmatrix}0\\0\\0\end{bmatrix} + (2)^5\begin{bmatrix}0\\0\\0\end{bmatrix} + (3)^5\begin{bmatrix}\frac{6}{11}\\0\\0\end{bmatrix} + (4)^5\begin{bmatrix}0\\ \frac{12}{25}\\0\end{bmatrix} + (5)^5\begin{bmatrix}0\\0\\ \frac{60}{137}\end{bmatrix}\right] = \begin{bmatrix}-\frac{13}{55}\\ -\frac{28}{125}\\ -\frac{10}{137}\end{bmatrix}$$

Definition2 (Error Constant): The term C_{p+1} is called the error constant and it implies that the local truncation error is given by:

$$LTE = C_{p+1}h^{p+1}y^{(p+1)}(x_n) + O(h^{p+3})$$

Hence, by the above definitions 1 and 2, we conclude that the method (15) is of order 5, with error constant given by:

$$C_6 = \begin{bmatrix}-\frac{13}{55}\\ -\frac{28}{125}\\ -\frac{10}{137}\end{bmatrix} \neq \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Definition 3: A linear multistep method LMM is said to consistent if and only if the following conditions are satisfied:

$$\left.\begin{aligned}\textstyle\sum_{j=o}^{k}\alpha_j &= 0\,,\\ \textstyle\sum_{j=o}^{k} j\alpha_j &= \textstyle\sum_{j=o}^{k}\beta_j.\end{aligned}\right\} \quad (19)$$

Lemma 1: The 3-point diagonally implicit block backward differentiation formula (3DBBDF) method:

$$\left.\begin{aligned} y_{n+1} &= \frac{2}{11}y_{n-2} - \frac{9}{11}y_{n-1} + \frac{18}{11}y_n + \frac{6}{11}hf_{n+1},\\ y_{n+2} &= -\frac{3}{25}y_{n-2} + \frac{16}{25}y_{n-1} - \frac{36}{25}y_n + \frac{48}{25}y_{n+1} + \frac{12}{25}hf_{n+2},\\ y_{n+3} &= \frac{12}{137}y_{n-2} - \frac{75}{137}y_{n-1} + \frac{200}{137}y_n - \frac{300}{137}y_{n+1} + \frac{300}{137}y_{n+2} + \frac{60}{137}hf_{n+3}.\end{aligned}\right\} \quad (20)$$

is consistent.

Proof:

To show that the 3-point diagonally implicit block BDF method is consistent, we need to show that the conditions in (19) are satisfied. Let $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ and $\beta_0, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5$ be as previously defined. Then

$$\sum_{j=0}^{5} \alpha_j = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$$

$$= \begin{bmatrix} -\frac{2}{11} \\ \frac{3}{25} \\ -\frac{12}{137} \end{bmatrix} + \begin{bmatrix} \frac{9}{11} \\ -\frac{16}{25} \\ \frac{75}{137} \end{bmatrix} + \begin{bmatrix} -\frac{18}{11} \\ \frac{36}{25} \\ -\frac{200}{137} \end{bmatrix} + \begin{bmatrix} 1 \\ -\frac{48}{25} \\ \frac{300}{137} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -\frac{300}{137} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{21}$$

Hence, the first condition in (19) is satisfied.

$$\sum_{j=0}^{5} (j\alpha_j) = ((0)\alpha_0 + (1)\alpha_1 + (2)\alpha_2 + (3)\alpha_3 + (4)\alpha_4 + (5)\alpha_5)$$

$$= \left[(0)\begin{bmatrix} -\frac{2}{11} \\ \frac{3}{25} \\ -\frac{12}{137} \end{bmatrix} + (1)\begin{bmatrix} \frac{9}{11} \\ -\frac{16}{25} \\ \frac{75}{137} \end{bmatrix} + (2)\begin{bmatrix} -\frac{18}{11} \\ \frac{36}{25} \\ -\frac{200}{137} \end{bmatrix} + (3)\begin{bmatrix} 1 \\ -\frac{48}{25} \\ \frac{300}{137} \end{bmatrix} + (4)\begin{bmatrix} 0 \\ 1 \\ -\frac{300}{137} \end{bmatrix} + (5)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right] = \begin{bmatrix} \frac{6}{11} \\ \frac{12}{25} \\ \frac{60}{137} \end{bmatrix} \tag{22}$$

$$\sum_{j=0}^{5} \beta_j = \beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{6}{11} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{12}{25} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{60}{137} \end{bmatrix} = \begin{bmatrix} \frac{6}{11} \\ \frac{12}{25} \\ \frac{60}{137} \end{bmatrix} \tag{23}$$

Hence, $\sum_{j=0}^{7} j\alpha_j = \sum_{j=0}^{7} \beta_j$

Thus, the second condition in (19) is also satisfied. The consistency conditions are therefore met. Hence, the method is consistent.

Stability Analysis of the Method

The stability of a Linear Multistep Method (LMM) is a crucial property that determines its reliability and accuracy in solving initial value problems. A stable LMM ensures that:

- Small errors or perturbations in the solution do not amplify excessively.
- The method maintains accuracy over time.
- The solution remains bounded and does not diverge.

In other words, a stable LMM prevents the growth of errors, ensuring that the numerical solution remains close to the exact solution. This is particularly important when solving stiff problems or problems with large timescales, where instability can lead to rapid error growth and inaccurate results (Yusuf et al, 2024).

There are different types of stability, including zero-stability, absolute stability, and A-stability, each with its own criteria and implications for the method's performance (Lambert, 1973). By analysing the stability properties of 3-point diagonally implicit block backward differentiation formula (3DBBDF) method, we can determine its suitability for solving various problems and ensure that it produces reliable and accurate results.

In this section, we present the stability properties of the method (15), we begin by defining a general k-step linear multistep method, zero and A-stability.

Definition 4 (Zero-Stability): The block numerical method (15) is said to be zero stable if all the roots of first characteristics polynomial have modulus less than or equal to unity and those roots with modulus unity are simple. (Alhassan et al, 2023a).

Definition 5 (A-Stability): The block numerical method (15) is said to be A-stable if the absolute stability region covers the whole left half plane (Musa et al, 2022). The stability properties of the method is determined by applying the linear test differential equation of the form

$$y' = \lambda y, \lambda < 0 \tag{24}$$

Where, λ is complex constant with $Re(\lambda) < 0$. The formulae (15) can be cast in matrix form as:

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}y_{n+1}\\y_{n+2}\\y_{n+3}\end{bmatrix}=\begin{bmatrix}\frac{2}{11}&-\frac{9}{11}&\frac{18}{11}\\-\frac{3}{25}&\frac{16}{25}&-\frac{36}{25}\\\frac{12}{137}&-\frac{75}{137}&\frac{200}{137}\end{bmatrix}\begin{bmatrix}y_{n-2}\\y_{n-1}\\y_n\end{bmatrix}+\begin{bmatrix}0&0&0\\\frac{48}{25}&0&0\\-\frac{300}{137}&\frac{300}{137}&0\end{bmatrix}\begin{bmatrix}y_{n+1}\\y_{n+2}\\y_{n+3}\end{bmatrix}+h\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}f_{n-2}\\f_{n-1}\\f_n\end{bmatrix}+$$

$$h\begin{bmatrix}\frac{6}{11}&0&0\\0&\frac{12}{25}&0\\0&0&\frac{60}{137}\end{bmatrix}\begin{bmatrix}f_{n+1}\\f_{n+2}\\f_{n+3}\end{bmatrix} \quad (25)$$

After rearranging and collecting the like terms, equation (25) becomes

$$\begin{bmatrix}1&0&0\\-\frac{48}{25}&1&0\\\frac{300}{137}&-\frac{300}{137}&1\end{bmatrix}\begin{bmatrix}y_{n+1}\\y_{n+2}\\y_{n+3}\end{bmatrix}=\begin{bmatrix}\frac{2}{11}&-\frac{9}{11}&\frac{18}{11}\\-\frac{3}{25}&\frac{16}{25}&-\frac{36}{25}\\\frac{12}{137}&-\frac{75}{137}&\frac{200}{137}\end{bmatrix}\begin{bmatrix}y_{n-2}\\y_{n-1}\\y_n\end{bmatrix}+h\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}f_{n-2}\\f_{n-1}\\f_n\end{bmatrix}+$$

$$h\begin{bmatrix}\frac{6}{11}&0&0\\0&\frac{12}{25}&0\\0&0&\frac{60}{137}\end{bmatrix}\begin{bmatrix}f_{n+1}\\f_{n+2}\\f_{n+3}\end{bmatrix} \quad (26)$$

Equation (26) can be represented in the following matrix form:

$$A_0Y_m = A_1Y_{m-1} + h(B_0F_{m-1} + B_1F_m) \quad (27)$$

where,

$$A_0=\begin{bmatrix}1&0&0\\-\frac{48}{25}&1&0\\\frac{300}{137}&-\frac{300}{137}&1\end{bmatrix}, A_1=\begin{bmatrix}\frac{2}{11}&-\frac{9}{11}&\frac{18}{11}\\-\frac{3}{25}&\frac{16}{25}&-\frac{36}{25}\\\frac{12}{137}&-\frac{75}{137}&\frac{200}{137}\end{bmatrix}, B_0=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}, B_1=\begin{bmatrix}\frac{6}{11}&0&0\\0&\frac{12}{25}&0\\0&0&\frac{60}{137}\end{bmatrix},$$

$$Y_m=\begin{bmatrix}y_{n+1}\\y_{n+2}\\y_{n+3}\end{bmatrix}, Y_{m-1}=\begin{bmatrix}y_{n-2}\\y_{n-1}\\y_n\end{bmatrix}, F_{m-1}=\begin{bmatrix}f_{n-2}\\f_{n-1}\\f_n\end{bmatrix}, F_m=\begin{bmatrix}f_{n+1}\\f_{n+2}\\f_{n+3}\end{bmatrix}$$

After applying equation (24) into equation (27), then equation (27) can also be written as:

$$A_0Y_m = A_1Y_{m-1} + \lambda h(B_0Y_{m-1} + B_1Y_m) \quad (28)$$

Substitute $\bar{h} = \lambda h$ in equation (28). This leads to

$$A_0Y_m = A_1Y_{m-1} + \bar{h}(B_0Y_{m-1} + B_1Y_m) \quad (29)$$

$$\Rightarrow \quad \left(A_0 - \bar{h}B_1\right)Y_m = \left(A_1 + \bar{h}B_0\right)Y_{m-1} \quad (30)$$

To determine stability polynomial of the method (15), the following equation is evaluated

$$det\left(\left(A_0 - \bar{h}B_1\right) - \left(A_1 + \bar{h}B_0\right)\right) = 0 \quad (31)$$

After the evaluation of equation (31), we obtain the following stability polynomial

$$R(t,\bar{h}) = -\frac{864}{7535}t^3\bar{h}^3 + \frac{26784}{37675}t^3\bar{h}^2 - \frac{864}{1507}t^2\bar{h}^2 - \frac{55134}{37675}t^3\bar{h} - \frac{13824}{7535}t^2\bar{h} - \frac{5508}{37675}t\bar{h} + t^3 - \frac{32178}{37675}t^2 - \frac{1083}{7535}t - \frac{82}{37675} = 0 \quad (32)$$

The stability boundary of the method (15) is determined by substituting $t = e^{i\theta}$into the stability polynomial (32).Thus, the region of absolute stability of the method is plotted in figure 2 below, indicating that the method is almost A-stable, as in Yusuf et al. (2024).

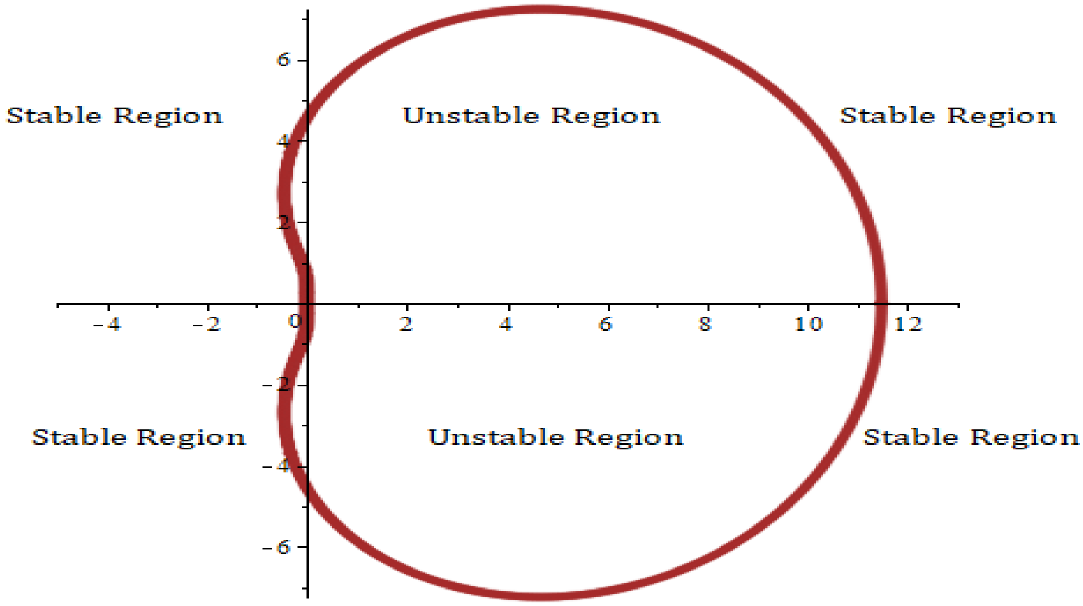

Figure 2: Stability region of 3-point diagonally implicit Block BDF method.
To show that the method is zero-stable, we substitute $\bar{h} = 0$ in (32) to obtain the first characteristics polynomial:

$$t^3 - \frac{32178}{37675}t^2 - \frac{1083}{7535}t - \frac{82}{37675} = 0 \quad (33)$$

Solving equation (33) for t, we obtain the following roots as:

$$t = 1, t = -\frac{5497}{75350} - \frac{9}{75350}\sqrt{220489}, t = -\frac{5497}{75350} + \frac{9}{75350}\sqrt{220489}$$

These roots implies that

$$t = 1, t = -0.1290386604, t = -0.01686711265$$

Therefore, in accordance with definition 4, the values of t mentioned earlier confirm that the method is zero-stable, since none of the roots have a magnitude greater than one and the root $t = 1$ has a multiplicity of one (i.e., it is a simple root).
Convergence of the Method
A crucial requirement for any Linear Multistep Methods (LMMs), including Adams, BDF, and Runge-Kutta methods, is convergence. A method that fails to converge has no practical significance, as it cannot provide a reliable approximation of the solution. Therefore, it is essential to establish the necessary and sufficient conditions for the convergence of LMMs (Butcher, 2016).
This section focuses on determining the conditions under which the LMM (15) converges. Specifically, we aim to identify the requirements that ensure the method produces a numerical solution that approaches the exact solution as the step size decreases. By establishing these conditions, we can guarantee the reliability and accuracy of the method for approximating stiff systems.
Theorem 1: The necessary and sufficient conditions for the linear multistep method (LMM) to be convergent are that it be consistent and zero-stable (Lambert, 1973).
Theorem 2: The 3-point diagonally implicit block backward differentiation formula converges.
Proof:

Having satisfied both the conditions of consistency and zero stability in the previous sections, therefore the diagonally implicit 3-point block backward differentiation formula (3DBBDF) converges and is suitable for the numerical integration of first order stiff systems of ordinary differential equations.

Implementation of the Method
Newton's iteration is used to implement the method. Let y_i and $y(x_i)$ represent the approximate and exact solutions, respectively, for first order stiff IVP of the form:

$y' = f(x, y), y(a) = y_0 a \leq x \leq b$ (34)

Then, the absolute error in the $(i)^{th}$ iteration is defined as follows:

$(error_i)_t = |(y_i)_t - y(x_i)_t|$(35)

And the maximum error is defined by:

$$MAXE = \underbrace{max}_{1 \le i \le T}\left(\underbrace{max(error_i)_t}_{1 \le i \le N}\right) \quad (36)$$

Where, N and T represent the number of equations and the total number of steps, respectively.
Let F_1, F_2 and F_3 be defined as follows:

$$\left.\begin{aligned} F_1 &= y_{n+1} - \frac{4}{7}hf_{n+1} - \epsilon_1 \\ F_2 &= -\frac{48}{25}y_{n+1} + y_{n+2} - \frac{12}{25}hf_{n+2} - \epsilon_2 \\ F_3 &= \frac{300}{137}y_{n+1} - \frac{300}{137}y_{n+2} + y_{n+3} - \frac{60}{137}hf_{n+3} - \epsilon_3 \end{aligned}\right\} \quad (37)$$

Where ϵ_1, ϵ_2 and ϵ_3 are the back values, define as follows:

$$\left.\begin{aligned} \epsilon_1 &= \frac{2}{11}y_{n-2} - \frac{9}{11}y_{n-1} + \frac{18}{7}y_n \\ \epsilon_2 &= -\frac{3}{25}y_{n-2} + \frac{16}{25}y_{n-1} - \frac{36}{25}y_n \\ \epsilon_3 &= \frac{12}{137}y_{n-2} - \frac{75}{137}y_{n-1} + \frac{200}{137}y_n \end{aligned}\right\} \quad (38)$$

Then, let $y_{n+j}^{(i+1)}$ denote the $(i+1)^{th}$ iterative values of y_{n+j} and define

$$e_{n+j}^{(i+1)} = y_{n+j}^{(i+1)} - y_{n+j}^{(i)}, \qquad j = 1,2,3 \quad (39)$$

Applying the Newton's iteration for the 3DBBDF method, we get;

$$e_{n+j}^{(i+1)} = -\left(F_j'(y_{n+j}^{(i)})\right)^{-1}\left(F_j(y_{n+j}^{(i)})\right), \quad j = 1,2,3 \quad (40)$$

This can be written as;

$$\left(F_j'(y_{n+j}^{(i)})\right)e_{n+j}^{(i+1)} = -\left(F_j(y_{n+j}^{(i)})\right), \quad j = 1,2,3 \quad (41)$$

Equation (41) can be written in matrix form as follows;

$$\begin{bmatrix} \left(1-\frac{6}{11}h\frac{\partial F_{n+1}^{(i)}}{\partial y_{n+1}^{(i)}}\right) & 0 & 0 \\ -\frac{48}{25} & \left(1-\frac{12}{25}h\frac{\partial F_{n+2}^{(i)}}{\partial y_{n+2}^{(i)}}\right) & 0 \\ \frac{300}{137} & -\frac{300}{137} & \left(1-\frac{60}{137}h\frac{\partial F_{n+3}^{(i)}}{\partial y_{n+3}^{(i)}}\right) \end{bmatrix}\begin{bmatrix} e_{n+1}^{(i+1)} \\ e_{n+2}^{(i+1)} \\ e_{n+3}^{(i+1)} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ \frac{48}{25} & -1 & 0 \\ -\frac{300}{137} & \frac{300}{137} & -1 \end{bmatrix}\begin{bmatrix} y_{n+1}^{(i)} \\ y_{n+2}^{(i)} \\ y_{n+3}^{(i)} \end{bmatrix} + \begin{bmatrix} \frac{6}{11} & 0 & 0 \\ 0 & \frac{12}{25} & 0 \\ 0 & 0 & \frac{60}{137} \end{bmatrix}\begin{bmatrix} f_{n+1}^{(i)} \\ f_{n+2}^{(i)} \\ f_{n+3}^{(i)} \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix} \quad (42)$$

A computer program in C language is developed to implement equation (42). The program takes in input parameters such as the exact solutions, initial conditions, and step sizes, and outputs the approximate solution at each step of the integrations.

The detailed derivation of coefficients of the 3-point explicit predicator method that are stored in the code can be found in (Yusuf et al., 2024)and are given by the following tables:

Table 1: coefficient of the first point

$\sigma_{0,1}$	$\sigma_{1,1}$	$\sigma_{2,1}$	$\sigma_{3,1}$	$\sigma_{4,1}$
1	−4	6	−4	1

Table 2: coefficient of the second point

$\sigma_{0,2}$	$\sigma_{1,2}$	$\sigma_{2,2}$	$\sigma_{3,2}$	$\sigma_{5,2}$
4	−15	20	−10	1

Table 3: coefficient of the third point

$\sigma_{0,3}$	$\sigma_{1,3}$	$\sigma_{2,3}$	$\sigma_{3,3}$	$\sigma_{6,3}$
10	−36	45	−20	1

Test Problems and Numerical Results

To evaluate the performance and efficiency of the 3DBBBDF method, several systems of first-order linear and nonlinear stiff ordinary differential equations (ODEs) with initial value problems are tested. These systems are chosen to assess the method's ability to handle a range of stiff problems:

Problem 1:

$$y'_1 = -(\epsilon^{-1}+2)y_1 + \epsilon^{-1}y_2^2, \qquad y_1(0)=1, \qquad 0 \le x \le 20,$$
$$y'_2 = y_1 - y_2(1+y_2), \qquad y_2(0)=1.$$

Exact solution:

$$y_1(x) = e^{-2x},$$
$$y_2(x) = e^{-x}.$$

Eigenvalues: $\lambda_1 = -1$ and $\lambda_2 = -(\epsilon^{-1}+2)$ and we set $\epsilon = 10^{-3}$

Source: Suleiman et al (2013).

Problem 2:

$$y'_1 = -3y_1 + 2y_2 + 3cos(x) - 3sin(x), \qquad y_1(0)=1, \qquad 0 \le x \le 20,$$
$$y'_2 = 2y_1 - 3y_2 - cos(x) + 3sin(x), \qquad y_2(0)=0.$$

Exact solution:

$$y_1(x) = cos(x),$$
$$y_2(x) = sin(x).$$

Eigenvalues: $\lambda_1 = -1$ and $\lambda_2 = -5$

Source: Aminikhah and Hemmatnezhad (2011).

Problem 3:

$$y'_1 = -2y_1 + y_2 + 2sin(x), \qquad y_1(0)=2, \qquad 0 \le x \le 10,$$
$$y'_2 = -(\epsilon^{-1}+2)y_1 + (\epsilon^{-1}+1)\big(y_2 - cos(x) + sin(x)\big), y_2(0)=3.$$

Exact solution:

$$y_1(x) = 2e^{-x} + sin(x),$$
$$y_2(x) = 2e^{-x} + cos(x).$$

Eigenvalues: $\lambda_1 = -1$and $\lambda_2 = \epsilon^{-1}$.

Source: Aminikhah and Hemmatnezhad (2011).

Problem 4:

$$y'_1 = (0)y_1 + y_2, \qquad y_1(0)=1.01, \qquad 0 \le x \le 10,$$
$$y'_2 = -100y_1 - 101y_2\,, \qquad y_2(0)=-2.$$

Exact solution:

$$y_1(x) = 0.01e^{-100x} + e^{-x},$$
$$y_2(x) = -e^{-100x} - e^{-x}.$$

Eigenvalues: $\lambda_1 = -100$and $\lambda_2 = -1$.

Source: Yaakub and Evans (2003).

The fully implicit 3-point Block Backward Differentiation Formula and diagonally implicit 3-point Block Backward Differentiation Formula methods are applied to solve the aforementioned problems. To evaluate the accuracy and efficiency of these methods, the maximum absolute errors for various step sizes (H) are presented for each problem. Additionally, the number of steps required to solve each problem and the computation time are also provided. To make referencing easy, below is the 3 – point BBDF method found in Ibrahim et al (2007):

$$\left.\begin{aligned} y_{n+1} &= \frac{1}{10}y_{n-2} - \frac{3}{4}y_{n-1} + 3y_n - \frac{3}{2}y_{n+2} + \frac{3}{20}y_{n+3} + 3hf_{n+1} \\ y_{n+2} &= -\frac{3}{65}y_{n-2} + \frac{4}{13}y_{n-1} - \frac{12}{13}y_n + \frac{24}{13}y_{n+1} - \frac{12}{65}y_{n+3} + \frac{12}{13}hf_{n+2} \\ y_{n+3} &= \frac{12}{137}y_{n-2} - \frac{75}{137}y_{n-1} + \frac{200}{137}y_n - \frac{300}{137}y_{n+1} + \frac{300}{137}y_{n+2} + \frac{60}{137}hf_{n+3} \end{aligned}\right\} \tag{44}$$

Andthe 3-point diagonally implicit BBDF method is expressed as:

$$\left.\begin{aligned} y_{n+1} &= \frac{2}{11}y_{n-2} - \frac{9}{11}y_{n-1} + \frac{18}{11}y_n + \frac{6}{11}hf_{n+1} \\ y_{n+2} &= -\frac{3}{25}y_{n-2} + \frac{16}{25}y_{n-1} - \frac{36}{25}y_n + \frac{48}{25}y_{n+1} + \frac{12}{25}hf_{n+2} \\ y_{n+3} &= \frac{12}{137}y_{n-2} - \frac{75}{137}y_{n-1} + \frac{200}{137}y_n - \frac{300}{137}y_{n+1} + \frac{300}{137}y_{n+2} + \frac{60}{137}hf_{n+3} \end{aligned}\right\} \tag{45}$$

The following notations are used in the tables below:

H: Step Size
METHOD: The Methods Used
TS: Total Number of Steps taken to complete the Integration
MAXE: Maximum Error
CPU TIME: Computation time in seconds
3BBDF: 3-point block backward differentiation formula
3DBBDF: 3-point diagonally implicit block backward differentiation formula

Table 4: Numerical Result for Problem 1.

H	METHOD	TS	MAXE	CPU TIME
10^{-2}	3BBDF	666	1.01454E+251	5.78100E-002
	3DBBDF	666	4.91435E+159	5.82200E-003
10^{-3}	3BBDF	6666	2.21008E+210	7.86000E-002
	3DBBDF	6666	5.72422E+168	6.42500E-003
10^{-4}	3BBDF	66666	1.10663E-004	8.34700E-001
	3DBBDF	66666	1.10662E-004	1.55600E-002
10^{-5}	3BBDF	666666	1.10748E-005	1.36400E+000
	3DBBDF	666666	1.10748E-005	1.41700E-001
10^{-6}	3BBDF	6666666	1.10756E-006	1.00900E+001
	3DBBDF	6666666	1.10755E-006	5.91300E-001

Table 5: Numerical Result for Problem 2.

H	METHOD	TS	MAXE	CPU TIME
10^{-2}	3BBDF	666	1.79395E-002	6.26800E-002
	3DBBDF	666	1.79396E-002	5.86900E-003
10^{-3}	3BBDF	6666	1.76790E-003	7.48400E-002
	3DBBDF	6666	1.76790E-003	7.72200E-003
10^{-4}	3BBDF	66666	1.76533E-004	2.64900E-001
	3DBBDF	66666	1.76533E-004	2.65600E-002
10^{-5}	3BBDF	666666	1.76511E-005	2.14000E+000
	3DBBDF	666666	1.76511E-005	2.09500E-001
10^{-6}	3BBDF	6666666	1.76511E-006	2.04000E+001
	3DBBDF	6666666	1.76512E-006	1.15700E+001

Table 6: Numerical Result for Problem 3.

H	METHOD	TS	MAXE	CPU TIME
10^{-2}	3BBDF	333	1.42161E-001	6.17200E-002
	3DBBDF	333	1.42198E-001	5.62000E-003
10^{-3}	3BBDF	3333	1.39302E-002	6.23100E-002
	3DBBDF	3333	1.39306E-002	6.84900E-003
10^{-4}	3BBDF	33333	1.39012E-003	2.42200E-001
	3DBBDF	33333	1.39012E-003	1.96100E-002
10^{-5}	3BBDF	333333	1.38983E-004	1.14000E+000
	3DBBDF	333333	1.38983E-004	1.05800E-001
10^{-6}	3BBDF	3333333	1.38982E-005	1.03000E+001
	3DBBDF	3333333	1.38982E-005	1.18700E+001

Table 7: Numerical Result for Problem 4.

H	METHOD	TS	MAXE	CPU TIME
10^{-2}	3BBDF	333	5.08510E+127	7.90600E-002
	3DBBDF	333	1.68135E+131	5.64500E-003
10^{-3}	3BBDF	3333	6.92468E-002	6.24200E-002
	3DBBDF	3333	7.18991E-002	6.99600E-003
10^{-4}	3BBDF	33333	1.07293E-002	1.14300E-001
	3DBBDF	33333	1.07266E-002	1.10900E-002
10^{-5}	3BBDF	333333	1.10089E-003	6.15300E-001
	3DBBDF	333333	1.10083E-003	8.78800E-002
10^{-6}	3BBDF	3333333	1.10363E-004	6.88200E-001
	3DBBDF	3333333	1.10362E-004	7.77700E-001

The graph below provides a visual representation of the method's efficiency, showing the relationship between the logarithm of the maximum error (Log(MAXE)) and the logarithm of the computational time (Log(Time)).

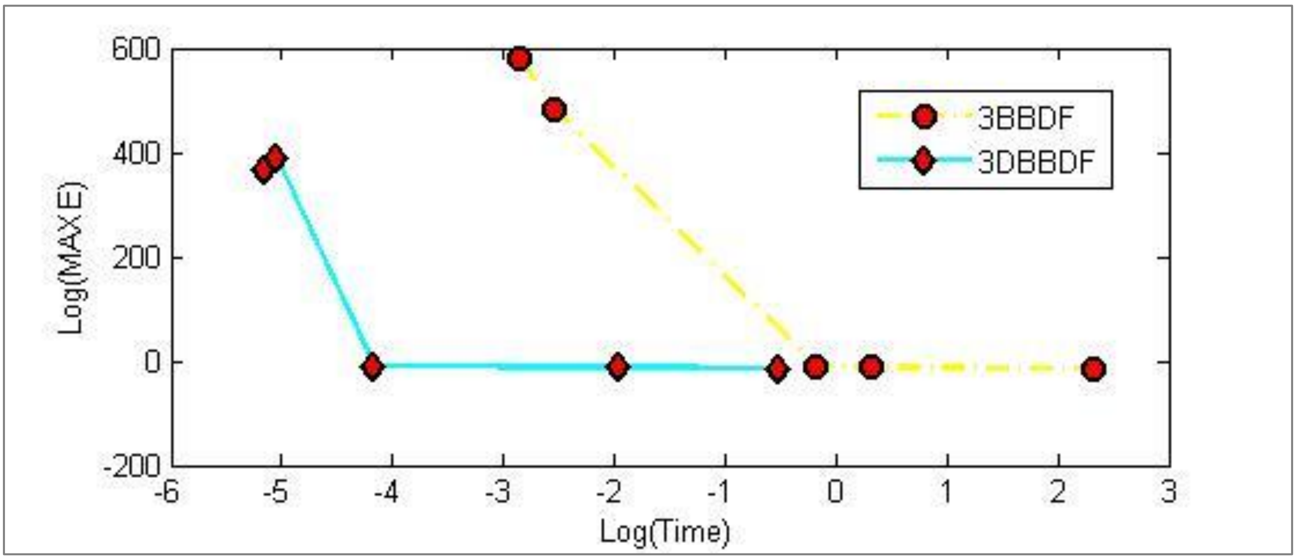

Figure 3: Efficiency Curves for Problem 1

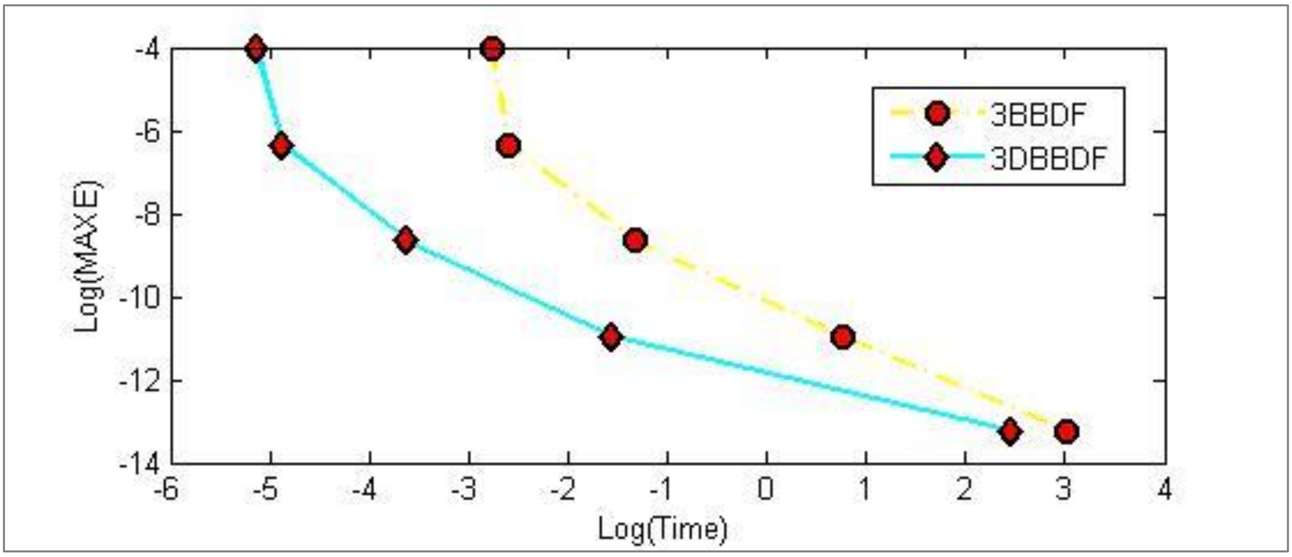

Figure 4: Efficiency Curves for Problem 2

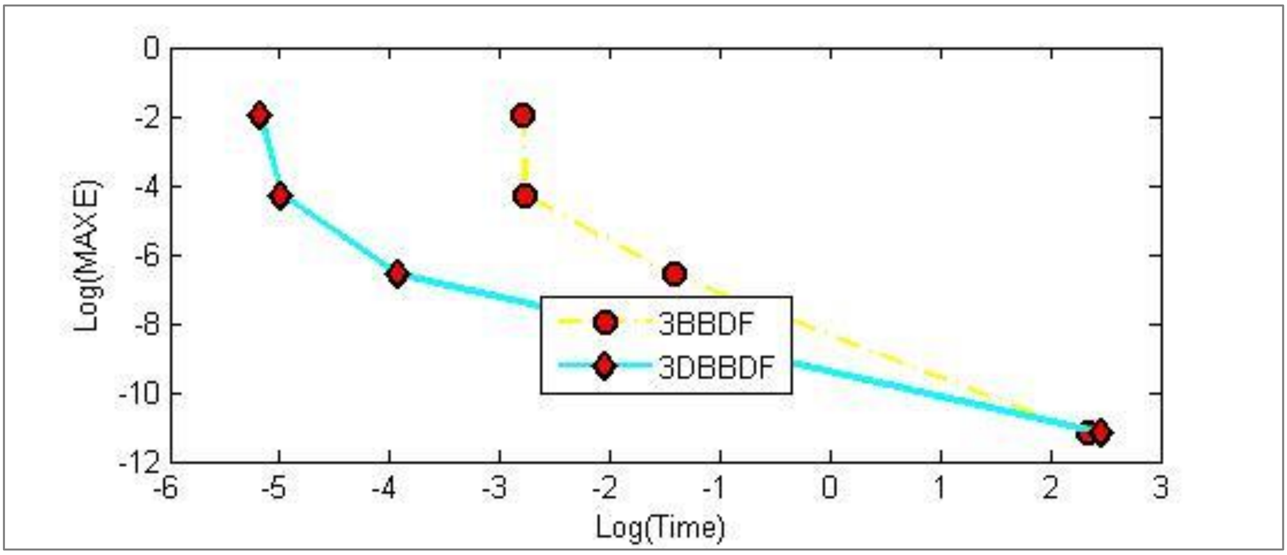

Figure 5: Efficiency Curves for Problem 3

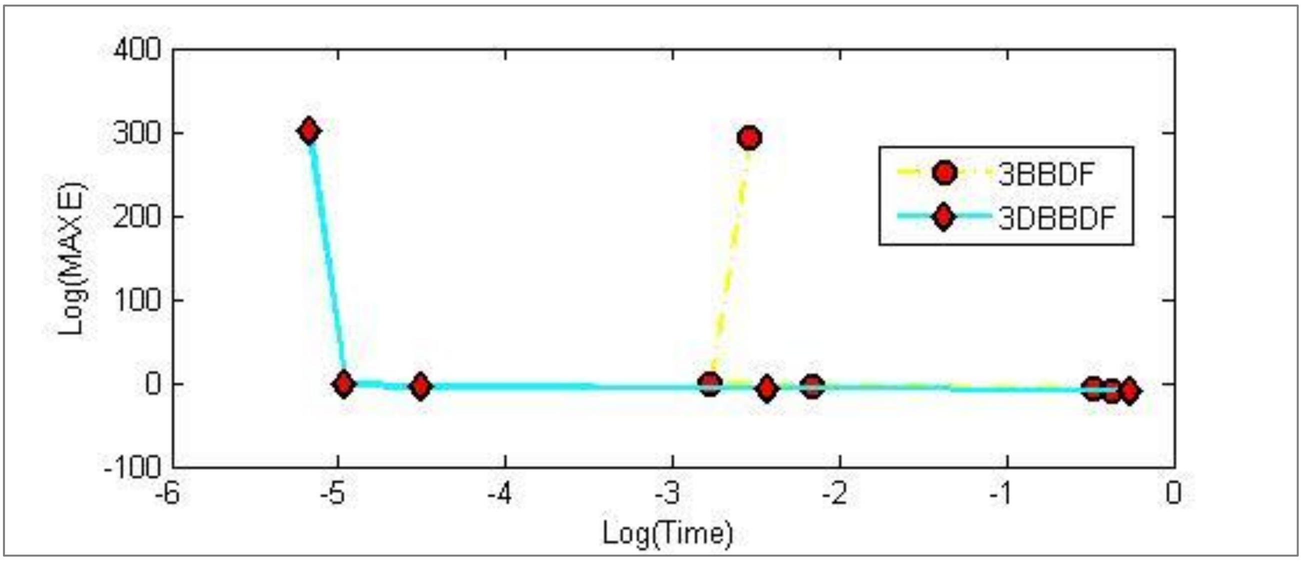

Figure 6: Efficiency Curves for Problem 4

DISCUSSION ON THE RESULTS

Based on the data presented in the tables and figures, the following observations and conclusions can be made:

- **Accuracy:** The 3DBBDF method consistently achieves smaller maximum absolute errors across nearly all step sizes compared to the 3BBDF method. This demonstrates the reliability of the 3DBBDF method for solving the given ODE problems.
- **Efficiency:** The 3DBBDF method generally shows shorter computational times for most step sizes, indicating it is more efficient then the 3BBDF method. This suggests that the 3DBBDF method uses less computational resources and is faster for solving the problems.
- **Step size Influence:** The results show that as the step size (H) decreases, the accuracy of both methods improves, but the computational time increases. This is expected, as smaller step sizes require more steps to solve the problems, leading to increased computational time.
- **Comparison:** The 3DBBDF method is competitive with the 3BBDF method in terms of both accuracy and efficiency, likely due to the improved numerical stability and consistency of the 3DBBDF method.

Therefore, based on the results, we conclude that the 3DBBDF method is an accurate and efficient numerical method for solving the given ODE problems, outperforming the 3BBDF method in several aspects.

CONCLUSION

The diagonally implicit 3-point block backward differentiation formula (3DBBDF) is derived using Lagrange interpolation polynomial for solving first-order stiff initial value problems. The stability properties of the method are thoroughly examined, and the necessary and sufficient conditions for its convergence are discussed.

The method's order of accuracy is also derived and found to be 5, indicating its high accuracy.

The numerical results consistently show that the 3DBBDF method achieves smaller maximum errors and shorter computational time, making it both more accurate and efficient. This enhanced performance is attributed to the improved numerical stability and consistency of the 3DBBDF method. Therefore, the 3DBBDF method is recommended as a more reliable and effective tool for the numerical integrating of first-order stiff initial value problems, offering significant advantages over the 3BBDF method.

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