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ORIGINAL RESEARCH ARTICLE

On some Asymptotic Properties of the Extended Cosine Burr XII Distribution

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ABSTRACT
In statistics, asymptotic functions provide a powerful framework for understanding the behavior of functions or statistical procedures within the limits of certain parameters. Also, in reliability, mean residual life assumes a critical role in evaluating the durability and effectiveness of diverse components. In addition, order statistics are essential tools in extreme value analysis. In this article, a new probability model is proposed based on the extended cosine-G family of distributions, called extended cosine Burr XII (ECSBXII) distribution. Some important basic properties are derived. We drive and study the asymptotic of mean residual life and asymptotic of the extreme order statistics of the ECSBXII distribution. An illustrative application of asymptotic MRL of ECBXII is given using simulation studies.

INTRODUCTION

In studies related to reliability and other statistical analyses, the concept of mean residual life (MRL) holds substantial importance, featuring a broad spectrum of applications, refer to (Chen et al., 2023). MRL assumes a critical role in evaluating the durability and effectiveness of diverse components, spanning from mechanical systems to software and extending to survival analysis in medical research. Its capability to predict the potential failure of a system underscores its significance. This discussion explores the importance and practical uses of mean residual life, highlighting how this concept empowers reliability engineers and statisticians to improve the quality, safety, and efficiency of various systems and processes. A recent study by (Hall & Wellner 2020) extensively delves into the estimation of mean residual life, contributing valuable insights to this field.

Definition 1. *The mean residual life (MRL) is the average time above t until failure, provided a component or subject has survived up to the time t. Given a random variable X, the MRL is expressed by*

$$M_X(t) = E(X - t|X > t),$$

$$M_X(t) = \int_0^{\infty} \frac{R(x+t)}{R(T)} dx, \qquad (1)$$

Where R(t) is the survival function X.

On the other hand, order statistics play a substantial role in theoretical studies and practice, particularly in dealing with the minimum, maximum, and range and in studying a particular X_i. Order statistics are crucial tools in extreme value study, analysis of random phenomena in quality control, and life testing, among other fields. One can see (Chua et al., 2010), (Abba et al., 2023), (Serinaldi et al., 2020), and (Koutras & Koutras 2020).

Definition 2. *Given an n-size random sample denoted by* $X_1, X_2, \ldots, X_n,\ i = 1, 2, \ldots n.$, *the order statistics are the* $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$, *expressed in function of* $X_1, X_2, \ldots, X_n$, *such that* $X_{(1)} = \inf(X_1, X_2, \ldots, X_n)$, $X_{(n)} = \sup(X_1, X_2, \ldots, X_n)$ *and* $P(X_{(1)} \leq X_{(2)} \leq \cdots, X_{(n)}) = 1$.

In this work, we derive a new probability model from so-called extended cosine Burr XII (ECS- BXII) distribution, and some basic properties are derived. We discussed the asymptotic of MRL and asymptotic of the extreme order statistics of the ECSBXII distribution by initially obtaining some asymptotic functions of the ECSBXII such as the asymptotic of the cumulative distribution function and asymptotic of the reliability function.

The rest of the work follows: in Section 2, we derive the ECSBXII. In Section 3, the asymptotic of MRL and asymptotic of the extreme order statistics of the ECS-BXII are discussed. Conclusion in section 4.

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THE ECSBXII DISTRIBUTION

(Chen et al., 2023) Proposed the general form of the CDF of the extended cosine-G (ECSG) family of distribution as

$$D(x) = 1 - \left(1 - cos\left(\frac{\pi}{2} s(x,\xi)\right)\right)^{\alpha} \quad (2)$$

The related probability density function (PDF) reliability function are respectively given as

$$d(x) = \alpha \frac{\pi}{2} g(x,\xi)\left(1 - cos\left(\frac{\pi}{2} s(x,\xi)\right)\right)^{\alpha-1}, \quad (3)$$

$$R(x) = \left(1 - cos\left(\frac{\pi}{2} s(x,\xi)\right)\right)^{\alpha}, \quad (4)$$

Where $g(x,\xi)$ and $s(x,\xi)$ are the PDF and reliability function of any valid distribution, ξ a vector parameters. In our case, we consider $g(x,\xi)$ and $s(x,\xi)$ to be from the BurrXII distribution proposed by (Al-Essa et al., 2023) with density functions and reliability functions:

$$g(x,\lambda,\theta) = \lambda\theta x^{\theta-1}(1+x^{\theta})^{-\lambda},$$

$$s(x,\lambda,\theta) = (1+x^{\theta})^{-\lambda-1}, \quad (5)$$

respectively, were $x\ \lambda,\theta > 0$. There are several extensions and applications of BXII in the literature; for example, exponentiated exponential Burr XII was proposed by (Badr & Ijaz, 2021), complementary exponentiated Burr XII Poisson Muhammad (2017), and generalized of the Burr XII-Poisson (Muhammad, 2016), among others. For some recent applications of BXII one can see (Du & Gui, 2022). For statistical inference of burr-xii distribution with adaptive type II progressive censored strategies with competing risks, and (Saini & Garg, 2022) for the reliability inference of multicomponent stress–strength model by employing the Burr XII distribution under progressively first-failure censored samples, etc.

Hence, the CDF of the ECSBXII is derived using (5) and the corresponding PDF and reliability functions as:

Hence, the CDF, PDF and reliability functions of the ECSBXII are derived as follows:

$$F(x) = \left(1 - cos\ cos\ \left(\frac{\pi}{2}(1+x^{\theta})^{-\lambda}\right)\right)^{\alpha} \quad (6)$$

$$f(x) = \frac{\alpha\lambda\theta\pi}{2} x^{\theta-1}(1+x^{\theta})^{-\lambda-1}\ sin\ sin\ \left(\frac{\pi}{2}(1+x^{\theta})^{-\lambda}\right)\left(1 - cos\ cos\ \left(\frac{\pi}{2}(1+x^{\theta})^{-\lambda}\right)\right)^{\alpha-1} \quad (7)$$

$$R(x) = 1 - \left(1 - cos\ cos\ \left(\frac{\pi}{2}(1+x^{\theta})^{-\lambda}\right)\right)^{\alpha} \quad (8)$$

The failure rate (FR) function of the ECBXII is derived as

$$h(x) = \alpha\lambda\pi x^{\theta-1}(1+x^{\theta})^{-\lambda-1} \frac{sin\ sin\ \left(\frac{\pi}{2}(1+x^{\theta})^{-\lambda}\right)\left(1 - cos\ cos\ \left(\frac{\pi}{2}(1+x^{\theta})^{-\lambda}\right)\right)^{\alpha-1}}{2\left(1 - \left(1 - cos\ cos\ \left(\frac{\pi}{2}(1+x^{\theta})^{-\lambda}\right)\right)^{\alpha}\right)} \quad (9)$$

Figure 1 shows the plot of the possible shape of the ECSBXII density functions and FR function for some parameter values.

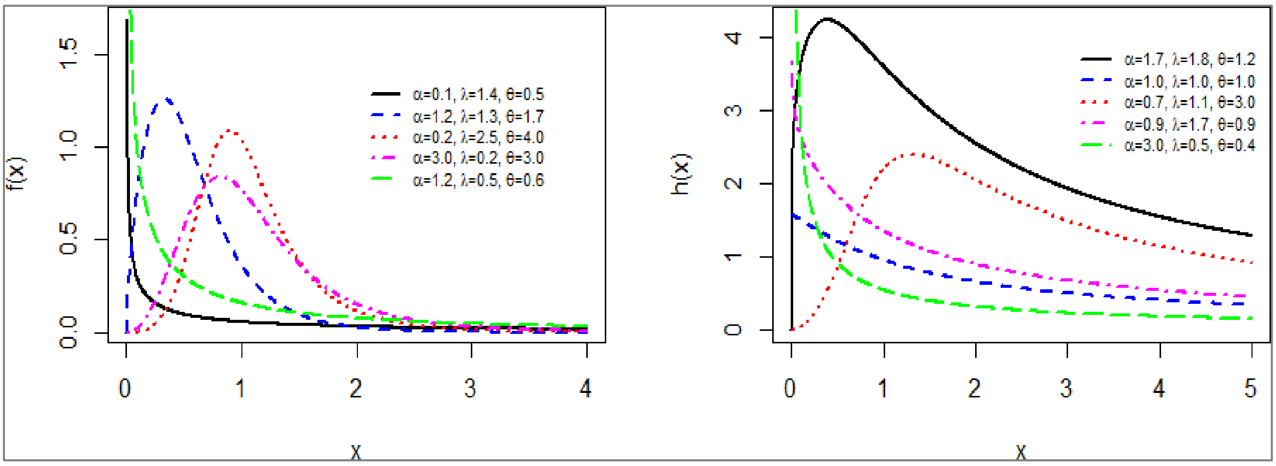

Figure 1: Plot of the PDF and FR of the ECSBXII for some parameter values

THE ASYMPTOTIC OF MRL AND ASYMPTOTIC OF THE EXTREME ORDER STATISTICS FOR THE ECSBXII

Asymptotic functions play a crucial role in both mathematics and statistics. For example, in statistics, asymptotic theory is employed to make inferences about population parameters based on sample data. Asymptotic results often provide insights into the behavior of estimators and test statistics as the sample size becomes large. In both mathematics and statistics, asymptotic functions provide a powerful framework for understanding the behavior of functions, algorithms, and statistical procedures in the limit as certain parameters, such as input size or sample size, approach infinity. In this section, we derive and discuss the asymptotic of mean residual life and asymptotic of the extreme order statistics for the ECSBXII. First, we provided some important Lemma as follows:

Lemma 3.1. (Muhammad et al., 2021).*The asymptotic of the CDF in (2) and reliability function in (4) are given by.*

$$F(x) \sim \alpha \cos\cos\left(\frac{\pi}{2}s(x,\xi)\right). \quad (10)$$

$$R(x) \sim \left(\frac{1}{2}\right)^{\alpha} Sin^{2\alpha}\left(\frac{\pi}{2}s(x,\xi)\right) \sim \left(\frac{1}{2}\right)^{\alpha}\left(\frac{\pi}{2}\right)^{2\alpha} s^{2\alpha}(x,\xi) \quad (11)$$

The asymptotic of MRL for the ECSBXII

The MRL of the ECSBXII distribution is no doubt complicated to derive because it contains expressions that require huge approximations and expansions, one can see (Muhammad et al., 2021), (Muhammad et al., 2021), (Muhammad et al., 2023) to verify. In some models, the MRL is not compatible by taking direct integral at any given parameter; Monte Carlo integral is needed to be employed to evaluate MRL using the sampling technique, one can see (Ahsanullah et al., 2013), (Arnold et al., 1992), Abba & Wang (2023). Related studies regarding the derivation of asymptotic MRL of some models can be found in some chapters of Muhammad (2023) & Muhammad & Liu (2021), among others.

Now, we utilized Lemma 3.1 and the given Lemma below to derive the asymptotic of MRL for the ECSBXII distribution:

Lemma 3.2. (Gradshteyn & Ryzhik, 2007, p. 315) *$u, t, \mu > 0$ be real numbers such that $\mu > \mu > 0$, then*

$$\int_t^{\infty} y^{-u}(y-t)^{\mu-1}dy = t^{\mu-u}B(u-\mu,\mu),$$

Where $B(\dots)$ is a beta function.

Theorem 3.3. *Let X be a random with ECSBXII, for $\alpha\theta\lambda > \frac{1}{2}$, and for a very large $t > 0$, i.e., as $t \to \infty$ the mean residual life is*

$$M_X(t) \sim tB(2\alpha\theta\lambda - 1, 1).$$

Proof. From the (11), the asymptotic of the reliability function of the ECSBXII as $x \to \infty$ is

$$R(x) \sim \left(\frac{1}{2}\right)^{\alpha}\left(\frac{\pi}{2}\right)^{2\alpha}(1 + x^{\theta})^{-2\alpha\lambda} \sim \left(\frac{1}{2}\right)^{\alpha}\left(\frac{\pi}{2}\right)^{2\alpha} x^{-2\alpha\theta\lambda} \quad (12)$$

Thus, we can derive the asymptotic of the $M_X(t)$ for the ECSBXII as $t \to \infty$. By letting $u = x + t$;

$$M_X(t) = \int_0^{\infty} \frac{R(x+1)}{R(t)}dx \sim t^{2\alpha\theta\lambda}\int_0^{\infty}(x+t)^{2\alpha\theta\lambda}dx$$

$$M_X(t) \sim t^{2\alpha\theta\lambda}\int_t^{\alpha} u^{-2\alpha\theta\lambda}du$$

by applying the Lemma 3.2,

$$M_X(t) \sim tB(2\alpha\theta\lambda - 1, 1).$$

The asymptotic of the extreme order statistics for the ECSBXII

Let $X_1, X_2, \dots, X_n, n \geq 1$, be an ordered sample from a probability distribution, then the PDF of the j^{th} − order statistic represented by $f_{j:n}(x)$, is expressed as

$$f_{j:n}(x) = \frac{n!}{(j-1)!\,(n-j)!} f(x)\big(F(x)\big)^{j-1}\big(1 - F(x)\big)^{n-j},$$

Where, for $F(x)$ and $f(x)$ are given in are given in (6) and (7), respectively.

Now, we can derive the extreme order statistics asymptotic distributions, i.e., $X_{1:n}$ and $X_{n:n}$ from $X_1, X_2, X_3, \dots, X_n$, follow ECSBXII. Based on the detailed information in (Al-Essa et al., 2023, p. 50130–50144.), let $d \to$ denote convergence in distribution, let W denote a random variable with cdf, G, then, claiming that the cdf, F is contained in the domain of maximal attraction of G is equally as saying $\frac{(X_{n,n} - a_n)}{b_n\, d \to W,}$ on condition that a sequence $\{a_n\}$ and $\{b_n > 0\}$ exist. Take W^* as a random variable with cdf, G^*, then, to claim that the cdf F is contained in the domain of minimal attraction of G^* is equally as saying $\frac{(X_{1,n} - a_n^*)}{b_n^*\, d \to W^*,}$ so long as the sequence $\{a_n^*\}$ and $\{b_n^* > 0\}$ exist.

Theorem 3.4. *Let* $\{X_i\}_{i=1}^{n}$ *be a random sample from ECSBXII, let* $B_n = \frac{(X_{n:n}-a_n)}{b_{n'}}$ then $B_n\ d \to B$ *indicated that*

$$P(B_n \leq x) = G(x) = e^{-x^{-2\alpha\theta\lambda}},$$

for every point $x \in R$ *of* $G(x)$ *for which* $G(x)$ *is continuous. The normalizing constant can be determined following Theorem 8.3.4 of* (Al-Essa et al., 2023, p. 50130–50144.) *as* $a_n = 0$ *and* $b_n = F^{-1}\left(1-\frac{1}{n}\right)$.

Proof. From the Theorem 8.3.2 of (Al-Essa et al., 2023, p. 50130–50144.) [4], and the asymptotic of *R*(*x*) in equation (12) above, we have

$$\frac{R(tx)}{R(t)} \sim \frac{(tx)^{-2\alpha\theta\lambda}}{t^{-2\alpha\theta\lambda}} = x^{-2\alpha\theta\lambda}.$$

Theorem 3.5. *Let* $\{X_i\}_{i=1}^{n}$ *be a random sample from ECSBXII, let* $B_n^* = \frac{(X_{1:n}-a_n^*)}{b_{n'}^*}$ *then* $B_n^*\ d \to B^*$ *is implies that*

$$P(B_n^* \leq x) = G^*(x;\theta) = 1 - e^{-x^{\theta}},$$

For every point $x \in R^+$ *of* $G^*(x;\theta)$ *for which* $G^*(x;\theta)$ *is continuous. The normalizing constant can be derived the Theorem 8.3.6 of* (Al-Essa et al., 2023, p. 50130–50144.) *[4], thus,* $a_n^* = 0$ *and* $F^{-1}\left(\frac{1}{n}\right)$.

Proof. From the Lemma 3.1 equation (11), and the Theorem 8.3.6 of (Al-Essa et al., 2023, p. 50130–50144.), we get,

$$\frac{F(F^{-1}(0)+tx)}{F(F^{-1}(0)+t)} \sim \frac{\cos\cos\left(\frac{\pi}{2}\left(1+(tx)^{\theta}\right)^{-\lambda}\right)}{\cos\cos\left(\frac{\pi}{2}(1+t^{\theta})^{-\lambda}\right)} = x^{\theta}.$$

SIMULATION

Next, we fitted the asymptotic of MRL of ECSW to some suitable simulated data set for demonstration. We used simulation due to the necessary condition on the parameters for the asymptotic of MRL of ECSW to exist, because finding suitable real data is not certain.

We simulated data in Table 2, 3, 4 and 5 from ECSW as follows:

Table 1: Data generating process

1. Choose the sample size *n* and the values of parameters α, θ and λ
2. Generate random sample $U_i \sim uniform(0,1)$ distribution, $i = 1,2,3,\ldots,n$.
3. Generate random sample $X_i \sim ECSBXII\ i = 1,2,3,\ldots,n$. Using $$X_i = \left[\left(\frac{\pi}{2} across\left(1-(1-u_i)^{\frac{1}{\alpha}}\right)\right)^{\frac{-1}{\lambda}} - 1\right]^{\frac{1}{\theta}}$$

Table 2: First simulated data set: We have drawn this data of size 50 by choosing α=0.08, θ=2.5 and λ=0.5, this satisfied 2αθλ>1.

1.0526863, 0.7592534, 0.6199662, 0.8458926, 1.4466212, 0.9477421, 3.0628151, 1.1563635, 1.0027766, 4.3769630,
1.3499269, 2.0325724, 4.0858710, 12.2951947, 1.5724289, 1.3628819, 2.9104412, 0.3677692, 0.9983797, 0.5418595,
0.6193770, 0.4555360, 0.9372574, 0.6227970, 1.6177902, 0.7998012, 0.6682441, 1.3015635, 1.1718817, 0.9652878,
0.2432195, 2.5558146, 0.7921677, 0.6560692, 1.8018975, 2.1471022, 0.7028552, 3.2714242, 0.6163472, 1.3733793,
1.0069490, 0.9576615, 1.6597047, 0.8709967, 0.5631992, 1.7340274, 0.7442131, 0.9937960, 0.6482892, 5.134060

Table 3: Second simulated data set: We have drawn this data of size 50 by selecting α=0.8, θ=0.9 and λ=1.9, also satisfied 2αθλ>1.

0.1487469904,	0.3497698184,	0.3270217450,	0.1587141255,	0.0001295949,	0.3274437413,	0.0431751702,
0.0883832411,	0.1159976864,	0.1833147067,	0.0304904596,	0.6717685272,	0.0574652540,	0.0124909890,
0.5219354780,	0.7783300370,	0.4892950387,	0.0131813038,	0.0464725191,	0.7688078808,	0.1408983738,
0.1744707690,	0.0247276823,	0.3090570888,	0.4993275507,	0.2267023790,	1.0682109251,	0.0198553691,
0.0637601647,	0.4667411720,	0.0441704883,	0.5777042682	,0.0956217975,	0.1482966344,	0.0344888000,
0.5799731889,	0.0765468260,	0.4882239330,	2.5517335519,	0.0316012322,	0.2174329972	0.3089546985,
0.2462696853,	0.3977333498,	0.1824579543,	0.4428262937,	0.0038487734,	0.8168070320,	0.1687887830,
0.7258260875						

Table 4: Third simulated data set: We simulated the data of size 50 by selecting α=1.0, θ=1.0 and λ=1.0, also satisfied 2αθλ>1.

0.317424722	0.436271523	0.283245708	0.211991164	0.412140211	0.747203967
0.415501799	0.395017732	0.824139527	0.404383786	0.369420448	1.450366517
0.007041701	0.628569242	0.145155315	1.029257875	0.285588171	0.031727335
0.299336555	0.108554273	0.611769540	1.724334625	1.548385695	0.281150165
0.180429692	0.997998484	0.290885684	0.760445347	0.414530449	2.694521631
1.072451123	1.913158077	0.099198306	0.447055353	2.289521445	0.403036248
0.197310956	0.659403472	0.430110769	0.647715953	1.882792993	0.748080428
0.176273293	0.126879159	0.224925179	0.293758877	0.063075250	6.380805968
0.334256376	0.618229284				

Table 5: Four simulated data set: We simulated the data of size 50 by selecting $\alpha = 0.9, \theta = 0.8$ and $\lambda = 1.6$, also satisfied $2\alpha\theta\lambda > 1$.

7.866669412	0.193544049	0.233349655	0.322319552	1.052919107	0.121931824
0.148202018	0.126403365	0.944103201	0.430205515	0.338933892	0.271518993
0.060368537	0.273225421	0.391951481	0.267072678	0.832361330	0.029514171
0.006185465	0.024960830	1.011453127	1.031239638	0.299797742	0.151558890
1.131417000	0.004871563	0.047579686	3.975409945	0.011196773	0.111160054
0.264921805	0.383905373	0.433469446	0.316959442	0.233411001	0.857271256
2.866480859	1.772993497	0.630951587	0.054277414	1.760295934	0.837123919
0.282658501	0.124953075	0.394427667	0.607135536	0.304025434	0.058698082
0.020184320	0.559280036				

Figures 2, 3, 4 and 5 are the plots of the MRL and the asymptotic of MRL of ECSW; it can be seen that the estimation was good and that the asymptotic of MRL of ECSW approximates the empirical MRL very well.

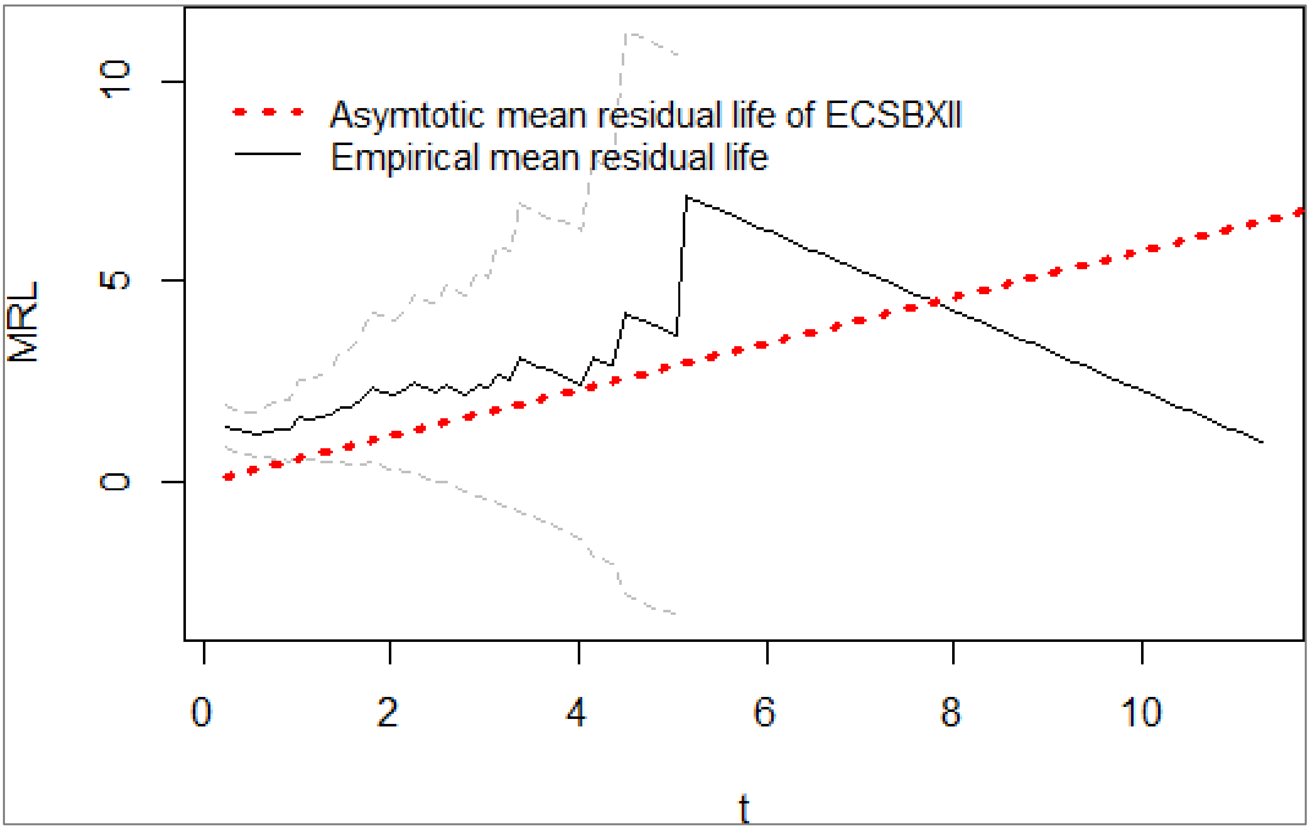

Figure 2: Plots of MRL and asymptotic of MRL of ECSW for the first simulated data

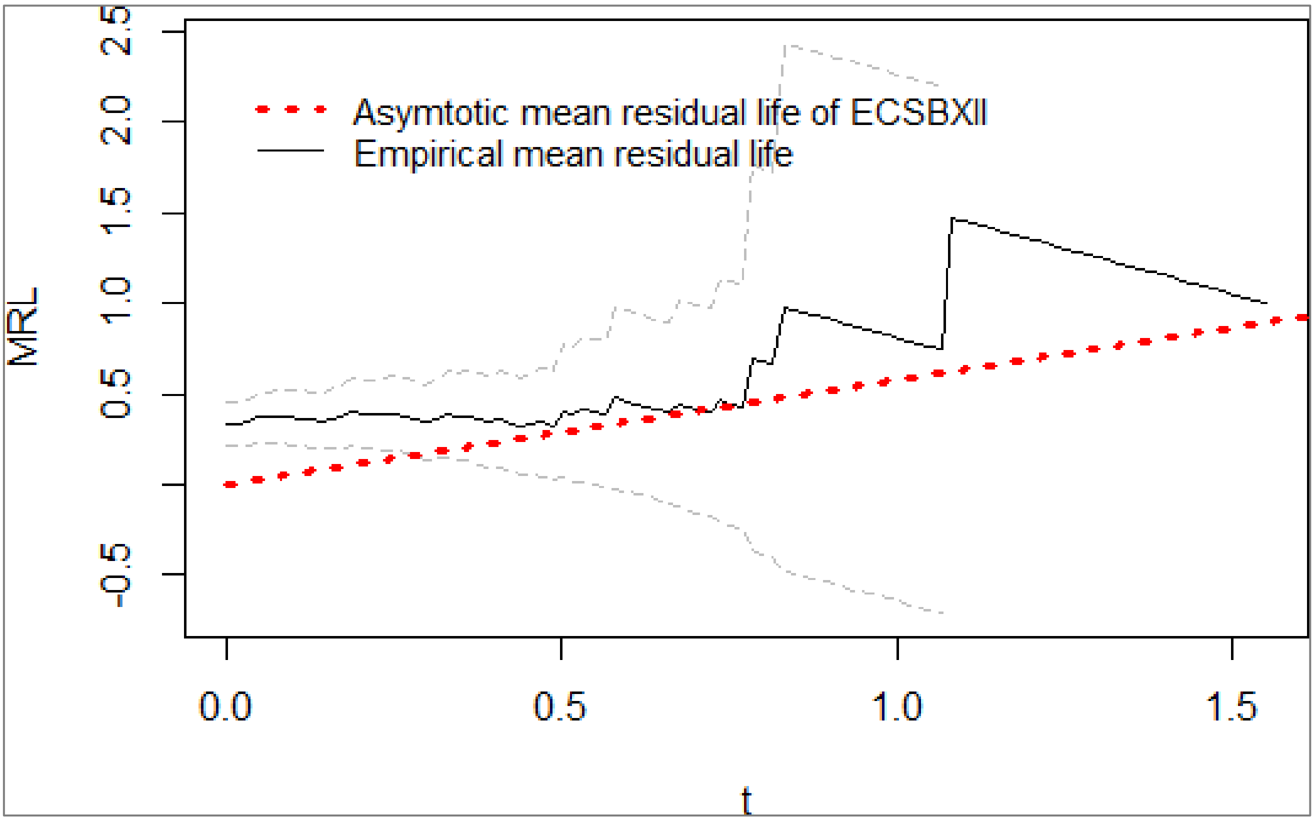

Figure 3: Plots of MRL and asymptotic of MRL of ECSW for the second simulated data

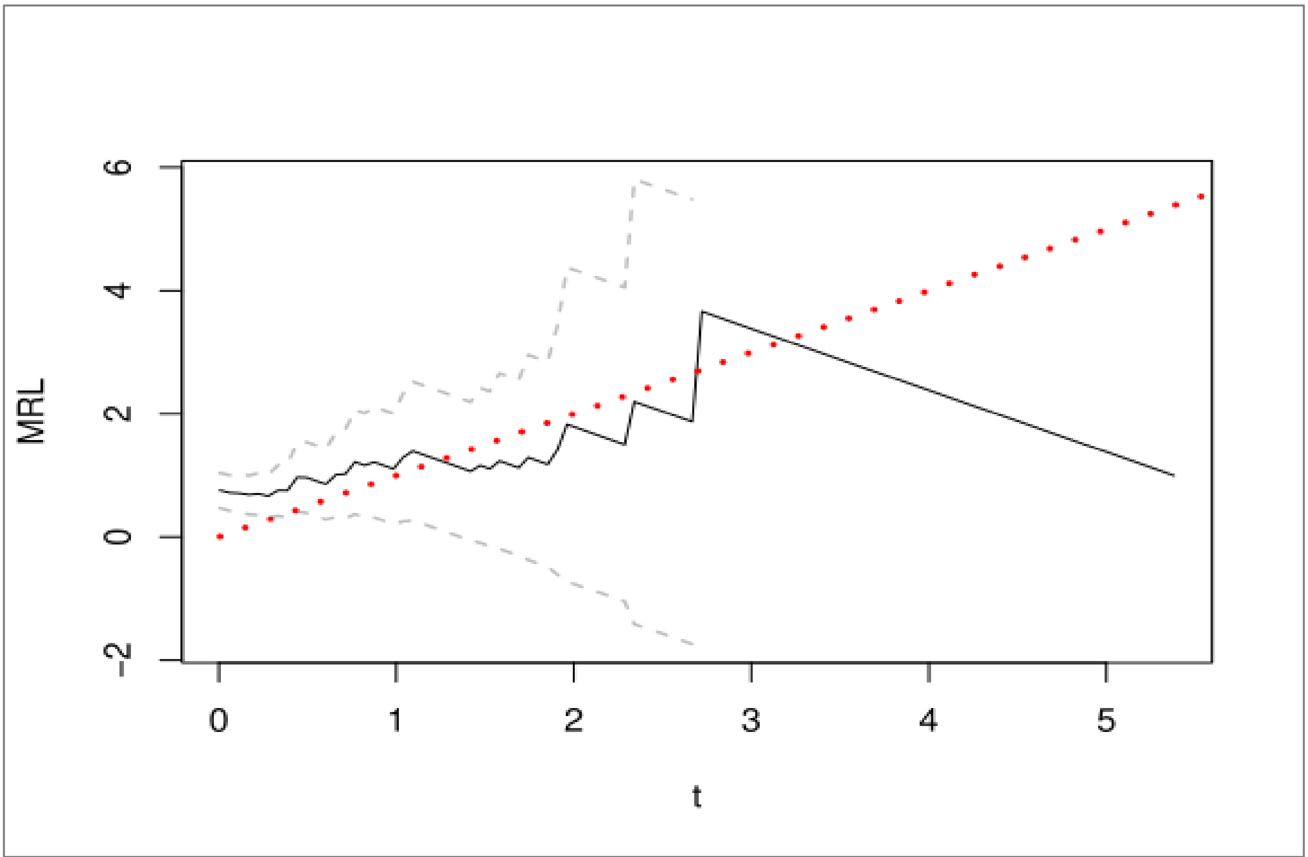

Figure 4: Plots of MRL and asymptotic of MRL of ECSW for the third simulated data

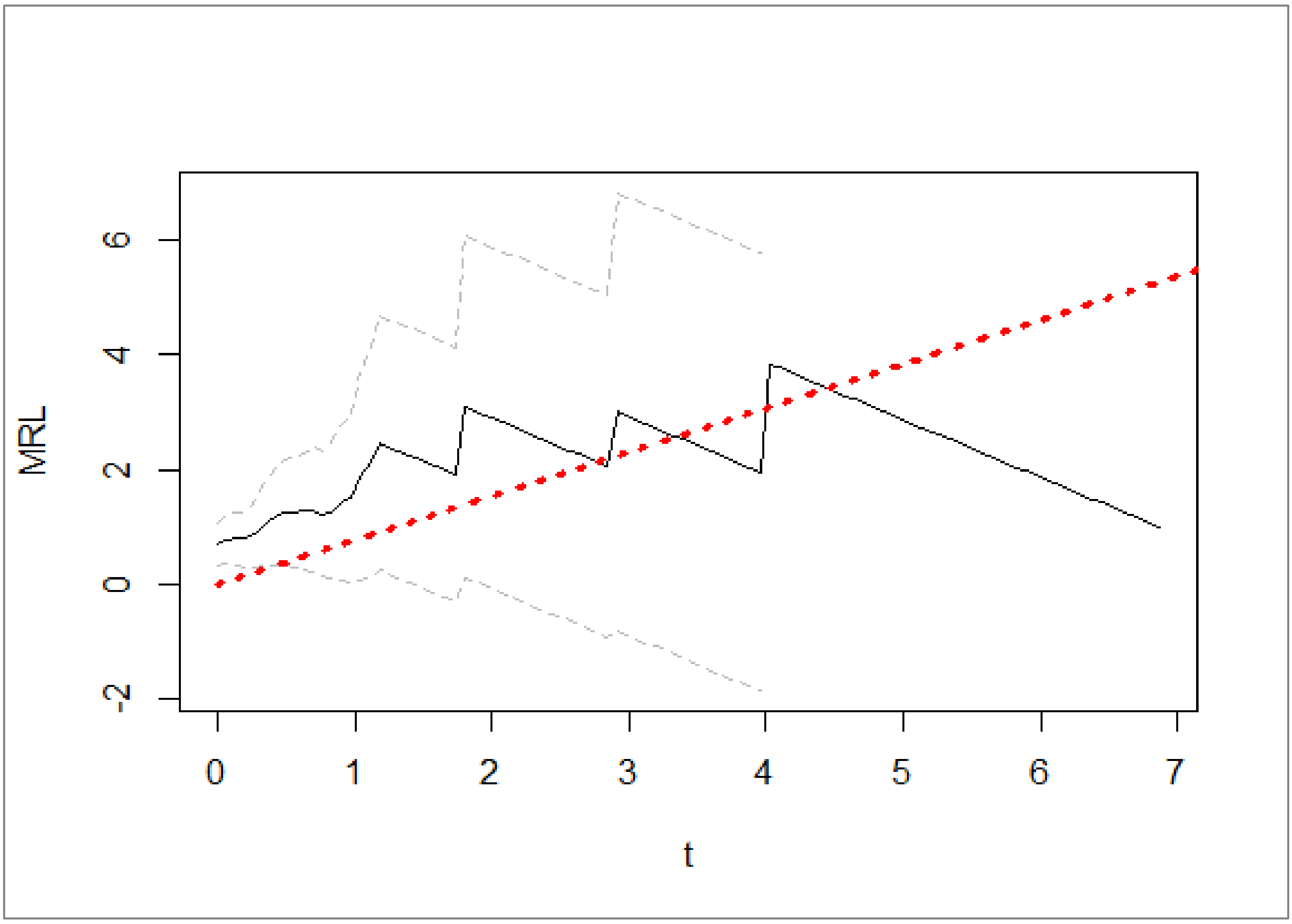

Figure 5: Plots of MRL and asymptotic of MRL of ECSW for the fourth simulated data

CONCLUSIONS

In this paper, a new probability model called extended cosine Burr XII distribution is proposed. Some important basic properties are derived. We drive and study the asymptotic of mean residual life and asymptotic of the extreme order statistics of the ECSBXII distribution. We demonstrated the asymptotic MRL of ECBXII performance using simulated data, and the result was satisfactory. These properties are important tools in statistical studies. Further studies can be conducted by applying these important tools in reliability analysis and extreme value analysis using the ECSBXII.

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REFERENCES

Abba, B., & Wang, H.(2023). A new failure times model for one and two failure modes system: A Bayesian study with Hamiltonian Monte Carlo simulation. *Proceedings of the Institution of Mechanical Engineers, Part O: Journal of Risk and Reliability*, page 1748006X221146367. **[Crossref]**

Abba, B., Wang, H., Muhammad, M., & Bakouch, H. S. (2023). A robust bathtub-shaped failure time model for a two-component system with applications to complete and censored reliability data. *Quality Technology & Quantitative Management*, 0(0):1–31.

Ahsanullah, M., Nevzorov V. B., & Shakil, M. (2013). *An introduction to order statistics*, volume 8. Springer. **[Crossref]**

Al-Essa, L. A., Muhammad, M., Tahir, M. H., Abba, B., Xiao,J., & Jamal, F. (2023) A new flexible four parameter bathtub curve failure rate model, and its application to right-censored data. *IEEE Access*, 11:50130–50144. **[Crossref]**

Arnold, B. C., Balakrishnan, N., & Nagaraja, H. N. (1992). *A first course in order statistics, vol. 54*.

Badr, M. & Ijaz, M. (2021). The exponentiated exponential burr xii distribution: Theory and application to lifetime and simulated data. *Plos one*, 16(3). **[Crossref]**

Bayramoglu, I. (2020). Joint distribution of a random sample and an order statistic: A new approach with an application in reliability analysis. *Reliability Engineering System Safety*, 193:106594. **[Crossref]**

Burr, I. W. (1942). Cumulative frequency functions. *The Annals of mathematical statistics*, 13(2):215–232. **[Crossref]**

Chen,C., Weng,S., Tsai, J., & Shen, P. (2023). The mean residual life model for the right-censored data in the presence of covariate measurement errors. *Statistics in Medicine*, 42(15):2557–2572. **[Crossref]**

Chua,K. C., Chandran,V., Acharya,U. J. & Lim, C. M.(2010). Application of higher order statistics/spectra in biomedical signals—a review. *Medical Engineering Physics*, 32(7):679–689. **[Crossref]**

Du, Y. & Gui, W. (2022). Statistical inference of burr-xii distribution under adaptive type ii progressive censored schemes with competing risks. *Results in Mathematics*, 77(2):81. **[Crossref]**

Gradshteyn, I. S. & Ryzhik I. M. (2007). Table of integrals series, and products. *New York: Academic Press.*

Hall, W. J., & Wellner, J. A. (2020). Estimation of mean residual life. *Statistical Modeling for Biological Systems: In Memory of Andrei Yakovlev*, pages 169–189. **[Crossref]**

Koutras, V. M., & Koutras, M. V. (2020). Exact distribution of random order statis- tics and applications in risk management. *Methodology and Computing in Applied Prob- ability*, 22:1539–1558. **[Crossref]**

Muhammad, M. (2016). A generalization of the burr xii-poisson distribution and its applications. *Journal of Statistics Applications & Probability*, 5(1):29–41. **[Crossref]**

Muhammad, M. (2017). The complementary exponentiated Burr XII Poisson distribution: Model, properties and application. *Journal of Statistics Applications & Probability*, 6(1):33–48. **[Crossref]**

Muhammad, M. (2023). A new three-parameter model with support on a bounded do- main: Properties and quantile regression model. *Journal of Computational Mathematics and Data Science*, 6:100077. **[Crossref]**

Muhammad, M., Alshanbari, H. M., Alanzi, A. R. A.,Liu, L., Sami, W., Chesneau, C. & Jamal, F. (2021). A new generator of probability models: The exponentiated sine-g family for lifetime studies. *Entropy*, 23(11). **[Crossref]**

Muhammad, M., Bantan,R. A. R., Liu, L., Chesneau, C., Tahir, M. H., Jamal, F., & Elgarhy, M. (2021). A new extended cosine-G distributions for lifetime studies. *Mathematics*, 9(21):2758, 2021. **[Crossref]**

Muhammad, M. & Liu, L. (2021). A new three parameter lifetime model: The complementary poisson generalized half logistic distribution. *IEEE Access,* 9:60089–60107. **[Crossref]**

Muhammad, M. & Liu, L., Abba, B. Muhammad, I., Bouchane, M., Zhang, H. & Musa S. (2023). A New Extension of the Topp–Leone-Family of Models with Applications to Real Data. *Annals of Data Science*, 10(1):225–250. **[Crossref]**

Saini, S. Tomer, S., & Garg, H. (2022). On the reliability estimation of multi-component stress–strength model for burr xii distribution using progressively first-failure censored samples. *Journal of Statistical Computation and Simulation*, 92(4):667–704. **[Crossref]**

Serinaldi, F., Lombardo, F., & Kilsby, C. G. (2020). All in order: Distribution of serially correlated order statistics with applications to hydrological extremes. Advances in Water Resources, 144:103686. **[Crossref]**