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ORIGINAL RESEARCH ARTICLE

A New Extension of Topp-Leone Distribution (NETD) Using Generalized Logarithmic Function

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ABSTRACT
In this study, we introduce a novel statistical model termed the New Extension of Topp-Leone Distribution (NETD), constructed using a generalized logarithm function. The derivation of the cumulative density function (CDF) and the probability density function (PDF) for this new distribution is thoroughly detailed. A validity test was conducted to confirm the legitimacy of the proposed NETD, and the results affirm that it is indeed a valid probability distribution. We investigate several key mathematical and statistical properties of the NETD, including the quantile function, moments, Renyi entropy, probability-weighted moments, and order statistics. These properties provide a comprehensive understanding of the distribution's behaviour and characteristics. To estimate the parameters of the NETD, we applied the Maximum Likelihood Estimation (MLE) method, which is known for its efficiency and asymptotic properties. The flexibility, versatility, and performance of the NETD were evaluated using four diverse real-world datasets. Through the application of model selection criteria such as the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Corrected Akaike Information Criterion (CAIC), and Hannan-Quinn Information Criterion (HQIC), we demonstrated that the NETD exhibits superior goodness-of-fit compared to existing variants of the Topp-Leone distribution. Our findings suggest that the NETD is a robust and adaptable distribution, capable of effectively modelling a wide range of empirical data. This new extension not only broadens the applicability of the Topp-Leone family of distributions but also enhances the toolkit available for statistical analysis and modelling in various fields.

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INTRODUCTION

The Topp-Leone (TL) distribution, introduced by Topp and Leone in 1955, is a flexible distribution used primarily in reliability engineering and survival analysis. It is known for its simplicity and ability to model data with increasing, decreasing, or constant failure rates. However, like many distributions, its flexibility can be limited in capturing the complexity of real-world data, prompting the need for extending the distribution. Extensions of classical distributions often lead to more flexible models capable of better capturing the characteristics of empirical data. By introducing additional parameters or functions, these extensions can provide improved fit and greater adaptability. One common method for achieving such extensions is through the use of generalized logarithmic functions, which can transform the base distribution in ways that enhance its flexibility. As mentioned earlier, the Topp-Leone (TL) model was presented by Topp and Leone (1955) as a lifetime model. Lifetime distributions are statistical distributions used in analyzing real-life problems centring on survival time. Topp-Leone (TL) distribution is one of the several statistical distributions developed to model lifetime data. In practical situations, many lifetime data sets used are not well handled by the Topp-Leone (TL) distribution. Being a single parameter distribution on a unit interval, its versatility and flexibility are constrained in handling lifetime data sets.

Accordingly, several extensions, modifications, and generalizations have remained developed by many statisticians to heighten desirable features (possessions) of the Topp-Leone (TL) distribution. We must acknowledge the exertion made to enhance the flexibility of the Topp-Leone (TL) model by several scholars, such as, Nadarajah and Kotz (2003), who premeditated the Topp-Leone (TL) distribution's characteristic function, moments, and properties, Ghittany *et al.* (2005) delivered some reliability approaches of the Topp-Leone (TL) distribution, Kotz and Seier (2007) studied the kurtosis of the Topp-Leone

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(TL) distribution, Ali *et al.* (2007) give insight into some exponentiated distributions, Vicaria *et al.* (2008) presented a two-sided (2-S) generalized description of the Topp-Leone (TL) distribution, Cordeiro and De Castro (2011) conversed a new family of generalized distributions, Al-Zahrani (2012) developed the goodness-of-fit (gof) test for the Topp-Leone (TL) model; Al-Shomrani *et al.* (2016) offered Topp-Leone (TL) comprehensive family of distributions (FOD), and Usman *et al.* (2023) constructed log Topp-Leone distribution. Breeding a new family (NF) of distributions involves two (2) main statistical mechanisms, which include a parent distribution and a generator (Alzaatreh *et al.*, 2013; Jones, 2004, Sadiq *et al.,* 2022, 2023a, 2023b and 2023c). Furthermore, some of the classical statistical distributions (SD) have precincts and deficiencies concerning their tractability and adaptability in modelling diverse types of datasets (Habu *et al.*, 2024). Some dataset characteristics indicate a great extent of kurtosis and multimodality, skewness, reversed J shapes, etc. Due to this, there has remained a growing attentiveness to refining the presentation of predominant statistical distributions (SD) making them extra supple and versatile in demonstrating real-life datasets. The main objective of this study is to suggest and put forward a new extension (NE) of the Topp-Leone (TL) distribution using a generalized logarithmic function. This involves deriving the CDF (cumulative density function) and the PDF (probability density function) of the proposed probability distribution, deriving some of its mathematical properties, estimating the parameters of the PD (proposed distribution) using Maximum Likelihood Estimation (MLE) and evaluating the recital of the proposed model by comparing it with other variants of the Topp-Leone distribution. Various methods have been proposed to extend classical distributions, often through the inclusion of additional parameters or by employing transformation functions. These extensions aim to provide greater flexibility and improve the fit of empirical data. One such approach involves using logarithmic functions, which can introduce new shapes and behaviours to the distribution. The proposed study is expected to contribute to the field of statistical distributions by providing a new, flexible extension of the TL (Topp-Leone) distribution. This new distribution can potentially offer better fit and modelling capabilities for various types of data, particularly in reliability engineering and survival analysis.

METHODOLOGY

In this segment, we discussed how our proposed model is constructed by introducing a link function into the TP (Topp-Leone) distribution developed by Topp and Leone in 1955, which is a one-parameter (1-P) distribution. Then, we obtained the cumulative density function (cdf) of our proposed distribution. After that, we derived our probability density function (pdf) by differentiating the cumulative density function (cdf) which we obtained. Subsequently, we show the mixture presentation or simplest form of our proposed distribution, we check the validity of our proposed model, and lastly, some features of the developed model are explained, and the parameters are estimated using maximum likelihood estimate (MLE).

NEW EXTENSION (NE) OF TOPP-LEONE (TL) DISTRIBUTION

The Topp-Leone (TL) distribution, which is a one-parameter (1-P) distribution constructed by Topp and Leone (1955), the CDF as follows:

$$F_{TL}(x) = x^{\alpha}(2-x)^{\alpha};\ 0 \le x \le 1 \qquad (1)$$

We are going to extend this one-parameter (1-P) Topp-Leone (TL) distribution to have a two-parameter model using the generalized logarithmic function as

$$-\log(1-x)^{\frac{1}{\beta}} = t \qquad (2)$$

It is very clear from equation (2) we obtained the following relations;

$$x = 1 - exp\{-(\beta t)\} \qquad (3)$$

Now, substituting equation (3) into equation (1), then we have a two-parameter cumulative distribution function (CDF) of the developed new extension of the Topp-Leone distribution as:

$$F_{NETD}(t;\alpha,\beta) = (1-exp\{-(\beta t)\})^{\alpha}\left(2-(1-exp\{-(\beta t)\})\right)^{\alpha} \qquad (4)$$

Equation (4) can further be conveniently simplified as;

$$F_{NETD}(t;\alpha,\beta) = (1-exp\{-(2\beta t)\})^{\alpha};\quad t,\ \alpha,\ \beta\ >\ 0 \qquad (5)$$

The study obtains the PDF (probability density function) of the developed new extension of the Topp-Leone distribution (NETD) by differentiating equation (5) with respect to t.

$$f_{NETD}(t;\alpha,\beta) = 2\alpha\beta\, exp\{-(2\beta t)\}\,(1-exp\{-(2\beta t)\})^{\alpha-1};\quad t,\ \alpha,\ \beta\ >\ 0 \qquad (6)$$

Test of Validity of the CDF of the NETD

If equation (5) is a valid CDF, it must satisfy the statistical properties of any continuous distributions:

i. $\lim_{t\to-\infty} F(t) = \lim_{t\to 0} F_{NETD}(t;\alpha,\beta) = 0$
ii. $\lim_{t\to\infty} F(t) = \lim_{t\to\infty} F_{NETD}(t;\alpha,\beta) = 1$

Proof of Case 1

$$\lim_{t\to-\infty} F(t) = \lim_{t\to 0} F(t;\alpha,\beta) = \lim_{t\to 0}(1-exp\{-(2\beta t)\})^{\alpha} = \left(1-exp\{-(2\beta(0))\}\right)^{\alpha} = 0$$

Proof of Case 2

$$\lim_{t\to\infty} F(t) = \lim_{t\to\infty} F(t;\alpha,\beta) = \lim_{t\to\infty}(1-exp\{-(2\beta t)\})^{\alpha} = \left(1-exp\{-(2\beta(\infty))\}\right)^{\alpha} = 1$$

Mixture Representation (MR) or Reduced Form (RF) of the CDF and PDF of the NETD

The given PDF in equation (6) and the given CDF in equation (5) of the developed extension of the Topp-Leone distribution will be reduced to mixture representation using the standard binomial and power series expansion (Sadiq *et al.,* 2023c, and Habu *et al.,* 2024) of the following forms;

$$(1-x)^{-\alpha} = \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma(\alpha+1)}{i! \Gamma(\alpha+1+i)} x^i \quad (7)$$

$$(1-x)^{\alpha-1} = \sum_{i=0}^{\infty} (-1)^i \binom{\alpha-1}{i} x^i \quad (8)$$

$$e^{-\alpha x} = \sum_{j=0}^{\infty} \frac{(-1)^j (\alpha x)^j}{j!} \quad (9)$$

Applying equations (8) and (9) to the PDF in equation (6), we obtained the reduced form of the PDF as;

$$f(t;\alpha,\beta) = 2\alpha\beta \sum_{i=0}^{\infty} (-1)^i \binom{\alpha-1}{i} exp\{-2\beta t(1+i)\} \quad (10)$$

An additional overview of equation (10) is as follows;

$$f(t;\alpha,\beta) = \psi_i \, exp\{-2\beta t(1+i)\} \quad (11)$$

where $\psi_i = 2\alpha\beta \sum_{i=0}^{\infty} (-1)^i \binom{\alpha-1}{i}$

Some Features of the Developed NETD

Here, we discuss some features of our developed distribution, including the quantile function, moments, Renyl entropy, MLE parameter estimate, and OS (order statistics).

Quantile Function (QF) of the Developed NETD

The study obtains the quantile function of the developed NETD. Suppose that U stands as a random variable (R.V) following a Rectangular or Uniform distribution bounded on the interval $(0,1)$ by relating the rectangular distribution with the inverse of the cumulative distribution (Sadiq *et al.,* 2023c and Habu *et al.,* 2024) of the NETD presented in the equation (5), we have as;

$$Q(u) = F(t)^{-1} = (1 - exp\{-(2\beta t)\})^{\alpha} = u$$
$$= 1 - exp\{-(2\beta t)\} = u^{\frac{1}{\alpha}}$$
$$= exp\{-(2\beta t)\} = 1 - u^{\frac{1}{\alpha}}$$

Taking the logarithm of both sides

$$-2\beta t = log\left(1 - u^{\frac{1}{\alpha}}\right)$$
$$2\beta t = -\,log\left(1 - u^{\frac{1}{\alpha}}\right)$$
$$t = \frac{1}{2\beta}\left[-\,log\left(1 - u^{\frac{1}{\alpha}}\right)\right]$$

$$Q(u) = \frac{1}{2\beta}\left[-\,log\left(1 - u^{\frac{1}{\alpha}}\right)\right] \quad (12)$$

where u follows uniform distribution on intervals 0 and 1.

Furthermore, we can obtain the median from equation (12) by setting $u = \frac{1}{2}$, and the third quartile range, set $u = \frac{1}{3}$, to have the fourth quartile range, we set $u = \frac{1}{4}$.

Moments

The moments for any given distribution are defined (Sadiq *et al.,* 2022, 2023a, 2023b and 2023c) as;

$$E(t^r) = \int_{-\infty}^{\infty} t^r f(t) dt \quad (13)$$

Therefore, the moment of the developed NETD using equations (11) and (13) is given by;

$$E(t^r) = \int_0^{\infty} t^r \psi_i \, exp\{-2\beta t(1+i)\} \, dt = \psi_i \int_0^{\infty} t^r \, exp\{-2\beta t(1+i)\} \, dt \quad (14)$$

Let $m = 2\beta t(1+i)$

$$\frac{dm}{dt} = 2\beta(1+i) \Rightarrow dt = \frac{dm}{2\beta(1+i)}$$

$$t = \frac{m}{2\beta(1+i)} \quad (15)$$

Substitute equation (15) into equation (14)

$$E(t^r) = \psi_i \int_0^{\infty} \left[\frac{m}{2\beta(1+i)}\right]^r exp\{-m\} \frac{dm}{2\beta(1+i)}$$

$$E(t^r) = \frac{\psi_i}{[2\beta(1+i)]^{r+1}} \int_0^{\infty} m^r \, exp\{-m\} \, dm$$

$$E(t^r) = \frac{\psi_i}{[2\beta(1+i)]^{r+1}} \int_0^{\infty} m^{r+1-1} \, exp\{-m\} \, dm$$

$$E(t^r) = \frac{\psi_i \Gamma(r+1)}{[2\beta(1+i)]^{r+1}} \quad (16)$$

Equation (16) is the rth moment about the origin of the developed NETD. We can also obtain the first moment about the mean, which is the mean of the proposed model, by setting $r = 1$ in equation (16) as;

$$E(t) = \frac{\psi_i}{[2\beta(1+i)]^2}; \text{ (Note that } \Gamma(2) = 1! = 1) \quad (17)$$

Equation (17) is the first moment about the mean of the developed NETD. Similarly, by setting $r = 2$, in equation (16), we also obtain the second moment about the mean as;

$$E(t^2) = \frac{\psi_i \Gamma(3)}{[2\beta(1+i)]^3} = \frac{\psi_i 2!}{[2\beta(1+i)]^3} = \frac{2\psi_i}{[2\beta(1+i)]^3} \quad (18)$$

The variance of the developed NETD is obtained by the following relations;

$$Var(t) = E(t^2) - [E(t)]^2 \quad (19)$$

$$Var(t) = \frac{\psi_i}{[2\beta(1+i)]^2} - \left[\frac{2\psi_i}{[2\beta(1+i)]^3}\right]^2 = \frac{\psi_i\left[(2\beta(1+i))^4 - 4\psi_i\right]}{[2\beta(1+i)]^6} \quad (20)$$

Renyi Entropy

The entropy (Renyi) is generally well-defined (Sadiq *et al.,* 2022, 2023a, 2023b and 2023c) as;

$$R_\sigma = \frac{1}{1-\sigma} log \int_0^\infty [f(t)]^\sigma dt; \sigma > 0 \text{ and } \sigma \neq 1 \quad (21)$$

Substituting equation (11) into equation (21), the entropy (Renyi) of the developed NETD is obtained as;

$$R_\sigma(t) = \frac{1}{1-\sigma} log \int_0^\infty \left[\psi_i e^{-2\beta t(1+i)}\right]^\sigma dt; \sigma > 0 \text{ and } \sigma \neq 1 \quad (22)$$

Order Statistics

The order statistics is generally defined (Sadiq *et al.,* 2022, 2023a, 2023b, and 2023c) as;

$$f_{i,n}(t) = Kf(t)[F(t)]^{i-1}[1 - F(t)]^{n-1} \quad (23)$$

where $K = \frac{n!}{(i-1)!(n-1)!}$ $f(t)$ is presented in equation (6) as pdf and $F(t)$ is also presented in equation (5) as cdf of developed distribution (NETD). Using generalized binomial expansion, equation (23) can be represented as:

$$f_{i,n}(t) = Kf(t)[F(t)]^{i-1} \sum_{j=0}^{\infty}(-1)^j \binom{n-i}{j} [F(t)]^j = Kf(t) \sum_{j=0}^{\infty}(-1)^j \binom{n-i}{j} [F(t)]^{i+j-1} \quad (24)$$

Substituting equations (5) and (6) into equation (24) and then simplifying it, however, the order statistics (OS) of the developed continuous probability distribution (NETD) is gotten as;

$$f_{i,n}(t) = \sum_{m=0}^{\infty} \sigma_m \, exp\{-2\beta t(1+m)\} \quad (25)$$

Where

$$\sigma_m = 2\alpha\beta K \sum_{j=0}^{\infty} (-1)^{j+m} \binom{n-i}{j} \binom{\alpha(i+j)-1}{m}$$

Parameter Estimation

The parameter of the developed probability model (NETD) is obtained using the method and principles of Maximum Likelihood Estimation (MLE). Supposed $x1, x2, \dots, xn$ denotes the RS (random sample) (Sadiq *et al.,* 2022, 2023a, 2023b and 2023c) haggard from the proposed distribution (NETD) with parameters α and β. The parameters estimates are obtained and estimated by taking the log-likelihood (L-L) function of equation (6) as;

$$log(L(t_i/\alpha,\beta)) = \Theta = n\,log\,2 + n\,log\,\alpha + n\,log\,\beta - 2\beta\sum_{i=1}^{n} t_i + (\alpha-1)\sum_{i=1}^{n} log(1 - e^{-2\beta t_i}) \quad (26)$$

Taking the partial derivative of equation (26) with respect to each parameter, equate the resultant derivatives to zero, and then solve for each parameter estimate, we have as;

$$\hat{\alpha} = -\frac{n}{\sum_{i=1}^{n\Sigma(1-exp\{-(2\beta t_i)\})} log} \quad (27)$$

$$\hat{\beta} = \frac{n}{2\sum_{i=1}^{n} t_i(2-\alpha)} \quad (28)$$

RESULTS AND DISCUSSIONS

Goodness-of-fit (GoF) tests are essential for assessing how well a statistical model fits a set of observations. For this study, four distinct datasets were used to measure and validate (Sadiq *et al.,* 2022, 2023a, 2023b, and 2023c) the recital of the offered extension of the Topp-Leone distribution. This section details the results of the GoF tests for each data set, providing a wide-ranging analysis of the model's adequacy and suitability. Data Set 1: Provide a brief description of the first data set, including its source, sample size, and any pertinent characteristics. Data Set 2: Similarly, describe the second data set. Data Set 3: Outline the third data set. Data Set 4: Describe the fourth data set.

Dataset 1: This data set is derived from the study conducted by Bekker *et al.* (2000) and includes the ST (survival times) (in years) of a set of patients who underwent chemotherapy medication alone. Specifically, this subset contains the ST (survival times) for 45 patients. The detailed ST (survival times) are as follows:

0.047, 0.115, 0.121, 0.132, 0.164, 0.197, 0.203, 0.260, 0.282, 0.296, 0.334, 0.395, 0.458, 0.466, 0.501, 0.507, 0.529, 0.534, 0.540, 0.641, 0.644, 0.696, 0.841, 0.863, 1.099, 1.219, 1.271, 1.326,1.447, 1.485, 1.553, 1.581, 1.589, 2.178, 2.343, 2.416, 2.444, 2.825, 2.830, 3.578, 3.658, 3.743, 3.978, 4.003, 4.033.

Table 1 Summary of the Goodness-of-Fit (Gof) Measures for the First (1st) Dataset using MLE

Models	Estimates	A.I.C	C.A.I.C	B.I.C	H.Q.I.C
NETD	α = 1.1125 β = 0.4008	120.192	120.4777	123.8053	121.539
LTD	θ = 2.3026	156.8924	156.9854	158.699	157.5659
ITD	δ = 0.0792	354.4252	354.5182	356.2318	355.0987
TWD	η = 0.9020 λ = 1.2371 τ = 0.4550	122.2142	122.7996	127.6342	124.2347

Dataset 2: The second data set originates from the study conducted by Linhart *et al.* (1986). This data set encompasses the results from endurance tests performed on deep groove ball bearings. Specifically, the observation represents the aggregate of million mutinies apiece of the 23 ball bearings underwent before failure during the life test. The detailed endurance data are as follows: 17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.80, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40.

Table 2 Summary of the Goodness-of-Fit (Gof) Measures for the Second (2nd) Dataset using MLE

Models	Estimates	A.I.C	C.A.I.C	B.I.C	H.Q.I.C
NETD	$\alpha = 2.4809$ $\beta = 0.0091$	235.9353	236.5353	238.2063	236.5065
LTD	$\theta = 0.2276$	6684.118	6684.308	6685.253	6684.403
ITD	$\delta = 0.2870$	297.5965	297.787	298.732	297.8821
TWD	$\eta = 0.8115$ $\lambda = 1.6448$ $\tau = 0.0285$	250.7611	252.0242	254.1676	251.6178

Dataset 3: This data set shows the dataset entailing of the dimension of interludes among the times at which automobiles pass a point taking place in the highway. The source of this data is given by Jorgensen (2012). The data are presented as: 2.50, 2.60, 2.60, 2.70, 2.80, 2.80, 2.90, 3.00, 3.00, 3.10, 3.20, 3.40, 3.70, 3.90, 3.90, 3.90, 4.60, 4.70, 5.00, 5.60, 5.70, 6.00, 6.00, 6.10, 6.60, 6.90, 6.90, 7.30, 7.60, 7.90, 8.00, 8.30, 8.80, 8.80, 9.30, 9.40, 9.50, 10.1, 11.0, 11.3, 11.9, 11.9, 12.3, 12.9, 12.9, 13.0, 13.8, 14.5, 14.9, 15.3, 15.4, 15.9, 16.2, 17.6, 20.1, 20.3, 20.6, 21.4, 22.8, 23.7, 23.7, 24.7, 29.7, 30.6, 31.0, 34.1, 34.7, 36.8, 40.1, 40.2, 41.3, 42.0, 44.8, 49.8, 51.7, 55.7, 56.5, 58.1, 70.5, 72.6, 87.1, 88.6, 91.7, 119.8.

Table 3 Summary of the Goodness-of-Fit (Gof) Measures for the Third (3rd) Dataset using MLE

Models	Estimates	A.I.C	C.A.I.C	B.I.C	H.Q.I.C
NETD	$\alpha = 1.0135$ $\beta = 0.0241$	688.1853	688.3335	693.047	690.1397
LTD	$\theta = 0.2276$	7392.532	7392.58	7394.963	7393.509
ITD	$\delta = 0.4950$	744.8644	744.9132	747.2952	745.8416
TWD	$\eta = 0.7802$ $\lambda = 1.4300$ $\tau = 0.0731$	691.1375	691.4375	698.43	694.069

Dataset 4: The fourth data set was reported by Van Montfort (1970). The data are presented as:

19.885, 20.940, 21.820, 23.700, 24.888, 25.460, 25.760, 26.720, 27.500, 28.100, 28.600, 30.200, 30.380, 31.500, 32.600, 32.680, 34.400, 35.347, 35.700, 38.100, 39.020, 39.200, 40.000, 40.400, 40.400, 42.250, 44.020, 44.730, 44.900, 46.300, 50.330, 51.442, 57.220, 58.700, 58.800, 61.200, 61.740, 65.440, 65.597, 66.000, 74.100, 75.800, 84.100, 106.600, 109.700, 121.970, 121.970, 185.560.

Table 4 Summary of the Goodness-of-Fit (Gof) Measures for the Fourth (4th) Dataset using MLE

Models	Estimates	A.I.C	C.A.I.C	B.I.C	H.Q.IC
NETD	$\alpha = 2.3619$ $\beta = 0.0172$	453.2048	453.4717	456.9472	454.619
LTD	$\theta = 0.2276$	9964.619	9964.706	9966.49	9965.326
ITD	$\delta = 0.3182$	576.1394	576.2263	578.0106	576.8465
TWD	$\eta = 0.8115$ $\lambda = 1.6448$ $\tau = 0.0285$	475.9925	476.5379	481.6061	478.1139

The parameter estimates presented in Tables 1 through 4 correspond to the first, second, third, and fourth datasets, respectively. These tables display the goodness-of-fit metrics for both the proposed model and alternative models across all datasets. The selection of the optimal model is based on minimizing the information criterion values. Across all datasets, the proposed model consistently demonstrates superior performance compared to alternative models, as indicated by its consistently lower information criterion values. Hence, it can be concluded that the proposed model demonstrates the best overall performance in modelling the datasets, as evidenced by its ability to minimize the information criteria more effectively than the competing models. This finding underscores the robustness and applicability of the proposed model in various data contexts.

From Figure 1, one can observe that the pdf of NETD distribution is unimodal, and the curves are right-skewed with a long tail. And has a reverse J shape. The curve is flat (constant) when α = 1 and β= 0.01.

From Figure 2, one can observe that the hrf of the NETD distribution is unimodal and right-skewed. The shape is constant when α = 1 and β= 0.01.

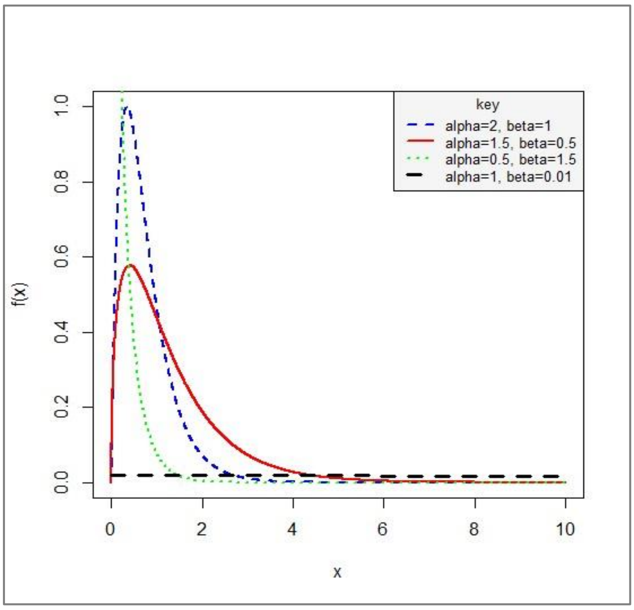

Figure 1: Plot of pdf of the NETD

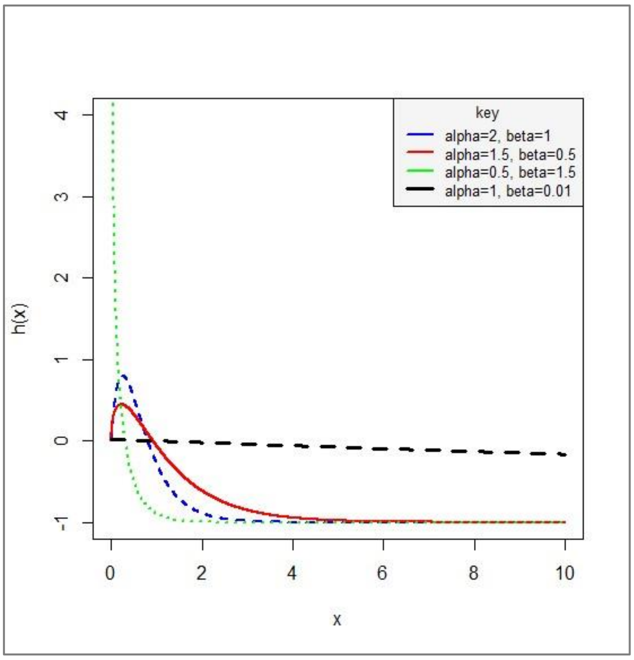

Figure 2: Plot of hazard of the NETD

CONCLUSION

In this study, we introduced a novel extension of the Topp-Leone distribution, denoted as the New Extension of Topp-Leone Distribution (NETD), utilizing a generalized logarithm function. The validity of this new statistical distribution was rigorously examined, confirming its status as a valid probability distribution. Our investigation encompassed a comprehensive analysis of various statistical properties inherent to the NETD, including the quantile function, moments, Renyi entropy, probability-weighted moments, order statistics, cumulative density function (CDF), and probability density function (PDF).To determine the estimators for the parameters of the NETD, we employed the Maximum Likelihood Estimation (MLE) method. This approach provided robust estimates that facilitated the characterization of the distribution's shape and parameters. To assess the performance and versatility of the NETD, we conducted an extensive evaluation using four distinct real-world datasets. The evaluation criteria included the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Corrected Akaike Information Criterion (CAIC), and Hannan-Quinn Information Criterion (HQIC). Through comparative analysis against alternative distributions such as the Log Topp-Leone distribution, Inverted Topp-Leone distribution, and Topp-Leone-Weibull distribution, we demonstrated that the NETD consistently exhibited superior goodness-of-fit across various metrics. The development and validation of the NETD hold significant implications for statistical modelling and analysis in diverse fields. Its enhanced flexibility and robust performance underscore its utility in accurately modelling a wide range of empirical data distributions. Researchers and practitioners can leverage the NETD to effectively model data exhibiting complex distributions, thereby enhancing the accuracy and reliability of statistical analyses. Future research could further explore extensions and applications of the NETD in specialized domains, such as finance, economics, and environmental sciences. Additionally, refining methodologies for parameter estimation and expanding the theoretical underpinnings of the NETD could enhance its applicability and broaden its scope across various disciplines. In summary, the New Extension of Topp-Leone Distribution (NETD) represents a significant contribution to statistical theory and practice, offering a robust framework for modelling and analyzing empirical data distributions with enhanced accuracy and flexibility.

CONTRIBUTION TO KNOWLEDGE

The paper provides a New Extension of Topp-Leone Distribution (NETD) built using a generalized logarithm function and the parameters were estimated and validated by the maximum likelihood estimation methods. This adds to current knowledge of distribution theory.

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