UMYU Scientifica, Vol. 3 NO. 3, September 2024, Pp 284 – 296
https://doi.org/10.56919/usci.2433.031
ISSN: 2955 – 1145 (print); 2955 – 1153 (online)
A periodical of the Faculty of Natural and Applied Sciences, UMYU, Katsina

ORIGINAL RESEARCH ARTICLE

An EOQ Model for Delayed Deteriorating Items: Analysis of Three-Stage Consumption, Dual Storage Facilities, and Variable Partial Backlogging Rates

Asmau Abdullahi Mani[1], Babangida Bature[1] and Mustapha Lawal Malumfashi[1]*

[1]Department of Mathematics and Statistics, Umaru Musa Yar'adua University, P.M.B. 2218, Katsina, Nigeria

ARTICLE HISTORY

Received June 14, 2024
Accepted September 15, 2024
Published September 18, 2024

KEYWORDS

Delayed deterioration, own and rented ware-houses, three-stage consumption rates, time-dependent partial backlogging

ABSTRACT

This paper presents an Economic Order Quantity (EOQ) model for items with delayed deterioration, incorporating three distinct stages of consumption rates, dual storage facilities, and shortages. The model assumes that the average consumption rates before and after the onset of deterioration, as well as during shortages, are constant but differ from one another. The shortages are partially backlogged and vary according to the waiting time until the next replenishment. The model aims to minimize the overall variable cost per unit of time by optimizing the cycle length, order quantity, and the timing at which the inventory in the owned warehouse is depleted. The paper establishes the necessary and sufficient conditions to ensure the existence and uniqueness of the optimal solution. A sensitivity analysis is conducted, and the findings are used to derive managerial insights.

INTRODUCTION

In the realm of inventory management, traditional models for delayed deteriorating items have typically relied on simplifying assumptions: a single warehouse with infinite capacity and constant average demand rates throughout various stages of the inventory lifecycle, such as before and after deterioration, as well as during periods of shortage. While these models provide a foundational understanding, they fall short of addressing the complexities encountered in modern, competitive markets. Retailers today are confronted with several factors that drive the need for larger inventory quantities. Stock-outs, which occur when inventory runs out, can lead to lost sales and customer dissatisfaction, prompting retailers to maintain higher stock levels. Additionally, price and quantity discounts offered by suppliers incentivize bulk purchases. Inflationary pressures and demand uncertainties further complicate inventory management, often leading retailers to secure larger quantities of goods to buffer against future cost increases and fluctuations in demand. These increased inventory levels can easily exceed the capacity of a retailer's primary storage facility Malumfashi et al. (2024). As a result, retailers often turn to additional rented warehouses to accommodate surplus stock. These rented warehouses, while offering the advantage of better preservation technologies that slow down deterioration, come with higher holding costs. Thus, it becomes economically advantageous to expedite the depletion of inventory stored in rented facilities to reduce holding expenses.

Moreover, the assumption of constant consumption rates, before and after deterioration and during shortages, does not align with real-world scenarios where demand and consumption can vary significantly. For example, demand may fluctuate based on seasonal trends or promotional activities, and consumption rates may change as inventory levels deplete or as products approach their expiration. Therefore, modern inventory models must integrate these dynamic factors. Incorporating variability in consumption rates, adjusting for the costs and benefits of using additional storage facilities, and accounting for the economic implications of holding inventory in rented spaces will provide a more accurate and practical framework for managing delayed deteriorating items. This approach will better reflect the realities faced by retailers and improve the effectiveness of inventory management strategies.

Previous research, including studies by Tiwari et al. (2016), Kumar et al. (2017), Chandra et al. (2017), Jaggi et al. (2017), Udayakumar and Geetha (2018), Chakrabarty et al. (2018), Sahoo et al. (2020), Gupta et al. (2020), Datta et al. (2022), and Das (2024), has primarily focused on inventory models involving two warehouses with a single-phase consumption rate. However, this

Correspondence: Mustapha Lawal Malumfashi. Department of Mathematics and Statistics, Umaru Musa Yar'adua University, P.M.B. 2218, Katsina, Nigeria. ✉ mustapha.lawal@umyu.edu.ng.

How to cite: Mani, A. A., Bature, B., & Malumfashi, M. L. (2024). An EOQ Model for Delayed Deteriorating Items: Analysis of Three-Stage Consumption, Dual Storage Facilities, and Variable Partial Backlogging Rates. *UMYU Scientifica*, 3(3), 284 – 296. https://doi.org/10.56919/usci.2433.031

single-phase demand rate assumption falls short of addressing the complexities encountered in real-world scenarios. To enhance practical applicability, it is essential to integrate variability in consumption rates before and after deterioration, as well as during shortages, into more comprehensive inventory models. This approach will better reflect the dynamic nature of inventory systems and improve the accuracy of inventory management strategies.

This study introduces an Economic Order Quantity (EOQ) model for delayed deteriorating items that account for three distinct consumption rates, two storage facilities, and shortages. The model assumes that the consumption rates before and after deterioration and during shortages are constant and incorporates time-dependent partially backlogged shortages, which fluctuate based on the waiting time for the next replenishment. The model aims to determine the optimal cycle length, order quantity, and the time at which the inventory level in the primary warehouse reaches zero, all to minimize the overall variable cost per unit of time. The study further examines the existence and uniqueness of the optimal solution by establishing the necessary and sufficient conditions. A sensitivity analysis is conducted, followed by the presentation of managerial insights.

MODEL DESCRIPTION AND FORMULATION

This section provides the model notation, assumptions, and formulation.

Notation

K The cost of order per unit order.

P The purchasing cost per unit per unit time (\$/unit/ year).

S_b Cost of shortage per unit for each unit of time.

C_o The unit cost of holding in own warehouse (\$/unit/ year).

C_r The cost of holding each item per unit time in the rented warehouse (\$/unit/ year).

ω_o The rate of deterioration in own warehouse, where $0 < \omega_o < 1$.

ω_r The rate of deterioration in a rented warehouse, where $0 < \omega_r < 1,\ \omega_r < \omega_o$

x_d The length of time in which the product exhibits no deterioration.

x_r Time at which the inventory level reaches zero in a rented warehouse.

x_o Time at which the inventory level reaches zero in the owned warehouse.

X The length of the replenishment cycle time (time unit).

R_m The maximum positive inventory level per cycle

R_d Capacity of the owned warehouse

$(R_m - R_d)$ Capacity of the rented warehouse

N_m The backorder level during the shortage period.

R The order quantity during the cycle length, where $R = (R_m + N_m)$.

$Q_O(x)$ Inventory level in the owned warehouse at any time x, where $0 \le x \le X$.

$Q_r(x)$ Inventory level in the rented warehouse at any time x, where $0 \le x \le X$.

$Q_s(x)$ Shortage level at any time x where $x_o \le x \le X$.

Assumptions

This model is established under the following assumptions.

1. Replenishment occurs instantaneously.
2. The model considers a single item with a non-instantaneous decay process.
3. The own warehouse has a fixed capacity of R_d units while the rented warehouse has a capacity of $(R_m - R_d)$.
4. The unit inventory holding cost per unit time is higher in the rented warehouse compared to the owned warehouse. Similarly, the deterioration rate in the rented warehouse is lower than that in the owned warehouse.
5. There is no replacement or repair of deteriorated goods during the period under consideration.
6. The consumption rate before deterioration begins is given by α.
7. The consumption rate within the deterioration period is given by β.
8. The consumption rate during a shortage is given by γ.
9. Shortages are permitted and partially backlogged. The backlogging rate is variable, depending on the waiting time for the next replenishment. Specifically, the backlogging rate decreases as the waiting time increases. The negative inventory backlog rate is calculated as $N(x) = \frac{1}{1+\delta(X-x)}$, δ is the backlogging parameter ($0 < \delta < 1$)and $(X - x)$ is waiting time $(x_o \le x \le X)$, $1 - N(x)$ is the remaining fraction lost.

FORMULATION OF THE MODEL

For non-instantaneous decaying commodities with two-phase demand rates, two storage facilities, and allowable payment delays, this model investigates the best replenishment plan. Payment delays can incentivize retailers to increase stock levels as they improve sales, enhance cash flow, reduce holding costs, attract new customers, or retain existing ones. When inventory exceeds the retailer's warehouse capacity, renting additional storage becomes an option for managing surplus. In this system, R_m units of a product arrive at the start of each cycle, with R_d units stored in the retailer's warehouse and the remaining $(R_m - R_d)$ units in the rented warehouse. To determine the optimal replenishment policy, the article examines two scenarios: one where $x_d < x_r$ and another where $x_d > x_r$.

Case I: When $t_d < t_r$ (Deterioration starts before the inventory level in the rented warehouse becomes zero)

Figure 1 depicts the operation of the inventory system. In the time span $[0,\ x_d]$, the inventory level $Q_r(x)$ in the rented warehouse decreases progressively due to market demand, modelled as a quadratic function of time x, while the inventory level in the owned warehouse remains stable. During the interval $[x_d,\ x_r]$, the inventory level $Q_r(x)$ in the rented warehouse declines further, driven by both the constant market demand rate λ and deterioration, whereas the inventory in the owned warehouse decreases only due to deterioration. In the period $[x_r,\ x_o]$, the inventory level $Q_o(x)$ in the owned warehouse depletes completely because of the combined impact of consumer demand and deterioration. Shortages occur at $x = x_o$ and are partially backlogged during the interval $[x_o,\ X]$. This inventory process repeats in cycles.

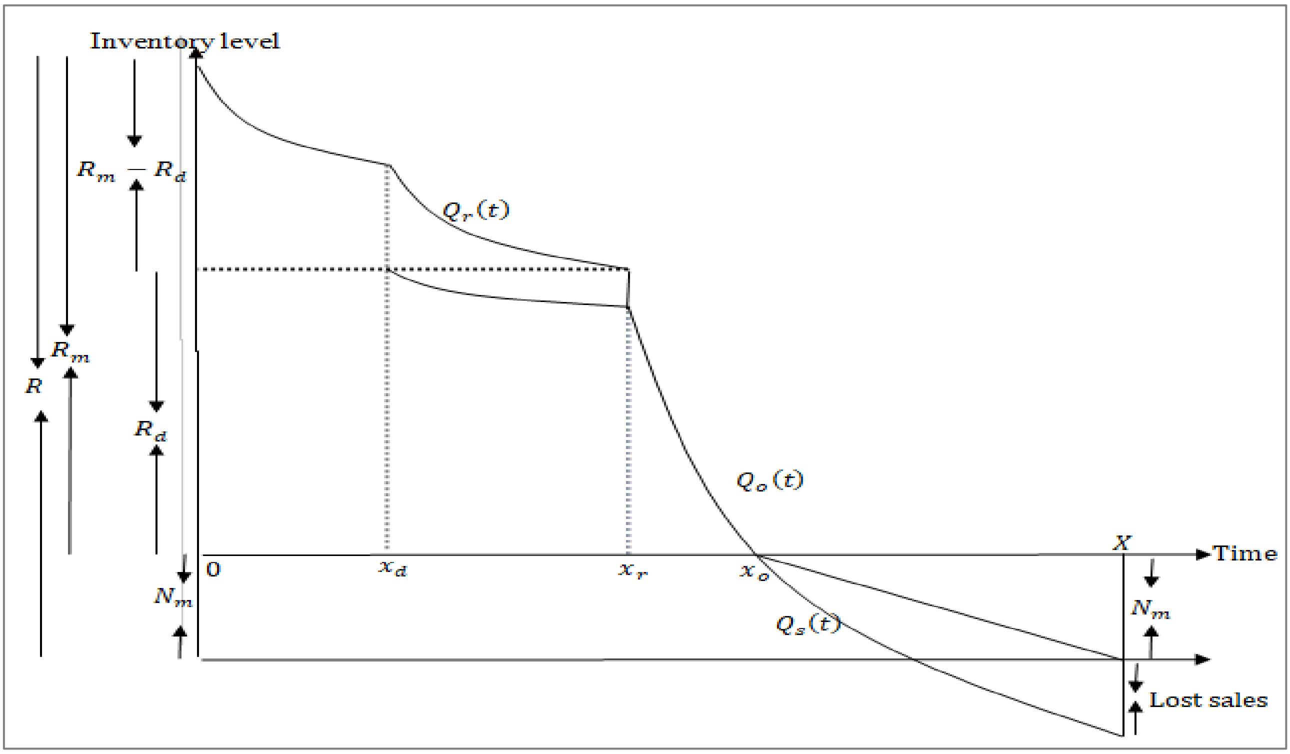

Figure 1: Two-ware-house inventory system when $x_d < x_r$

The differential equations that govern the inventory levels in both the rented and owned warehouses at any time x within the interval $[0,\ X]$ are expressed as follows:

$$\frac{dQ_r(x)}{dx} = -\alpha, \qquad 0 \le x \le x_d \tag{1}$$

$$\frac{dQ_r(x)}{dx} + \omega_r Q_r(x) = -\beta, \quad x_d \le x \le x_r \tag{2}$$

$$\frac{dQ_o(x)}{dx} + \omega_o Q_o(x) = 0, \quad x_d \le x \le x_r \tag{3}$$

$$\frac{dQ_o(x)}{dx} + \omega_o Q_o(x) = -\beta, \quad x_r \le x \le x_o \tag{4}$$

$$\frac{dQ_s(x)}{dx} = -\frac{\gamma}{1 + \delta(X - x)}, \qquad x_o \le x \le X \tag{5}$$

with boundary conditions $Q_r(0) = R_m - R_d$, $Q_r(x_r) = 0$, $Q_o(x_d) = R_d$, $Q_o(x_o) = 0$ and $Q_s(x_o) = 0$.

The solutions of equations (1), (2), (3), (4) and (5) are as follows

$$Q_r(x) = R_m - R_d - (\alpha x), \quad 0 \le x \le x_d \tag{6}$$

$$Q_r(x) = \frac{\beta}{\omega_r}\left(e^{\omega_r(x_r - x)} - 1\right), \quad x_d \le x \le x_r \tag{7}$$

$$Q_o(x) = R_d e^{\omega_o(x_d - x)}, \quad x_d \le x \le x_r \tag{8}$$

$$Q_o(x) = \frac{\beta}{\omega_o}\left(e^{\omega_o(x_o - x)} - 1\right), \quad x_r \le x \le x_o \tag{9}$$

$$Q_s(x) = -\frac{\gamma}{\delta}\left[ln[1 + \delta(X - x_o)] - ln[1 + \delta(X - x)]\right], \quad x_o \le x \le X \tag{10}$$

Considering continuity of $Q_o(x)$ at $x = x_r$, it follows from equations (8) and (9) that

$$R_d = \frac{\beta}{\omega_o}\left(e^{\omega_o(x_o - x_d)} - e^{\omega_o(x_r - x_d)}\right), \quad x_o \le x \le X \tag{11}$$

Considering continuity of $Q_r(x)$ at $x = x_d$, it follows from equations (6) and (7) that

$$R_m = \frac{\beta}{\omega_o}\left(e^{\omega_o(x_o - x_d)} - e^{\omega_o(x_r - x_d)}\right) + (\alpha x_d) + \frac{\beta}{\omega_r}\left(e^{\omega_r(x_r - x_d)} - 1\right), \quad x_o \le x \le X \tag{12}$$

The maximum backordered units, N_m, are reached at $x = X$, and from equation (10), it can be derived that:

$$N_m = -Q_s(X) = \frac{\gamma}{\delta}\left[ln[1 + \delta(X - x_o)]\right] \tag{13}$$

As a result, the order size over the entire period $[0,\ X]$ is:

$$R = R_m + N_m = \frac{\beta}{\omega_o}\left(e^{\omega_o(x_o - x_d)} - e^{\omega_o(x_r - x_d)}\right) + (\alpha x_d) + \frac{\beta}{\omega_r}\left(e^{\omega_r(x_r - x_d)} - 1\right) + \frac{\gamma}{\delta}\left[ln[1 + \delta(X - x_o)]\right] \tag{14}$$

The total variable cost per unit of time is given by

$TVC_1(x_o, X) = \frac{1}{X}${Sum of inventory holding cost for rented warehouse, inventory holding cost for owned ware-house, Ordering cost, cost of deterioration, cost of backordered, lost sales cost, interest charge – interest earned}

$$= \frac{1}{X}\Bigg\{K + C_r\left[\int_0^{x_d} Q_r(x)dx + \int_{x_d}^{x_r} Q_r(x)dx\right] + C_o\left[\int_0^{x_d} Q_o(x)dx + \int_{x_d}^{x_r} Q_o(x)dx + \int_{x_r}^{x_o} Q_o(x)dx\right]$$
$$+ P\left[\omega_r \int_{x_d}^{x_r} Q_r(x)dx + \omega_o \int_{x_d}^{x_r} Q_o(x)dx + \omega_o \int_{x_r}^{x_o} Q_o(x)dx\right] + S_b\left[\int_{x_o}^{X} -Q_s(x)dx\right]$$
$$+ L_\pi \int_{x_o}^{X}\left(1 - \frac{\gamma}{1 + \delta(X - x)}\right)dx\Bigg\}$$

$$= \frac{1}{X}\left\{\frac{1}{2}\mu_1 x_o^2 - \sigma_1 x_o + \pi_1 + \frac{\gamma(S_b + L_\pi\delta)}{2}X^2 - \gamma(S_b + L_\pi\delta)x_o X\right\} \tag{16}$$

but

$\mu_1 = [\beta C_o[\omega_o x_d + 1] + \beta P\omega_o + \gamma(S_b + L_\pi\delta)], \sigma_1 = \beta[C_o\omega_o x_d^2 + P x_d \omega_o]$ *and*

$$\pi_1 = \Bigg[K + C_r\left[\left(\alpha\frac{x_d^2}{2}\right) + \frac{\beta}{2}\{x_r^2 + \omega_r(x_r - x_d)^2 x_d\}\right] + C_o\left[\frac{\beta}{2}\{\omega_o(2x_r x_d^2 - x_r^2 x_d) - x_r^2\}\right]$$
$$+ P\left[\frac{\beta}{2}\{\omega_r(x_r - x_d)^2\} + \frac{\beta}{2}\{\omega_o(2x_r x_d - x_r^2)\}\right]\Bigg].$$

Optimal Decision

The necessary and sufficient conditions for identifying the optimal ordering policies that minimize the total variable cost per unit time are formulated. In particular, the necessary condition for minimizing the total variable cost per unit time, $TVC_1(x_o, X)$, is as follows:

$\frac{\partial TVC_1(x_o,X)}{\partial x_o} = 0$ and $\frac{\partial TVC_1(x_o,X)}{\partial X} = 0$ when $x_r > x_d$. The value of (x_o, X) obtained from $\frac{\partial TVC_1(x_o,X)}{\partial x_o} = 0$ and $\frac{\partial TVC_1(x_o,X)}{\partial X} = 0$ and for which the sufficient condition $\left\{\left(\frac{\partial^2 TVC_1(x_o,X)}{\partial x_o^2}\right)\left(\frac{\partial^2 TVC_1(x_o,X)}{\partial X^2}\right) - \left(\frac{\partial^2 TVC_1(x_o,X)}{\partial x_o \, \partial X}\right)^2\right\} > 0$ is satisfied which gives a minimum for the total variable cost per unit time $TVC_1(x_o, X)$.

The necessary conditions for the total variable cost in equation (16) to be the minimum are $\frac{\partial TVC_1(x_o,X)}{\partial x_o} = 0$ and $\frac{\partial TVC_1(x_o,X)}{\partial X} = 0$, which give

$$\frac{\partial TVC_1(x_o, X)}{\partial x_o} = \frac{1}{X}\{\mu_1 x_o - \sigma_1 - (S_b + L_\pi \delta)X\} = 0 \qquad (20)$$

and

$$X = \frac{1}{(S_b + L_\pi \delta)}(\mu_1 x_o - \sigma_1) \qquad (21)$$

Note that

$$\mu_1 x_o - \sigma_1 = [C_o(x_d \omega_o(x_o - x_d) + x_o) + P\omega_o(x_o - x_d) + (S_b + L_\pi \delta)x_o] > 0$$

since $(x_o - x_d) > 0$

Similarly

$$\frac{\partial TVC_1(x_o, X)}{\partial X} = -\frac{1}{X^2}\left\{\frac{1}{2}\mu_1 x_o^2 - \sigma_1 x_o + \pi_1 - \frac{X^2}{2}(S_b + L_\pi \delta)\right\} = 0 \qquad (22)$$

X from equation (21) is substituted into equation (22) which yields

$$\mu_1\big(\mu_1 - \gamma(S_b + L_\pi \delta)\big)x_o^2 - 2\sigma_1\big(\mu_1 - \gamma(S_b + L_\pi \delta)\big)x_o - (2\gamma(S_b + L_\pi \delta)\pi_1 - \sigma_1^2) = 0 \qquad (23)$$

Let $\Delta_1 = \mu_1\big(\mu_1 - \gamma(S_b + L_\pi \delta)\big)x_d^2 - 2\sigma_1\big(\mu_1 - \gamma(S_b + L_\pi \delta)\big)x_d - (2\gamma(S_b + L_\pi \delta)\pi_1 - \sigma_1^2)$, then the outcome shown below is attained.

Lemma 3.1

(I) If $\Delta_1 \leq 0$ then, the solution of $x_o \in [x_d, \infty)$ (say x_{o1}^*) which satisfies equation (23) not only exists but also is unique.

(II) If $\Delta_1 > 0$ then, the solution of $x_o \in [x_d, \infty)$ which satisfies equation (23) does not exist.

Proof of (I): From equation (23), a new function $\varphi_1(x_o)$ is defined as follows

$$\varphi_1(x_o) = \mu_1\big(\mu_1 - \gamma(S_b + L_\pi \delta)\big)x_o^2 - 2\sigma_1\big(\mu_1 - \gamma(S_b + L_\pi \delta)\big)x_o - (2\gamma(S_b + L_\pi \delta)\pi_1 - \sigma_1^2), \quad x_o \in [x_d, \infty) \qquad (24)$$

Taking the first-order derivative of $\varphi_1(x_o)$ with respect to $x_o \in [x_d, \infty)$ yields

$$\frac{\varphi_1(x_o)}{dx_o} = 2(\mu_1 x_o - \sigma_1)\big(\mu_1 - \gamma(S_b + L_\pi \delta)\big) > 0$$

Because $(\mu_1 x_o - \sigma_1) > 0$ and $\left(\mu_1 - \gamma(S_b + L_\pi \delta)\right) = \beta[C_o[\omega_o x_d + 1] + P\omega_o] > 0$

Hence $\varphi_1(x_o)$ is a strictly increasing of x_o in the interval $[x_d, \infty)$. Moreover, $\lim_{x_o \to \infty} \varphi_1(x_o) = \infty$ and $\varphi_1(x_d) = \Delta_1 \leq 0$. Therefore, by applying intermediate value According to the theorem, there exists a unique x_o, denoted as $x_1^* \in [x_d, \infty)$, such that $\varphi_1(x_{o1}^*) = 0$. Therefore, x_{o1}^* is the unique solution to equation (23). Consequently, the value of x_o (denoted by x_{o1}^*) can be determined from equation (23) and is given by:

$$x_{o1}^* = \frac{\sigma_1}{\mu_1} + \frac{1}{\mu_1}\sqrt{\frac{(2\mu_1\pi_1 - \sigma_1^2)\gamma(S_b + L_\pi\delta)}{\left(\mu_1 - \gamma(S_b + L_\pi\delta)\right)}} \tag{25}$$

After determining x_{o1}^*, the value of X (denoted by X_1^*) can be calculated using equation (21) and is expressed as:

$$X_1^* = \frac{1}{\gamma(S_b + L_\pi\delta)}(\mu_1 x_{o1}^* - \sigma_1) \tag{26}$$

Equations (25) and (26) yield the optimal values for x_{o1}^* and X_1^* for the cost function described in equation (16), but only if σ_1 meets the criterion outlined in equation (27).

$$\sigma_1^2 < 2\mu_1\pi_1 \tag{27}$$

Proof of (II): If $\Delta_1 > 0$, Therefore, from equation (24), $\varphi_1(x_o) > 0$. Given that $\varphi_1(x_o)$ is a strictly increasing function over the interval $[x_d, \infty)$, it follows that $\varphi_1(x_o) > 0$ for all x_o in this range. Consequently, there is no value of x_o in $[x_d, \infty)$ for which $\varphi_1(x_o) = 0$. This concludes the proof.

Theorem 3.1

(I) If $\Delta_1 \leq 0$ then, the total variable cost $TVC_1(x_o, X)$ is convex and approaches its global minimum at the point $(x_{o1}^*,\ X_1^*)$, which is a point satisfies equations (23) and (20).

(II) If $\Delta_1 > 0$, then the total variable cost $TVC_1(x_o, X)$ has a minimum value at the point $(x_{o1}^*,\ X_1^*)$ where $x_{o1}^* = x_d$ and $X_1^* = \frac{1}{\gamma(S_b + L_\pi\delta)}(\mu_1 x_d - \sigma_1)$

Proof of (I): When $\Delta_1 \leq 0$, it is noted that x_{o1}^* and X_1^* are the unique solutions to the equations (23) and (20) as established in Lemma 3.1(I). By taking the 2nd derivative of $TVC_1(x_o, X)$ with respect to x_o and X, and evaluating these functions at the point $(x_{o1}^*,\ X_1^*)$, we obtain:

$$\left.\frac{\partial^2 TVC_1(x_o,\ X)}{\partial x_o^2}\right|_{(x_{o1}^*,\ X_1^*)} = \frac{1}{X_1^*}\mu_1 > 0$$

$$\left.\frac{\partial^2 TVC_1(x_o,\ X)}{\partial x_o \partial X}\right|_{(x_{o1}^*,\ X_1^*)} = -\frac{1}{X_1^*}\gamma(S_b + L_\pi\delta)$$

$$\left.\frac{\partial^2 TVC_1(x_o,\ X)}{\partial X^2}\right|_{(x_{o1}^*,\ X_1^*)} = \frac{1}{X_1^*}\gamma(S_b + L_\pi\delta) > 0$$

and

$$\left(\left.\frac{\partial^2 TVC_1(x_o,\ X)}{\partial x_o^2}\right|_{(x_{o1}^*,\ X_1^*)}\right)\left(\left.\frac{\partial^2 TVC_1(x_o,\ X)}{\partial X^2}\right|_{(x_{o1}^*,\ X_1^*)}\right) - \left(\left.\frac{\partial^2 TVC_1(x_o,\ X)}{\partial x_o \partial X}\right|_{(x_{o1}^*,\ X_1^*)}\right)^2 = \frac{\gamma(S_b + L_\pi\delta)}{X_1^{*2}}\beta[C_o[\omega_o x_d + 1] + P\omega_o] > 0 \tag{28}$$

It follows from equation (28) and Lemma 3.1 that $TVC_1(x_{o1}^*,\ X_1^*)$ is convex, and $(x_{o1}^*,\ X_1^*)$ represents the global minimum point of $TVC_1(x_o,\ X)$. Therefore, the values of x_o and X given in equations (25) and (26) are optimal.

Proof of (II): When $\Delta_1 > 0$, then $\varphi_1(x_o) > 0$ for all $x_o \in [x_d,\ \infty)$. Thus, $\frac{\partial TVC_1(x_o,\ X)}{\partial X} = \frac{\varphi_1(x_o)}{X^2} > 0$ for all $x_o \in [x_d, \infty)$ which implies $TVC_1(x_o,\ X)$ is an increasing function of T. Thus $TVC_1(x_o,\ X)$ has the minimum value when T is minimum. Therefore, $TVC_1(x_o,\ X)$ has a minimum value at the point $(x_{o1}^*,\ X_1^*)$ where $x_{o1}^* = x_d$ and $X_1^* = \frac{1}{\gamma(S_b + L_\pi \delta)}(\mu_1 x_d - \sigma_1)$. This completes the proof.

EOQ^* =The maximum inventory +the backordered units during the shortage period.

$$= \frac{\beta}{\omega_o}\left(e^{\omega_o(x_o^* - x_d)} - e^{\omega_o(x_r - x_d)}\right) + \alpha x_d + \frac{\beta}{\omega_r}\left(e^{\omega_r(x_r - x_d)} - 1\right) + \frac{\gamma}{\delta}\left[ln[1 + \delta(X^* - x_o^*)]\right] \tag{50}$$

Numerical Examples

This section presents several numerical examples to demonstrate the application of the established model.

Example 3.1.1 (Case I)

Consider an inventory model with the input parameters: $K = \$500$/order, $P = \$65$/unit/year, $C_o = \$9$/unit/year, $C_r = \$15$/unit/year, $\omega_o = 0.08$ units/year, $\omega_r = 0.05$ units/year, $\alpha = 1080$ units, $\beta = 450$ units, $\gamma = 105$ $x_d = 0.1998$ year, $x_r = 0.2224$, $S_b = \$30$/unit/year, $L_\pi = \$15$/unit/year, $\delta = 0.7$, $\Delta_1 = -88.899 < 0$, $\sigma_1^2 = 0.7998$, $2\mu_1\pi_1 = 114.119$ and hence $\sigma_1^2 < 2\mu_1\pi_1$.

Using the above parameter values in equations (25), (26), (16) and (50), the optimal time at which the inventory level reaches zero in the owned warehouse, cycle length, total variable cost and EOQ are obtained as $x_{o1}^* = 0.523$ year, $X_1^* = 0.789$ year, $TVC_1(x_{o1}^*,\ X_1^*) = \$2345.767$ per year and $EOQ_1^* = 578.977$ units per year respectively.

Case II: when $t_d < t_r$ (Deterioration starts after the inventory level in the rented warehouse becomes zero)

Figure 2 shows the inventory system's behavior. During the time interval $[0,\ x_r]$, the inventory level $Q_r(x)$ in the rented warehouse gradually decreases due to market demand, following a quadratic function of time x, while the inventory level in the owned warehouse stays constant. In the interval $[x_r,\ x_d]$, the inventory level $Q_o(x)$ in the owned warehouse diminishes due to consumer demand, also described by a quadratic function of time x. In the interval $[x_d,\ x_o]$, the inventory in the owned warehouse dropped to zero as a result of both consumer demand and deterioration. Shortages occur at $x = x_o$ and are partially backlogged during the interval $[x_o, X]$. This entire inventory cycle is then repeated.

The differential equations that detail the inventory levels in both the rented and owned warehouses at any time x throughout the time interval $[0,\ X]$ are given by:

$$\frac{dQ_r(x)}{dx} = -\alpha, \qquad 0 \le x \le x_r \tag{51}$$

$$\frac{dQ_o(x)}{dx} = -\alpha, \qquad x_r \le x \le x_d \tag{52}$$

$$\frac{dQ_o(x)}{dx} + \omega_o Q_o(x) = -\beta, \qquad x_d \le x \le x_o \tag{53}$$

$$\frac{dQ_s(x)}{dx} = -\frac{\gamma}{1 + \delta(X - x)}, \qquad x_o \le x \le X \tag{54}$$

with boundary conditions $Q_r(x_r) = 0$, $Q_o(x_r) = R_d$, $Q_o(x_o) = 0$ and $Q_s(x_o) = 0$.

The solutions of equations (51), (52), (53) and (54) are as follows

$$Q_r(x) = \alpha(x_r - x), \qquad 0 \le x \le x_r \tag{55}$$

$$Q_o(x) = R_d + \alpha(x_r - x), \qquad x_r \le x \le x_d \tag{56}$$

$$Q_o(x) = \frac{\beta}{\omega_o}\left(e^{\omega_o(x_o - x)} - 1\right), \qquad x_d \le x \le x_o \tag{57}$$

$$Q_s(x) = -\frac{\gamma}{\delta}\left[ln[1 + \delta(X - x_o)] - ln[1 + \delta(X - x)]\right], \qquad x_o \le x \le X \tag{58}$$

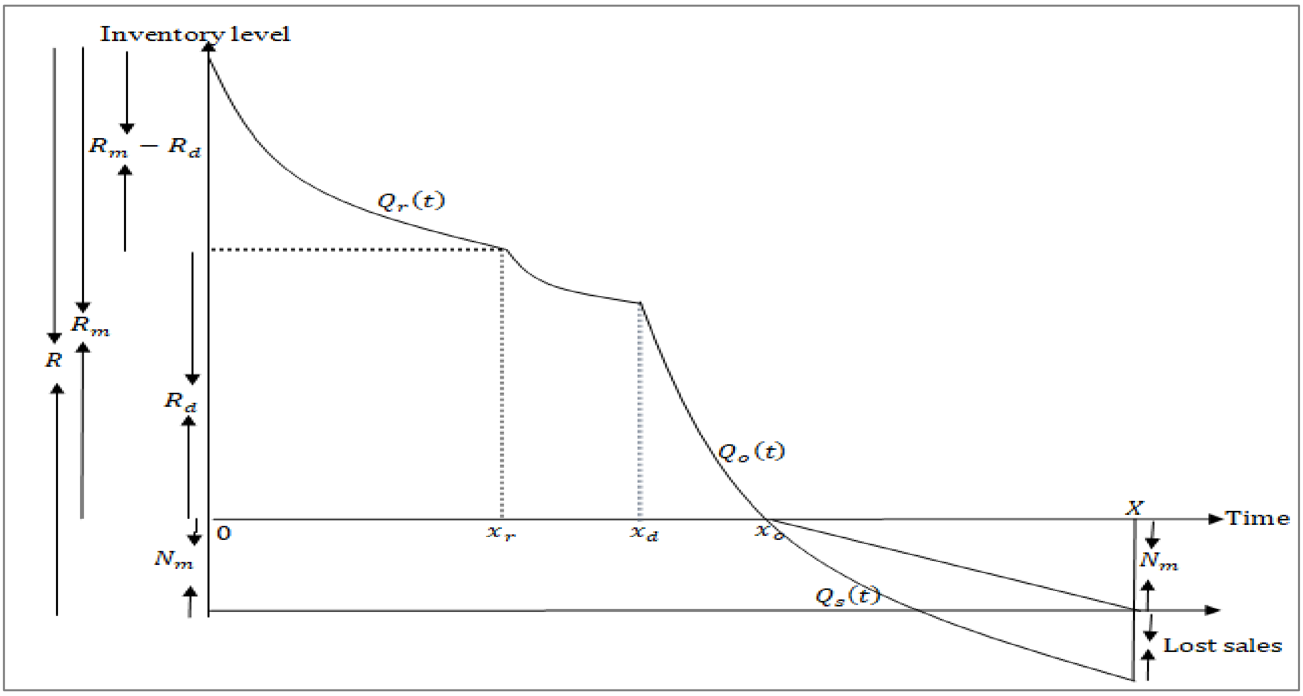

Figure 2: Two-ware-house inventory system when $x_d > x_r$

Considering the continuity of $Q_o(x)$ at $x = x_d$, it follows from equations (56) and (57) that

$$R_d = \alpha(x_d - x_r) + \frac{\beta}{\omega_o}\left(e^{\omega_o(x_o - x_d)} - 1\right) \tag{59}$$

Now, at $x = 0$ when $Q_r(x) = R_m - R_d$ and solving equation (55) to get the maximum inventory level per cycle as

$$R_m = \alpha x_d + \frac{\beta}{\omega_o}\left(e^{\omega_o(x_o - x_d)} - 1\right) \tag{60}$$

The maximum backordered units N_m is reached at $x = X$, and from equation (58), it can be derived that:

$$N_m = -Q_s(X) = \frac{\gamma}{\delta}\left[ln[1 + \delta(X - x_o)]\right] \tag{61}$$

Thus the order size within complete time interval $[0, X]$ is

$$R = R_m + N_m = \alpha x_d + \frac{\beta}{\omega_o}\left(e^{\omega_o(x_o - x_d)} - 1\right) + \frac{\gamma}{\delta}\left[ln[1 + \delta(X - x_o)]\right], \qquad x_o \le x \le X \tag{62}$$

The total variable cost per unit time is given by

$TVC_2(x_o, X) =$ (Sum of o inventory holding cost for rented warehouse, inventory holding cost for owned warehouse, ordering cost, backordered cost, deterioration cost, interest charge – interest earned)

$$= \frac{1}{X}\left\{K + C_r\left[\int_0^{x_r} Q_r(x)dx\right] + C_o\left[\int_0^{x_r} Q_o(x)dx + \int_{x_r}^{x_d} Q_o(x)dx + \int_{x_d}^{x_o} Q_o(x)dx\right] + P\left[\omega_o \int_{x_d}^{x_o} Q_o(x)dx\right] + S_b\left[\int_{x_o}^{X} -Q_s(x)dx\right] + L_\pi \int_{x_o}^{X}\left(1 - \frac{\gamma}{1+\delta(X-x)}\right)dx\right\}$$

$$= \frac{1}{X}\left\{\frac{1}{2}\mu_2 x_o^2 - \sigma_2 x_o + \pi_2 + \frac{(S_b + L_\pi\delta)}{2}X^2 - (S_b + L_\pi\delta)x_o X\right\} \quad (64)$$

where

$\mu_2 = [\beta C_o[\omega_o x_d + 1] + \beta P\omega_o + \gamma(S_b + L_\pi\delta)], \sigma_2 = \beta[C_o\omega_o x_d^2 + P\omega_o x_d]$ and

$$\pi_2 = \left[K + C_r\left\{\frac{\alpha}{2}x_r^2\right\} + C_o\left[\frac{\alpha}{2}(x_d^2 - x_r^2) + \frac{\beta\omega_o x_d^3}{2} - \frac{\beta}{2}x_d^2\right] + P\frac{\beta}{2}\omega_o x_d^2\right].$$

Optimal Decision

The necessary and sufficient conditions are established to find the optimal ordering policies that minimise the total variable cost per unit of time. The necessary conditions for the total variable cost per unit time $TVC_2(x_o, X)$ to be minimum are $\frac{\partial TVC_2(x_o,X)}{\partial x_o} = 0$ and $\frac{\partial TVC_2(x_o,X)}{\partial X} = 0$ when $x_d > x_r$. The value of (x_o, X) obtained from $\frac{\partial TVC_2(x_o,X)}{\partial x_o} = 0$ and $\frac{\partial TVC_2(x_o,X)}{\partial X} = 0$ and for which the sufficient condition $\left\{\left(\frac{\partial^2 TVC_2(x_o,X)}{\partial x_o^2}\right)\left(\frac{\partial^2 TVC_2(x_o,X)}{\partial X^2}\right) - \left(\frac{\partial^2 TVC_2(x_o,X)}{\partial x_o\,\partial X}\right)^2\right\} > 0$ is satisfied, it guarantees that the total variable cost per unit time $TVC_2(x_o, X)$ is minimized.

The necessary conditions for minimising the total variable cost $TVC_2(x_o, X)$ in equation (64) to be the minimum are $\frac{\partial TVC_2(x_o,X)}{\partial x_o} = 0$ and $\frac{\partial TVC_2(x_o,X)}{\partial X} = 0$, which give

$$\frac{\partial TVC_2(x_o, X)}{\partial x_o} = \frac{1}{X}\{\mu_2 x_o - \sigma_2 - \gamma(S_b + L_\pi\delta)X\} = 0 \quad (68)$$

and

$$X = \frac{1}{\gamma(S_b + L_\pi\delta)}(\mu_2 x_o - \sigma_2) \quad (69)$$

Note that

$$\mu_2 x_o - \sigma_2 = [\beta C_o(x_d\omega_o(x_o - x_d) + x_o) + \beta P\omega_o(x_o - x_d) + \gamma(S_b + L_\pi\delta)x_o] > 0$$

since $(x_o - x_d), > 0$

Similarly

$$\frac{\partial TVC_2(x_o, X)}{\partial X} = -\frac{1}{X^2}\left\{\frac{1}{2}\mu_2 x_o^2 - \sigma_2 x_o + \pi_2 - \frac{\sigma^2}{2}(S_b + L_\pi\delta)\right\} = 0 \quad (70)$$

Replacing X from equation (69) into equation (70) yields

$$\mu_2\big(\mu_2 - \gamma(S_b + L_\pi\delta)\big)x_o^2 - 2\sigma_2\big(\mu_2 - \gamma(S_b + L_\pi\delta)\big)x_o - (2\gamma(S_b + L_\pi\delta)\pi_2 - \sigma_2^2) = 0 \quad (71)$$

Let $\Delta_2 = \mu_2\big(\mu_2 - \gamma(S_b + L_\pi\delta)\big)x_r^2 - 2\sigma_2\big(\mu_2 - \gamma(S_b + L_\pi\delta)\big)x_r - (2\gamma(S_b + L_\pi\delta)\pi_2 - \mathrm{X}_2^2)$, then the following result is obtained.

Lemma 3.2.1

(i) When $\Delta_2 \leq 0$ then, the solution of $x_o \in [x_r, \infty)$ (known as x_{o2}^*) which satisfies equation (71) does not only exist but is also unique.

(ii) When $\Delta_2 > 0$ then, the solution of $x_1 \in [x_r, \infty)$ which satisfies equation (71) does not exist.

Proof: The proof follows a similar process to Lemma 3.1.1.

Therefore, the value of x_o (denoted by x_{o2}^*) can be determined from equation (71) and is expressed as:

$$x_{o2}^* = \frac{X_2}{\mu_2} + \frac{1}{\mu_2}\sqrt{\frac{(2\mu_2\pi_2 - X_2^2)\gamma(S_b + L_\pi\delta)}{\left(\mu_2 - \gamma(S_b + L_\pi\delta)\right)}} \tag{72}$$

After determining x_{o2}^*, the value of X (denoted by X_2^*) can be found using equation (69) and is given by:

$$X_2^* = \frac{1}{\gamma(S_b + L_\pi\delta)}(\mu_2 x_{o2}^* - \sigma_2) \tag{73}$$

The optimal values of x_{o2}^* and X_2^* for the cost function in equation (33) are determined by equations (72) and (73), given that σ_2 satisfies the requirement outlined in equation (74).

$$X_2^2 < 2\mu_2\pi_2 \tag{74}$$

Theorem 3.2.1

(i) When $\Delta_2 \leq 0$ then, the total variable cost $TVC_2(x_o, X)$ is convex and reaches its global minimum at the point (x_{o2}^*, X_2^*), where (x_{o2}^*, X_2^*) is the point which satisfies equations (71) and (68).

(ii) When $\Delta_2 > 0$ then, the total variable cost $TVC_2(x_o, X)$ has a minimum value at the point (x_{o2}^*, X_2^*) where $x_{o2}^* = x_r$ and $X_2^* = \frac{1}{\gamma(S_b+L_\pi\delta)}(\mu_2 x_r - \sigma_2)$

Proof: The proof follows a similar process to Theorem 3.1.1.

Therefore, the economic order quantity (EOQ) associated with the optimal cycle length X^* can be calculated as follows:

EOQ^* =The maximum inventory + the backordered units during the shortage period.

$$= \alpha x_d + \frac{\beta}{2}x_d^2 + \frac{\gamma}{3}x_d^3 + \frac{\lambda}{\omega_o}\left(e^{\omega_o(x_o^* - x_d)} - 1\right) + \frac{\gamma}{\delta}\left[ln[1 + \delta(X^* - x_o^*)]\right] \tag{96}$$

Numerical Examples

This section includes several numerical examples to demonstrate the model that has been developed.

Example 3.2.1

In addition to $x_d = 0.2476$ year, the parameter values are the same as in Example 3.1.1. It is observed that $\Delta_2 = -76.766 < 0$, $X_2^2 = 0.898$, $2\mu_2\pi_2 = 157.656$ and hence $\sigma_2^2 < 2\mu_2\pi_2$. By substituting the parameter values into equations (72), (73), (64), and (96), the optimal time at which the inventory level in the owned warehouse reaches zero, the cycle length, the total variable cost, and the economic order quantity can be determined as follows: $x_{o2}^* = 0.517$ year, $X_2^* = 0.776$ year, $TVC_2(x_{o2}^*, X_2^*) = \$2286.465$ per year and $EOQ_2^* = 588.807$ units per year.

SENSITIVITY ANALYSIS

The sensitivity analysis examines how variations in different parameters impact the inventory system. Each parameter is adjusted individually by ±20% while keeping other parameters constant. The analysis assesses the effects on several aspects, including the time it takes for the inventory level in the owned warehouse to reach zero, the cycle length, the total variable cost, and the economic order quantity per cycle. The results for all examples in Case I and Case II are summarized in the tables below.

Table 1: Effect of change in credit period (δ) on decision Variables

Cases	% change in δ	% change in t_o^*	% change in T^*	% change in EOQ^*	% change in $Z(t_o^*, T^*)$
1	-20%	-0.6676	1.0988	-1.0878	-4.87888
2		-0.5165	0.8987	-0.8998	-3.8787
1	-10%	-0.3784	0.6565	0.5758	-0.3881
2		-0.3111	0.5998	0.5512	-0.4002
1	10%	0.6565	-0.6741	0.6654	3.0883
2		0.4166	-0.6287	-0.6211	0.5326
1	20%	0.8981	-1.3654	1.2876	5.8977
2		0. 8358	-1.3657	-1.0129	1.0907

Table 2: Effect of change in shortage cost (C_b) on decision Variables

Cases	% change in δ	% change in t_o^*	% change in T^*	% change in EOQ^*	% change in $Z(t_o^*, T^*)$
1	-20%	-6.5325	7.9987	4.6538	-10.0886
2		-5.9989	7.9901	4.4611	-6.8932
1	-10%	-2.4878	3.4098	1. 9906	-4.3098
2		-2.5001	3.5086	1.9878	-2.9876
1	10%	1.9806	-2.5098	-1.4349	3.4943
2		1.8998	-2. 5609	-1.5087	2.9908
1	20%	3.5451	-4.4333	-2.6098	5. 9885
2		3. 6701	-4.6903	-2.9001	4.9877

Table 3: Effect of change in cost of lost sales (C_π) on decision Variables

Cases	% change in δ	% change in t_o^*	% change in T^*	% change in EOQ^*	% change in $Z(t_o^*, T^*)$
1	-20%	-0.9902	1.2577	0.8007	-5.6009
2		-0.9881	1.2775	0.8665	-1.2098
1	-10%	-0.6901	0.5612	0.3253	-2.7622
2		-0.7487	0.5703	0.3766	-2.9095
1	10%	0.6607	-0.7209	-0.4175	2.9983
2		0.6986	-0.5757	-0.4907	0.6744
1	20%	0.8951	-1.4376	-0.7979	5.9944
2		0.8637	-1.5485	-0.8082	6.2886

RESULTS AND DISCUSSION

From the computational results displayed in the tables above, the following insights for management can be derived:

(i) From the data presented in Table 1, it is evident that an increase in the backlogging parameter (δ) leads to significant changes in the model's optimal parameters. Specifically, as (δ) increases, both the optimal time at which the inventory level reaches zero in the owned warehouse (x_o^*), the economic order quantity (EOQ^*), and the total variable cost $(TVC(x_1^*, X^*))$ all increase. Conversely, the optimal cycle length (X^*) decreases.

This pattern reflects a common real-world scenario: as the backlogging rate increases, the order quantity must also rise to accommodate the higher level of backlogged orders, which in turn raises the total variable cost. Therefore, to balance the total variable cost and profit, it is crucial to manage the backlogging rate carefully. A well-calibrated backlogging parameter can help optimize both cost efficiency and profitability, emphasizing the need for a balanced approach in inventory management.

(ii) From the observations presented in Table 2, it is clear that an increase in the shortage cost (S_b) has notable effects on the model's optimal parameters. Specifically, as (S_b) rises, both the optimal time at which the inventory level reaches zero in the owned warehouse (x_o^*) and the total variable cost $(TVC(x_o^*, X^*))$ increase. Conversely, the optimal cycle length (X^*) and the economic order quantity (EOQ^*) decrease.

This relationship indicates that higher shortage costs lead to a higher total variable cost and a substantial reduction in the number of back-ordered goods. This reduction in back-ordered goods subsequently results in a decreased order quantity. Thus, when shortage costs increase, the inventory system adjusts by reducing the order quantity and thereby increasing the total variable cost, demonstrating the need to carefully balance shortage costs to optimize overall inventory performance.

(iii) From the analysis presented in Table 3, it is evident that an increase in the cost of lost sales (L_π) has a significant impact on the model's optimal parameters. Specifically, as (L_π) rises, both the optimal time at which the inventory level reaches zero in the owned warehouse (x_o^*) and the total variable cost $(TVC(x_o^*, X^*))$ also increase. Conversely, the optimal cycle length (X^*) and the economic order quantity (EOQ^*) decrease.
This inverse relationship underscores the critical importance of managing the cost of lost sales to achieve an optimal inventory system. As the cost of lost sales increases, the model indicates that it becomes necessary to adjust the inventory management parameters to mitigate the associated higher costs. Therefore, maintaining a minimum cost of lost sales is essential for minimizing the total variable cost and optimizing inventory performance. This insight highlights the need for careful consideration of lost sales costs in inventory planning and decision-making processes.

CONCLUSION

This research has introduced an (EOQ) model tailored for inventory systems involving delayed deteriorating items, featuring three distinct consumption rates, two separate storage facilities, and shortages. The model uniquely addresses the scenario where the average consumption rates are not uniform across different stages, specifically before and after the deterioration and during the shortage period. These consumption rates are treated as constants within their respective phases. A notable aspect of our model is its incorporation of partially backlogged shortages, where the backlogging rate is not static but varies according to the waiting time for the next inventory replenishment. This dynamic approach allows for a more realistic representation of how backlogging behaviour might fluctuate with changes in inventory replenishment schedules. Through this model, we have identified the optimal timing for when the inventory level reaches zero in the owned warehouse, the optimal cycle length, and the optimal order quantity that collectively minimize the total variable cost associated with the inventory system. To substantiate our model, we provide several numerical examples that illustrate the practical implications and outcomes of the proposed model under various scenarios. Additionally, a sensitivity analysis has been conducted to examine how variations in key model parameters affect the optimal solutions. This analysis provides insights into the robustness of the model and highlights the influence of different factors on the overall cost efficiency. Looking ahead, the model can be further refined by incorporating additional realistic assumptions. These may include variable deterioration rates that reflect more complex deterioration patterns, inflationary effects impacting cost structures, reliability considerations for goods, and linear holding costs that more accurately capture inventory-carrying expenses. Such enhancements would contribute to a more comprehensive and applicable inventory management framework. In conclusion, this study offers a valuable contribution to inventory management theory by addressing the complexities of deteriorating items and variable backlogging rates. The findings provide a foundation for further research and practical applications, such as incorporating shortages and stock-dependent demand rates aimed at optimizing inventory control and minimizing costs in real-world scenarios.

REFERENCES

Chakrabarty, R., Roy, T. and Chaudhuri, K. S. (2018). A two-warehouse inventory model for deteriorating items with capacity constraints and back-ordering under financial considerations. International Journal of Applied and Computational Mathematics, **4**(58), **[Crossref]**

Chandra, K. J., Sunil, T. and Satish K. G. (2017). Credit financing in economic ordering policies for non-instantaneous deteriorating items with price-dependent demand and two-storage facilities. Annal of Operational Research, **248**(11), 253–280. **[Crossref]**

Das SC (2024). Low Financing Trade Credit Inventory for High Deterioration under Time Dependent Holding Cost. Indian Journal of Science and Technology 17(30): 3093-3099. **[Crossref]**

Datta, D. A., Kumar, Y. R., Gupta, N., Singh, Y. R., Rathee, R., Boadh, R. and Kumar, A. (2022). Selling price, time-dependent demand and variable holding cost inventory model with two-storage facilities. Materials Today: Proceedings, **56**, 245–251. **[Crossref]**

Gupta, M., Tiwari, S. and Jaggi, C. K. (2020). Retailer's ordering policies for time-varying deteriorating items with partial backlogging and permissible delay in payments in a two-warehouse environment. Annals of Operations Research, **295,** 139–161. , **[Crossref]**.

Jaggi, C. K. Cardenas-Barron, L. E., Tiwaria, S. and Shafi, A. A. (2017). A two-warehouse inventory model for deteriorating items with imperfect quality under the conditions of permissible delay in payments. Scientia Iranica E, **24**(1), 390-412. **[Crossref]**

Kumar, A. and Chanda, Y. (2018). A two-warehouse inventory model for deteriorating items with demand influenced by innovation criterion in growing technology market. Journal of Management Analytics, **[Crossref]**

Kumar, N. K., Raj, R., Chandra, S. and Chaudhary, H. (2017). Two warehouse inventory models for deteriorating items with exponential demand rate and permissible delay in payment. Yugoslav Journal of Operations Research, **27** (1), 109-124. **[Crossref]**

Malumfashi, M. L., Asma, A. M., Kane, I. L., Babangida, B., & Yusuf, T. A. (2024). Production System for Delayed Deterioration Inventories with Two-Level Production Rate, Stock-Dependent Demand Rate and Linear Holding Cost Under Trade Credit Policy. *International Journal of Research and Innovation in Applied Science*, *9*(1), 81-104. **[Crossref]**

Sahoo, C. K., Paul, K. C. and Kumar, S. (2020). Two warehouses EOQ inventory model of degrading matter having exponential decreasing order, limited suspension in price including salvage value. SN Computer Science **1** **(**334). , **[Crossref]**.

Tiwari, S., Cardenas-Barron, L.E., Khanna, A., Jaggi, C.K. (2016). Impact of trade credit and inflation on retailer's ordering policies for non-instantaneous deteriorating items in a two-warehouse environment. International Journal of Production Economics, **176**(C), 154–169. **[Crossref]**

Udayakumar, R. and Geetha, K. V. (2018). An EOQ model for non-instantaneous deteriorating items with two levels of storage under trade credit policy. International journal of industrial engineering, **14**(2), 343- 365. **[Crossref]**