UMYU Scientifica, Vol. 3 NO. 4, December 2024, Pp 386 – 398
https://doi.org/10.56919/usci.2434.033
ISSN: 2955 – 1145 (print); 2955 – 1153 (online)
A periodical of the Faculty of Natural and Applied Sciences, UMYU, Katsina

ORIGINAL RESEARCH ARTICLE

ARTICLE HISTORY
Received September 05, 2024
Accepted December 13, 2024
Published December 24, 2024

KEYWORDS
Alternating Sign Matrix, Restricted permutation, Pattern avoidance

Permutation Pattern Avoidance in the Alternating Sign Matrices

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ABSTRACT
We represented permutations using matrices. In the alternating sign matrices, the idea of pattern avoidance in permutation matrices with reference to Aunu Permutations is examined. The primary outcome is that the Catalan numbers count the set of alternating sign matrices related to Aunu Permutations that do not follow pattern 213. Both a bijective and an algebraic proof are given.

INTRODUCTION

We represented permutations using matrices. There are many benefits to the concept of representing permutations with matrices. The fact that permutations are composed of matrix multiplication, that a permutation's inverse is the matrix transpose, and that patterns in permutations are viewed as submatrices that remain after rows and columns are removed from the permutation matrix are some benefits of using this method.

The study of pattern avoidance in permutation matrices is actually quite natural. Since every permutation matrix is also an alternating sign matrix, alternating sign matrices can be thought of as generalizations of permutations. The topic of this study is alternating sign matrix pattern avoidance.

Since the 1980s, alternating sign matrices have shown to be useful in a variety of contexts. They are intriguing combinatorial objects. One of the most active areas of mathematics research nowadays is pattern avoidance in permutations. Combining these two disparate mathematical ideas has been really thrilling, especially since it can lead to some intriguing discoveries.

A matrix is a rectangular array of numbers with an arbitrary number of rows and columns. We shall only talk about square matrices with elements 0s,1sV-1s in this work. Several mathematical phenomena can be represented using a matrix. For example, in linear algebra, a matrix can be used to describe a transformation, such as a rotation or a projection in the plane, and graph theory, it can be used to represent a graph. Matrix analysis is a highly helpful tool in combinatorics for characterizing permutations.

2. General Notations and Preliminaries:

Let's utilize totally different notations throughout the article to keep things clear and avoid confusing the reader.

Let us consistently denote a set of all natural numbers with prime order (cardinality) by $S=\{1,2,3,\ldots,n\}$, where $|S|=n$ (n is the order of S).

Let S_n be a symmetric group on the letters $1,2,\ldots,n$. Denote the permutation $\alpha \in S_n$ by the sequence $[\alpha(1),\alpha(2),\ldots,\alpha(n)]$. That is, the set of bijections on $\{1,2,\ldots,n\}$ is denoted by S_n and set of all permutations of length n that avoids α by $S_n(\alpha)$ where $\alpha \in S_n$.

A permutation q is said to contain a permutation τ if there exists a subsequence of q that has the same relative order as τ, and in this case, τ is said to be a pattern of q. Otherwise, q is said to avoid the pattern τ. For example, the permutation q = 14253 contains a pattern τ = 132 because its subsequence 142 and 253 have the same relative order as τ but avoids a pattern 321.

An arrangement of items or components of a finite set in a certain order without duplication or omission is called a permutation. The set [n]={1,2,...,n} is used to represent any set of n things because it doesn't matter which objects

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How to cite: Abba, S., Sanusi, U., & Sani, S. (2024). Permutation Pattern Avoidance in the Alternating Sign Matrices. *UMYU Scientifica*, 3(4), 386 – 398. https://doi.org/10.56919/usci.2434.033

are placed in a particular set. A permutation is, therefore, a bijection from $[n][n]$. The set of all permutations on $[n]$ is denoted by S_n. Thus, we denote a set of all permutations on $[n]$ by $S_n = \{\pi \vee \pi: [n] \rightarrow [n]\}$. The set of all permutations on a finite set $[n]$ has order $n!\,(i.e., \vee S|⬚|n \vee n!)$.

The word form is the most widely used representation of a permutation. Given that, $\pi = 15432 \in S_5$ is a way to represent a permutation from the set S_5, which means that $\pi(1) = 1, \pi(2) = 5, \pi(3) = 4, \pi(4) = 3$ and $\pi(5) = 2$.

When a permutation of length n is represented by an n×n matrix with a 1 entry on row I and column J if and only if π(i)=j, it is called a permutation matrix. The remaining components are all zero. Permutation matrices are used extensively in this paper. An example is illustrated in Figure 1.

1				
				1
			1	
		1		
	1			

Figure 1: The permutation matrix corresponding to $\pi = 15432$

A pair of elements (i, j) in a permutation π, read from left to right, such that i > j, is called an inversion. Two elements that relate to one another as "up and to the right" constitute an inversion in a permutation matrix. For example, $\pi = 15432 \in S_5$ contains six inversions, $(5,4), (5,3), (5,2), (4,3), (4,2)$ and $(3,2)$. Thus $I(\pi) = \{(i,j): i > j, \wedge\, i, j \in [n]\}$

The total number of inversions in a particular permutation is its inversion number, π, represented as I(π)∨. This means that I(15432) ∨ 6, since π(15432)={(5,4),(5,3),(5,2),(4,3),(4,2),(3,2)} A measure of how "far" a permutation deviates from the identity permutation is the inversion number. 12⋯n, as it indicates how many neighbouring row shifts are necessary for a permutation matrix to get to the identity matrix.

On the other hand, the number of inversions of a given permutation π is given by $i_\pi = \{(i,j): \pi(i) > \pi(j), 1 \leq i < j \leq n\} \vee$. The signature of π is given by $gn(\pi) = (-1)^{i_\pi}$. We say π is an even permutation [respectively; odd permutation] if sign(π) = 1 [respectively; sign(π) = -1]. In other words, we say π is an *even permutation* [respectively; *odd permutation*] if π is a permutation together with even [respectively; odd] number of inversions. For example, consider the Aunu permutations 15423; 15342 and 14532 in S_5 . Each of the Aunu permutations $\pi \in S_5$ has an odd number of inversions, for instance

$For\pi = 15423, i_\pi = |\{(2,3), (2,4), (2,5), (3,4), (3,5)\}| = 5$,;

Therefore, $sign(\pi) = (-1)^5 = -1 \Rightarrow \pi = 15423$ is Odd.

$For\pi = 15342, i_\pi = |\{(2,3), (2,4), (2,5), (3,5), (4,5)\}| = 5$;

Therefore, $sign(\pi) = (-1)^5 = -1 \Rightarrow \pi = 15342$ is Odd.

$For\pi = 14532, i_\pi = |\{(2,4), (2,5), (3,4), (3,5), (4,5)\}| = 5$.

Therefore, $sign(\pi) = (-1)^5 = -1 \Rightarrow \pi = 14532$ is Odd.

A determinant of a square matrix M , which is donated by MM or V, is a number associated with the matrix M. It can be applied to describe different matrix properties. The difference between the diagonal terms' products provides the determinant for a 2 × 2 matrix. Section 3.2 provides a formula for larger determinants.

3. The alternating sign matrices

We present a conjecture on the inversion number of a permutation π in this section. Additionally, we describe the origins of alternating sign matrices (ASM) and provide an introduction to them. The proof of the ASM conjecture is discussed.

3.1 Introduction

Definition As illustrated in Figure 2 below, an alternating sign matrix is a square matrix of 0's, 1's, and −1's where the sum of the entries in each row and column is 1 and the non-zero entries of each row and column alternate in sign.

1				
-1	1			1
				1
1	-1	1		
	1		1	-1

Figure 2: An example of an Alternating sign matrix

It should be noted that alternating matrices can be thought of as generalizations of permutation matrices, as every permutation matrix is also an alternating matrix. A_n is the symbol for the set of matrices with alternating signs.

3.2 Determinants and permutations

Since a determinant of a square matrix of dimension n is determined by adding up all n! permutations, determinants and permutations are closely linked concepts.

A determinant of a matrix M, can be calculated as.

$$\lfloor M \rfloor = \sum_{A \in S_n} (-1)^{I(A)} \prod_{i,j=1}^{n} a_{i,j}^{A_{i,j}}$$

3.3 Some Enumerative results:

Theorem 1: Let $S = \{1,2,\ldots,n\}$ be a finite set and $G = \{1,-1\}$ a multiplicative group. Then show that the mapping $f: S_n \to G$ of a Symmetric group S_n onto the multiplicative group $G = \{1,-1\}$, defined by

$$f(\alpha) = \begin{cases} 1 if \alpha is an even permutation, \forall \alpha \in S_n \\ -1 if \alpha is an Odd permutation, \forall \alpha \in S_n \end{cases},$$

Is a Homomorphism of S_n onto G.

Proof; Let $S = \{1,2,\ldots,n\}$ be a finite set. Then $S_n = \{\alpha \vee \alpha: S \to S\}$ is the set of all permutations on S.

Given that the mapping $f: S_n \to G$ of a Symmetric group S_n onto the multiplicative group $G = \{1,-1\}$, defined by

$$f(\alpha) = \begin{cases} 1 if \alpha is an even permutation, \forall \alpha \in S_n \\ -1 if \alpha is an Odd permutation, \forall \alpha \in S_n \end{cases},$$

Goal: we are to show that $f(\alpha\beta) = f$.

In order to accomplish this, we use the following four procedures;

i. If α, β are both even, then $f(\alpha\beta) = 1 = (1)(1) = f$;
ii. If α, β are both odd, then $f(\alpha\beta) = 1 = (-1)(-1) = f$;
iii. If α is even, and β is odd, then $f(\alpha\beta) = -1 = (1)(-1) = f$;
iv. If α is odd, and β is even, then $f(\alpha\beta) = -1 = (-1)(1) = f$;

Thus, in any case, we have $f(\alpha\beta) = f$.

Also, it is obvious that $f: S_n \to G$ is an onto (surjective) mapping since $Range(f) = f(S|⬚|n) = G$

Hence $f: S_n \to G$ is a Homomorphism of S_n onto G.

3.4. Counting occurrences of the patterns $\sigma \in S_3$ in the Aunu-permutations $\pi \in S_n$.

The counting of occurrences of a pattern σ in a permutation πis the number of distinct subsequences in the permutation π, which are order isomorphic to the pattern σ. Let $\pi = a_1 a_2 \ldots a_t$, $\tau = b_1 b_2 \ldots b_S$ be finite sequences of integers; then a subsequence of π of the same length as τ is said to be an occurrence of τ if its entries occur in the same relative order as in τ . More precisely, given indices $i_1 < i_2 < \ldots < i_S$, the subsequence $a_{i_1} a_{i_2} \ldots a_{i_S}$ of π is an occurrence of τ if and only if for all *j, k* we have $a_{ij} < a_{i_k} \Leftrightarrow b_i < b_j$. Thus, for example, the patterns in $\sigma \in S_3$ occur in the following selected permutations $\pi \in S_5$; for $\pi = 15423$, there are $\binom{5}{3} = 10$ distinct three-length subsequence in π, $154, 152, 153, 142, 143, 123, 542, 543, 524, 423$. Then

$|N_{[123]}(\pi)| = 1, |N_{[132]}(\pi)| = 5, |N_{[213]}(\pi)| = |N_{[231]}(\pi)| = 0, |N_{[312]}(\pi)| = |N_{[321]}(\pi)| = 2$; and, of course $\sum_{\sigma \in S_3} |N_\sigma(\pi)| = 10$. For $\pi = 32451$, 10 three-length subsequence of $\pi = 32451$ are

$324, 325, 321, 345, 341, 351, 245, 241, 251, 451$. Then

$|N_{[123]}(\pi)| = |N_{[213]}(\pi)| = 2, |N_{[321]}(\pi)| = 1, |N_{[132]}(\pi)| = |N_{[312]}(\pi)| = 0, |N_{[231]}(\pi)| = 5$;

For $\pi = 51243$, 10 three-length subsequence of $\pi = 51243$ are

$512, 514, 513, 524, 523, 543, 124, 123, 143, 243$. Then

$|N_{[123]}(\pi)| = |N_{[132]}(\pi)| = 2, |N_{[213]}(\pi)| = |N_{[231]}(\pi)| = 0\, |N_{[321]}(\pi)| = 1, |N_{[312]}(\pi)| = 5$;

For $\pi = 14532$, 10 three-length subsequence of $\pi = 14532$ are

$145, 143, 142, 153, 152, 132, 453, 452, 432, 532$. Then

$$|N_{[123]}(\pi)|=1, |N_{[132]}(\pi)|=5, |N_{[213]}(\pi)|=|N_{[312]}(\pi)|=0, |N_{[231]}(\pi)|=|N_{[321]}(\pi)|=2.$$

Note that the number $|N_\sigma(\pi)|$ is independent of π for example,

- $for\ |N_\sigma(15423)|\ when\ \sigma=213,231,123,312,321,132,\ we\ have\ 0,0,1,2,2,5;$
- $for\ |N_\sigma(32451)|\ when\ \sigma=132,312,321,123,213,231,\ we\ have\ 0,0,1,2,2,5;$
- $for\ |N_\sigma(51243)|\ when\ \sigma=213,231,321,123,132,312,,\ we\ have\ 0,0,1,2,2,5;$
- $for\ |N_\sigma(14532)|\ when\ \sigma=213,312,123,231,321,132,\ we\ have\ 0,0,1,2,2,5.$

Observe that on one hand, the permutation π the in second, third and fourth rows (32451, 51243 and 14532) are the reversal, complementation and inversion of the permutation in the first raw (15423). On the other hand, the corresponding patterns, specifically those bearing the same values $|N_\sigma(\pi)|$ are related similar. For example, the patterns with values of $|N_\sigma(\pi)|=0$ in the 2nd, 3rd, and 4th rows (i.e. $\{312,132\},\{231,213\}, and \{213,312\}$) are reversal, complementation and inversion, respectively, of the corresponding patterns in the 1st raw (i.e. $\{213,231\}$), etc. Similarly, the patterns 321,321 and 123, each bearing the same value $|N_\sigma(\pi)|=1$, are reversal, complementation and inversion of the corresponding pattern 123 in the 1st raw; the pairs of patterns, each bearing the value $|N_\sigma(\pi)|=2$, {213, 123}, {132, 123}, and {231,321} in the 2nd, 3rd, and 4th rows, relate in the same manner to the pair {312,321} in the 1st raw; and, lastly, the patterns 231,312 and 132 in the 2nd, 3rd, and 4th rows relate in the same manner to the pattern 132 in the 1st raw, each bearing $|N_\sigma(\pi)|=5$. We summarized the above in Table 1 below;

Table. 1.(a). Patterns occurrences in some permutations via standard bijections

Permutation $\pi\in S_5$	Pattern $\sigma\in S_3$	$|N_\sigma(\pi)|$ Number of occurrences of σ in permutation π
15423	$\{213,231\},\{312,132\},\{231,213\},\{213,312\}$	0
32451	{123},{321}	1
51243	{123,213},{321,312},{321,231}	2
14532	{132},{231},{312}	5

Note: Table 1(a) above shows that there are easy correspondences which explain why

$$|F_n(132)| = |F_n(213)| = |F_n(231)| = |F_n(312)|,$$

and why

$$|F_n(123)| = |F_n(321)|.$$

Definition 3.4.1. (Trivial bijections of S_n). For a permutation, say, *π*, we define the three standard bijections between sets of the types $F_n^k(\pi)$ and $F_n^k(\tau)$, ∀*π*, *τ*ϵ for general *k;* the reverse $r:S_n\to S_n$, the complement $c:S_n\to S_n$, and the inverse $I:S_n\to S_n$ to be the permutation β such that

$$\beta=r(\pi)\ iff\ \beta_i=\pi_{n+1-i}\ \ for\ 1\le i\le n,$$
$$\beta=r(\pi)\ iff\ \beta_i=n+1-\pi_i\ for\ 1\le i\le n,$$
$$\beta=I(\pi)\Rightarrow\beta_i=\pi^{-1}(j)=i\ \ iff\ \ \pi(i)=j\ for\ 1\le i,j\le n,$$

And *I* is the usual inverse operation on the symmetric group S_n. For example, if $\pi = 15423 = (2534) \in S_5$ then $r(\pi) = 32451 = (1345)$, $c(\pi) = 51243 = (1532)$, and $i(\pi) = 14532 = (2435)$. We call these operations *trivial bijections* (three standard bijections) from S_n to itself. We denote the group generated by the trivial bijections on the symmetric group S_n by Ω_p.

Note that these three operations of reversal, complementation and inversion are involutions; that is,

$$r(r(\pi)) = \pi,\ c(c(\pi)) = \pi, \text{ and } I(I(\pi)) = \pi.$$

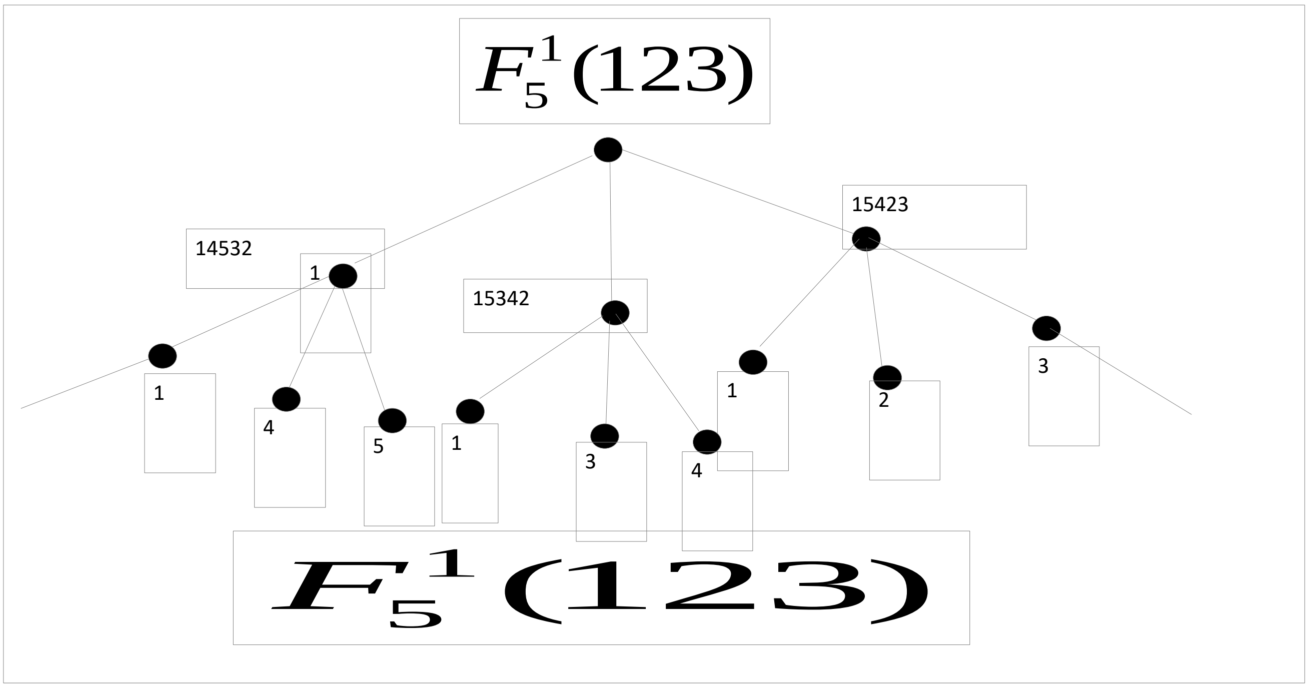
As will be seen again from Lemma 2.2, there are easy correspondences which explain why

$$|F_n(132)| = |F_n(213)| = |F_n(231)| = |F_n(312)|,$$

and why

$$|F_n(123)| = |F_n(321)|.$$

A tree diagram for Aunu permutations of length 5 containing exactly one Occurrence of 123 pattern



Lemma.3.4.2. If $\pi \in F_n^k(\tau), \tau = \{\tau_1, \tau_2, \ldots, \tau_k\} \subseteq S_3, \tau \leq \pi$, then

$(a)\ \pi^r \in F_n^k(\tau^r),\ where\ \tau^r \in \{\tau_1^r, \tau_2^r, \ldots, \tau_k^r\} \subset S_3,\ \tau_i^r = \tau_{n+1-i}\ \ for\ 1 \leq i \leq k;$

$(b)\ \pi^c \in F_n^k(\tau^c),\ where\ \tau^c \in \{\tau_1^c, \tau_2^c, \ldots, \tau_k^c\} \subset S_3,\ \tau_i^c = n+1-\tau_i\ \ for\ 1 \leq i \leq k;$

$(c)\ \pi^{-1} \in F_n^k(\tau^{-1}),\ where\ \tau^{-1} \in \{\tau_1^{-1}, \tau_2^{-1}, \ldots, \tau_k^{-1}\} \subset S_3,\ \pi^{-1}(j) = i\ \ iff\ \ \pi(i) = j,\ for\ 1 \leq i, j \leq k.$

Proof.(see Lemma. 3.4.2.(a,b,c)) The proof is trivial in the sense that the operations (reversal, complementation and inversion) are trivial but essential to the following lemmata.3.3.4. (a, b, c) which can be considered as a generalization of the above definition. The lemma are an important aspect of the theory of forbidden subsequence.

Lemma. 3.4.2.(a). If $\pi \in F_n^k(\sigma), then \pi^r \in F_n^k(\sigma^r), \forall \sigma \leq \pi$.

Proof. Consider an arbitrary subsequence of $\pi \in S_n$, say, $\sigma \in S_m,\ and\ m \le n$ such that $\sigma = (\pi(i_1)\pi(i_2)...\pi(i_m))$, where $1 \le i_1 < i_2 < ... < i_m \le n$. Then, if σ is of type τ, it follows that $\sigma^r = (\pi(i_m)\pi(i_{m-1})...\pi(i_1))$ it is equally of type τ^r. But since $\sigma \le \pi$, it is obvious that σ^r is a subsequence of π^r, namely $(\pi^r((n+1)-i_m), \pi^r((n+1)-i_{m-1}),...,\pi^r((n+1)-i_1))$. Since σ is of type τ, then π also contains a subsequence of type τ. And, if π contains the subsequence τ, it is obvious that π^r contains the subsequence τ^r. Since the operation of reversal is an involution, the argument works equally in the opposite direction, and we can replace "if" by "if and only if" in the previous sentence. Finally, since a permutation avoids a pattern τ exactly when it does not contain a subsequence of type τ, the proof is complete. ■

Lemma. 3.4.2.(b). . If $\pi \in F_n^k(\sigma), then \pi^C \in F_n^k(\sigma^C), \forall \sigma \le \pi$.

Proof. If, for $1 \le i_1 < i_2 < ... < i_k \le n$, a subsequence $\sigma = (\pi(i_1)\pi(i_2)...\pi(i_k))$ is of type τ, then the subsequence $(\pi^c(i_1)\pi^c(i_2)...\pi^c(i_k))$ is isomorphic to the reverse of σ. That is, it is of type τ^c. The remaining part of the argument is identical in its essential to the previous proof. Hence, the proof is complete. ■

Lemma. . 3.4.2.(c). If $\pi \in F_n^k(\sigma), then \pi^{-1} \in F_n^k(\sigma^{-1}), \forall \sigma \le \pi$.

Proof. Suppose π has a subsequence of type τ, namely $(\pi(i_{\tau(1)})\pi(i_{\tau(2)})...\pi(i_{\tau(k)}))$, where $1 \le i_{\tau(1)} < i_{\tau(2)} < ... < i_{\tau(k)} \le n$ and $(\pi(i_1) < \pi(i_2) < ... < \pi(i_k))$. In light of the last set of inequalities, it is clear that one subsequence of π^{-1} is $(\pi^{-1}(\pi(i_1)), \pi^{-1}(\pi(i_2)),...,\pi^{-1}(\pi(i_k))) = (i_1 i_2 ... i_k)$. This is a subsequence of type τ^{-1}. Since π^{-1} contains a subsequence of type τ^{-1}, precisely when π contains the pattern τ, the inverse permutation π^{-1} contains τ^{-1}, Precisely when π contains τ. ■

As a further observation of the relation between the three operations of inverse, complementation and inversion, note that $(\pi^C)^{-1} = (\pi^{-1})^R$.

For example, consider a permutation $\pi = 14532 \in S_5$ and a pattern $\tau = 123 \in S_3$. Then, it is clear that

$$\pi^r = 23541\ and\ \tau^r = 321,$$
$$\pi^c = 52134\ and\ \tau^c = 321,$$
$$\pi^{-1} = 15423\ and\ \tau^{-1} = 123.$$

Here, the pattern $\tau = 123 \in S_3$ has exactly one occurrence in the permutation $\pi = 14532 \in S_5$, in its subsequence $\pi_1\pi_2\pi_3 = 145$, while the pattern 213 has zero occurrences in π (i.e. π avoids 213 patterns); the pattern $\tau^r = 321$ has exactly one occurrence in the permutation $\pi^r = 23541$, in its subsequence $\pi_3^r\pi_4^r\pi_5^r = 321$; the pattern $\tau^c = 321$ has exactly one occurrence in the permutation $\pi^c = 52134$, in its subsequence $\pi_1^c\pi_2^c\pi_3^c = 321$; finally, the pattern $\tau^{-1} = 123$ has also one occurrence in the permutation $\pi^{-1} = 15423$, in its subsequence $\pi_1^{-1}\pi_4^{-1}\pi_5^{-1} = 123$. It is, therefore easy to see that π avoids the subsequence 312, and 213. It follows that π^r avoids the subsequence 213 and 312, that π^c avoids the subsequence 132 and 231, and that π^{-1} avoids the subsequence 231 and 213. The following proposition follows from Simion and Schmidt [1985].

Proposition. 3.4.3. We have that Ω_p is isomorphic to the dihedral group D_8.

Proof. It is easy to see that $r^2 = c^2 = (r\circ c)^2 = 1, c\circ r = r\circ c, i^2 = (r\circ i)^4 = (c\circ i)^4 = 1$ and $i\circ r\circ i = c$. So, Ω_p is isomorphic to D_8. ■

More generally, for a set of patterns T, we define $g(T) = \{g(\tau) \mid \tau \in T\}$ for any $g \in \Omega_p$. For example, if $T = \{123, 132\}$ and $g = r$, then $g(T) = \{321, 231\}$. The following proposition was given by Simion and Schnidt (1985).

4. Classification of Permutations in Cyclic notation form

A brief overview of permutation pattern avoidance is provided. A bijection between classes of pattern-avoiding permutation matrices and lattice pathways is used to provide an equivalence result. There is discussion on recent studies.

Definition 4.1.

Let π be any permutation in S_n. The number of inversions of π is given by $i_\pi = |\{(i,j) : \pi_i > \pi_j, 1 \le i < j \le n\}|$. The signature of π is given by $\text{sign}(\pi) = (-1)^{i_\pi}$. We say π is an even permutation [respectively; odd permutation] if sign(π) = 1 [respectively; sign(π) = -1]. In other words, we say π is an *even permutation* [respectively; *odd permutation*] if π is a permutation together with even [respectively; odd] number of inversions. For example, consider the Aunu permutations 15423; 15342 and 14532 in S_5. Each of the Aunu permutations $\pi \in S_5$ has an odd number of inversions, for instance

$$15423 = (2534) = (25)(23)(24),$$

$$\text{Here } i_\pi = 3 \Rightarrow sign(\pi) = (-1)^3 = -1.$$

We denote by E_n [respectively; O_n] the set of all even [respectively; odd] Aunu permutations in S_n.

The Table 2 below shows the classification of some Aunu permutations $\pi \in F_n^1(123)$ in cyclic form.

From the two different approaches (Tables 1 and 2), we made the following observations:

- $i_\pi = |\{(i,j) : \pi_i > \pi_j, 1 \le i < j \le n, \ \forall\ i, j, n \in \mathbb{N}\}|$;
- $i_\pi \in \mathbb{N} \cup 0, \quad 0 \le i_\pi < \infty$;
- $\text{sign}(\pi) = (-1)^{i_\pi}$;
- $\text{sign}(\pi) = \begin{cases} +ve. & if\ i_\pi = 0\ or\ an\ even\ number \\ -ve. & if\ i_\pi\ \ is\ an\ odd\ number \end{cases}$

4.3.1. Enumerative results:

Following up on our previous enumeration, we discovered—apparently for the first time—the connection between even (odd) Aunu permutations and the pattern-occurrence problem.

$$|E_n^1(123)| = |O_n^1(123)| = (n-2)\,|sign(\pi)|;\ where\ n \ge 3,\ \pi \in S_n$$

$$and\ sign(\pi) = \begin{cases} 1, & if\ \ \pi\ is\ even, \\ -1, & if\ \ \pi\ is\ odd. \end{cases}$$

A sign of a permutation can be viewed with this enumeration as a mapping of a symmetric group Sn into a set {+1,-1}. Consequently, we can characterize a permutation's sign as a group homomorphism.

$$sgn\ f : S_n \to \{+1, -1\}.$$

(i.e., a group homomorphism from the symmetric group S_n into a multiplicative group {+1, –1}, where +1 is *e*, the multiplicative identity/neutral element).

Generally speaking, $|E_n| = |O_n| = \frac{1}{2}n!\ for\ all\ n \geq 2$. The following lemma holds immediately by definitions.

Definition 4.3.1. If a permutation contains an odd (or even) number of inversions, it is referred to as odd (or even). The following proposition demonstrates that while discussing permutations, we must use caution when using the terms "odd" and "even." If a cycle has an even (resp. odd) number of elements, we'll refer to it as even (resp. odd).

Table 2. Classification of Aunu Even and Odd permutations $\pi \in F_n^1(123)$ in Cyclic notation form

Aunu permutation $\pi \in F_n^1(123), n \geq 3$	Transposition of $\pi \in F_n^1(123)$	N0. of inversions in $\pi \in F_n^1(123)$	Sign(π)	Classification of $\pi \in F_n^1(123)$
(1)(2)(3)	None	0	+	Even
(2435)	(24)(23)(25)	3	-	Odd
(25)	(25)	1	-	Odd
(2534)	(25)(23)(24)	3	-	Odd
(2637)(45)	(26)(23)(27)(45)	4	+	Even
(27)(3546)	(27)(35)(34)(36)	4	+	Even
(27)(26)	(27)(26)	2	+	Even
(27)(3645)	(27)(36)(34)(35)	4	+	Even
(2736)(45)	(27)(23)(26)(45)	4	+	Even
(2(11))(3(10))(49)(5867)	(2(11))(3(10))(49)(58)(56)(57)	6	+	Even
(2(11)3(10))(49)(58)(67)	(2(11))(23)(2(10))(49)(58)(67)	6	+	Even
(2(11))(3(10))(4859)(67)	(2(11))(3(10))(48)(45)(49)(67)	6	+	Even
(2(11))(3(10))(49)(58)	(2(11))(3(10))(49)(58)	4	+	Even
(2(11))(2(10)49)(58)(67)	(2(11))(3(10))(34)(39)(58)(67)	6	+	Even
(2(11))(394(10))(58)(67)	(2(11))(39)(34)(3(10))(58)(67)	6	+	Even
(2(11))(3(10))(49)(5768)	(2(11))(3(10))(49)(57)(56)(58)	6	+	Even
(2(11))(3(10))(4958)(67)	(2(11))(3(10))(49)(45)(48)(67)	6	+	Even
(2(10)3(11))(49)(58)(67)	(2(10))(23)(2(11))(49)(58)(67)	6	+	Even
	$\{(i,j): \pi_i > \pi_j, 1 \leq i < j \leq n\}$	i_π	$(-1)^{i_\pi}$	

Proposition 4.

An odd permutation is one that has precisely one even cycle. An even permutation has exactly one odd cycle in it. Cycle is peculiar. An even permutation is one that has precisely one odd cycle.

Proof

We prove the claim by induction on the length *n* of the only cycle of our permutation π. For $n=1$ and $n=2$, the statement is trivially true. Now let $n \geq 3$, and consider the cycle $(i_1 i_2 \ldots i_{n-1} i_n)$. It is straightforward to verify that $(i_1 i_2 \ldots i_{n-1} i_n)\ (i_1 i_2 \ldots i_{n-1} i_n) = (i_1 i_2 \ldots i_{n-1})(i_{n-1} i_n)$. The multiplication by $(i_{n-1} i_n)$ at the end simply swaps the last two entries of $(i_1 i_2 \ldots i_{n-1})$, and therefore, either increases the number of inversions by one or decreases it by one. So in either case, it changes the parity of the number of inversions. The proof is then immediate by the induction hypothesis ∎

4.3.2. The Catalan numbers

In contemporary combinatorics, the Catalan numbers are a number sequence that is frequently encountered. These numbers are used to count a variety of objects. A document on Catalan numbers, which was continuously updated by Stanley and Richard (1999), included 136 combinatorial interpretations of these numbers as of May 2006. Combinatorial counting issues abound, and the Catalan and Aunu numbers often provide the answer. A book called "Enumerative Combinatorics" by Stanley (2006) contains exercises that explain 66 distinct interpretations of Catalan numbers that match similarly to Aunu numbers, according to Wikipedia, a free encyclopedia.

The first Catalan number for $n = 0,1,2,3,\ldots$ *are*

$1,1,2,5,14,42,132,429,1430,4862,16796,58786,208012,742900,2674440,9694845,35357670,$
$129644790,477638700,1767263190,6564120420,\ 24466267020,\ 91482563640,\ 343059613650,$
$1289904147324,2861946401452,\ldots$

This sequence appears in the Online Encyclopaedia of Integer Sequences as A000108.

An alternative expression for C_n is

$$C_n = \binom{2n}{n} - \binom{2n}{n+1}; \quad for\ n \geq 0,$$

Which is equivalent to the expression given above because

$$\binom{2n}{n+1} = \frac{n}{n+1}\binom{2n}{n}.$$

This shows that C_n is an integer, which is not immediately obvious from the first formula given.

4.4. Pattern's avoidance in Permutations

Definition 4.3.2. (Permutation pattern). The two terms 'permutation' and 'pattern' mean the same thing, only in different semantics. The scenario is synonymous with the concept of 'a set' and 'a subset'. A pattern refers to a smaller permutation that is being contained in a bigger permutation. Let a, b and c be any three entries of the permutation π which are arranged in that order from left to right, not necessarily consecutive. If $a < b < c$, the entries a, b and c is said to form a $123 - pattern$. For example, the permutation $\pi = 15423 \in S_5$ contains only one $123 - pattern$, namely, $\pi_1\pi_4\pi_5$. If $a < c < b$, the entries a, b and c is said to form a 132-pattern. For example, the permutation $\pi = 35214 \in S_5$ has exactly one occurrence of $132 - pattern$, namely, $\pi_1\pi_2\pi_5$, which is formed by the elements $3{,}5$ and 4. etc. This definition is analogous for any pattern σ of arbitrary length and order.

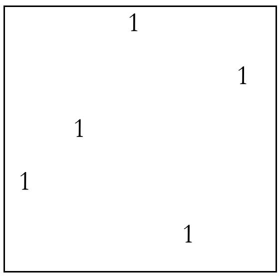


Figure 3 the permutation matrix corresponding to $\pi = 35214 \in S_5$ containing 132-pattern

4.4.1. Pattern's avoidance in Permutation Matrix.

If σ is a sub-matrix of π, then a pattern σ is contained in a permutation π. That is, if it is feasible to eliminate the rows and columns that correspond to π from the permutation matrix, leaving the remaining matrix to match the permutation σ.

4.4.2. Aunu Permutations avoiding 123-pattern;

Aminu (2007) conducted a study that led to the development of Aunu Permutation.

Interestingly, one of the most notable characteristics of Aunu permutation is "first entry unity," which means that in Aunu patterns, the sequence's arrangements always have the first element equal to 1 and the last element greater than 1. For instance, in a set $S=\{1,2,3,4,5\}$, the possible number of permutations by that pattern are as follows:

For (132)-avoiding, we have the permutation $\pi = 12345 \in S_5$;

For (123)-avoiding we have the permutation $\pi = 15432 \in S_5$.

As a result, there are four possible combinations in total. Based on the aforementioned (Aunu pattern), it is simple to conclude that there exists a number of Aunu permutations in a given finite set (sequence) of natural numbers with n order (cardinality).

Table 3: Some special (123)-avoiding sub words for samples of size 5 to 17

Length of number (n)	$Sequence\ as\ (132)-avoiding$	$Sequence\ as\ (123)-avoiding$
5	12345	14253; 15432
7	1234567	1526374; 1642753; 1765432
11	123456789(10)11	1728394(10)5(11)6; 184(11)73(10)6295; 1963(11)852(10)74; 1(10)8642(11)9753; 1(11)(10)98765432
13	123456789(10)(11)(12)13	18293(10)4(11)5(12)6(13)7; 194(12)72(10)5(13)83(11)6; 1(10)62(11)73(12)84(13)95; 1(11)852(12)963(13)(10)74; 1(12)(10)8642(13)(11)9753; 1(13)(12)(11)(10)98765432.
17	123456789(10)(11)(12)(13)(14)(15)(16)(17)	1(10)2(11)3(12)4(13)5(14)6(15)7(16)8(17)9; 1(11)4(14)7(17)(10)3(13)6(16)92(12)5(15)8; 1(12)6(17)(11)5(16)(10)4(15)93(14)82(13)7; 1(13)83(15)(10)5(17)(12)72(14)94(16)(11)6; 1(14)(10)62(15)(11)73(16)(12)84(17)(13)95; 1(15)(12)963(17)(14)(11)852(16)(13)(10)74; 1(16)(14)(12)(10)8642(17)(15)(13)(11)975; 1(17)(16)(15)(14)(13)(12)(11)(10)98765432.

4.5. A Comparison of Aunu numbers to Catalan numbers:

4.5.1. Their Comparison on Hankel matrix;

The $n \times n$ Hankel Matrix whose (i, j) entry is the Catalan number C_{i+j-2} has determinant 1 regardless of the value of n. For example, for $n=5$, we have

$$\det\begin{bmatrix} 1 & 1 & 2 & 5 & 14 \\ 1 & 2 & 5 & 14 & 42 \\ 2 & 5 & 14 & 42 & 132 \\ 5 & 14 & 42 & 132 & 439 \\ 14 & 42 & 132 & 429 & 1430 \end{bmatrix} = 1$$

Moreover, if the indexing is "shifted" so that the (i, j) entry is filled with the Catalan number C_{i+j-1} then the determinant is still 1, regardless of the value of n. For example, for $n = 5$ we have

$$\det\begin{bmatrix} 1 & 2 & 5 & 14 & 42 \\ 2 & 5 & 14 & 42 & 132 \\ 5 & 14 & 42 & 132 & 439 \\ 14 & 42 & 132 & 429 & 1430 \\ 42 & 132 & 429 & 1430 & 4862 \end{bmatrix} = 1$$

Taken together, these two conditions uniquely define the Catalan numbers.

Conversely, the $n \times n$ Hankel Matrix whose (i, j) entry is the Aunu number A_{i+j-2} does not have determinant 1 regardless of the value of n. For example, for $n = 5$, we have

$$\det\begin{bmatrix} 1 & 2 & 3 & 5 & 6 \\ 2 & 3 & 5 & 6 & 8 \\ 3 & 5 & 6 & 8 & 9 \\ 5 & 6 & 8 & 9 & 11 \\ 6 & 8 & 9 & 11 & 14 \end{bmatrix} = -324$$

.

However, it is interesting to note that square matrices by Aunu as well as Catalan numbers form symmetric matrices (i.e. $|A| = |A^T|$) about the main diagonal. Besides, it is important to note therefore, that Aunu numbers as well as Catalan numbers possess the following unique properties;

Some basic/general rules/properties of Matrices satisfied by square matrices whose $[i, j]$ entries are Aunu and Catalan numbers:

1. $(A+B)^T = A^T + B^T$
2. $|A| = |A^T|$
3. $(AB)^T = B^T A^T$

In fact, it is a much more useful and interesting to study the following object:

$S_n(Q) = \{\pi \in S_n \mid \pi \; avoids \; q \; for \; all \; q \in Q\}$:

From the symmetries of the square, we have $|S_n(\{123\})| = |S_n(\{321\})|$ and $|S_n(\{132\})| = |S_n(\{231\})| = |S_n(\{213\})| = |S_n(\{312\})|$. Simion and Schmidt (1985) provided a bijection between $\{132\} - avoiding$ permutations and $\{123\} - avoiding$ permutations, and moreover showed that

$|S_n(\{\pi_3\})| = C_n = \binom{2n}{n}\frac{1}{n+1}$ where π_3 is any permutation of length 3, and C_n denotes the n^{th} Catalan number. Besides, Sani and Aminu (2014) showed that $|S_n(132)| = |S_n(123)| = C_n = \binom{2n}{n}\frac{1}{n+1}$. Conversely, in restricted Aunu Permutations, we showed that $|A_3(123)| = |A_3(132)| = 1 = C_0$, Sani and Aminu (2014)

4.5.2 On their parenthesis:

Theorem

$(n+1)$ factors can be parenthesized in P different ways.

Proof

Let P be the number of different ways $F = (n+1)$ factors can be completely parethesized (or the number of ways of associating **n** applications of a binary operator), where **n** is a positive integer and $n \geq 2$. Then

For $n = 2$ for example, we have the following two different parenthesizations of three factors;

$(ab)c$ and $a(bc)$;

For $n = 3$, we have the following five different parenthesized of four factors;

$((ab)c)d,\ (a(bc))d,\ (ab)(cd),\ a((bc)d)\ \&\ a(b(cd))\ etc.$

Table 4 shows the number of parenthesized (P) and the corresponding factors **F** which is the same thing as $(n+1)$ for every $n \geq 2$. And Table 5 compares the three

parameters $(n, P \wedge F)$.

Table 4. Parenthesized of $(n+1)$ factors

$n \geq 2$	P	$F = (n+1)$
2	2	3
3	5	4
4	14	5
5	22	6
6	32	7
$\vdots$	$\vdots$	$\vdots$
n	$n(n-1)+2$	

Table 5 Relationship between $p, F \wedge n$

$n \geq 2$	P	$F = (n+1)$	$(P-n)$	$\Omega_n = (P-F)$
2	**2**	**3**	**0**	**-1**
3	**5**	**4**	**2**	**1**
4	**14**	**5**	**10**	**9**
5	**22**	**6**	**17**	**16**
6	**32**	**7**	**26**	**25**
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$n(n-1)+2$			n^2

What is more interesting here is the relationship between $P, n \wedge F$:

The Table 5 shows relationship between P (number of ways of parenthesizing $(n+1)$ factors) and n (number of applications of a binary operator). Consequently, two new integer sequences were formed in columns 2 and 5 of Table 5 denoted respectively by P_n and Ω_n thus:

$$P_n = \begin{cases} 2, & if \quad n=2; \\ 5, & if \quad n=3; \\ n(n-1)+2, & if \quad n \geq 4. \end{cases} \quad \ldots\ldots\ldots \quad (4)$$

$$\Omega_n = \begin{cases} -1, & if \quad n=1; \\ 1, & if \quad n=2; \\ n^2, & if \quad n \geq 3. \end{cases} \quad \ldots\ldots\ldots \quad (5)$$

Hence the proof.

CONCLUSION

We have so far developed some Algebraic and Geometric Number Theoretic characteristics for Aunu numbers in relation to Catalan numbers. Following our comparison, we saw the following outcomes:

Among other significant results obtained is a Formula / relation for generating P_n the number of different ways $F=(n-1)$ factors can be fully parethesized (or the number of ways of associating n applications of a binary operator) as shown by equation 4 as well as a formula/generating function for new integer sequence Ω_n (the difference between $P \wedge F$ as shown by equation 5. The aforementioned comparison demonstrates that there is some correlation between Catalan and Aunu numbers. Additionally, more research on Aunu sets could be done, particularly with regard to algebraic features.

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