UMYU Scientifica, Vol. 4 NO. 2, June 2025, Pp 032 – 048
https://doi.org/10.56919/usci.2542.005
ISSN: 2955 – 1145 (print); 2955 – 1153 (online)
A periodical of the Faculty of Natural and Applied Sciences, UMYU, Katsina

ORIGINAL RESEARCH ARTICLE

An Order Quantity Model for Delayed Deteriorating Items with Time-Varying Demand Rate and Holding Cost, Time Dependent Partial Backlogging Rate, and Two-Level Pricing Strategies under Trade Credit Policy

[1]Aisha Ahmed Madugu, [2]*Zulkifilu Muazu, [3]Yusuf Jibril, [4]Aminu Hamisu, [5]Danjuma Muhammad Auwal and [6]Babangida Bature
[1,2,4,5,6]Department of Mathematics, Umaru Musa Yar'adua University, Katsina, Nigeria
[3]Department of Mathematics and Statistics, Federal Polytechnic Nasarawa, Nasarawa State, Nigeria

ARTICLE HISTORY
Received January 09, 2025
Accepted March 28, 2025
Published May 11, 2025

ABSTRACT
In this study, an economic order quantity model is developed for non-instantaneous deteriorating items with two-level pricing strategies under trade credit policy, two-phase demand rates, linear holding costs, and time-dependent partial backlog rates; the demand rate is assumed to be continuous and time-dependent quadratic before deterioration sets in, and to be constant until the inventory is fully depleted; the holding cost is assumed to be linear and time-dependent, and the unit selling price is higher before deterioration sets in is higher than that after, shortages and time-dependent partial backlogs are acceptable. The model's objective is to determine the optimal period for positive inventory, cycle length, and order amount to maximize the inventory system's overall profit. The necessary and sufficient conditions for optimal solutions to exist and be unique have been identified. Numerical experiments have been used to illustrate the model's theoretical result. Together with sensitivity analysis of specific model parameters on the choice factors, recommendations for optimizing the overall profit were also given.

KEYWORDS
Economic Order Quantity, Non- instantaneous deteriorating item, two-phase demand rates, Linear Holding, two-level pricing strategies, time dependent partial backlogging rate, Trade Credit Policy.

INTRODUCTION

In traditional economic order quantity (EOQ) models, it is implicitly assumed that the demand rate is constant, inventory units have fixed costs, ordering and holding costs are fixed, and objects have an indefinite lifespan. Nonetheless, the demand rate for a lot of things (like computers, airplanes, fashion items, photographic films, televisions, computer chips, and so forth) might be in a dynamic state because the age of inventory affects demand negatively through depletion, spoiling, loss of quality, and diminished market potential. Harris created the first classical EOQ model in 1913. Later, a number of scholars, including Yahaya et al. (2019), Giri et al. (2000), Kar et al. (2001), and Chakrabarti et al. (1998), and others altered the standard EOQ model's assumptions in the situation of time-dependent linear demand rate, which assumed a constant increase or decrease in demand rate per unit of time—something that is rarely observed for many products. The assumption of the classical EOQ model is also modified by some researchers, such as Dash et al. (2014) and Ahmed and Musa (2016), in the case of a time-dependent exponential demand rate. This is also uncommon for any product, as the demand rate of the majority of products may not change with the higher rate of change as exponential.

Subsequently, scholars like Khanra et al. (2011), Uthayakumar and Karuppasamy (2017), and Priya and Senbagam (2018) created inventory models that included time-dependent quadratic demand rate, which is a quadratic function of time that best represents an accelerated or delayed rise or fall in demand rate. Demand rates typically increase more quickly for new things like gadgets, stylish clothing, and so forth.

The traditional EOQ model, which Harris created in 1913, believed that goods had an endless lifespan and that the only reason inventories were depleted was a steady rate of demand. But sometimes, degradation causes inventory items to run out. Therefore, it is impossible to overlook the impacts of deterioration on inventory items. The inventory model for fashionable products that deteriorate at the end of the recommended storage time was initially studied by Whitin (1957). Some relevant research on inventory models that assume deterioration begins as soon as things are received can be found in Jaggi et al. (2019),

Correspondence: Zulkifilu Muazu. Department of Mathematics, Umaru Musa Yar'adua University, Katsina, Nigeria. muazu.zulkifilu@umyu.edu.ng.

How to cite: Ahmed, A. M., Muazu, Z., Jibril, Y., Hamisu, A., Muhammad, D. A., & Bature, B. (2025). An Order Quantity Model for Delayed Deteriorating Items with Time-Varying Demand Rate and Holding Cost, Time Dependent Partial Backlogging Rate, and Two-Level Pricing Strategies under Trade Credit Policy. *UMYU Scientifica*, 4(2), 32 – 48. https://doi.org/10.56919/usci.2542.005

Mandal and Venkataraman (2019), Baraya and Sani (2016), and others.

The traditional EOQ models made the assumption that retailers ought to cover the purchase price as soon as the goods arrive. But the majority of the time, the supplier gives the store a grace period to pay for the purchase, and the shop can make money by selling goods and collecting interest. Haley and Higgins (1973) were the first to introduce the idea of trade credit in the inventory literature. Subsequent scholars have developed an EOQ model with trade credit under a variety of assumptions, including Babangida and Baraya (2020), Jaggi et al. (2015), Shaikh et al. (2018), Musa and Sani (2012), and others.

The premise of constant holding costs is used in the development of several inventory models. However, because the time value of money and price index fluctuate in real-world scenarios, many products' holding costs may be dynamic. For the majority of the items in stock, the holding cost is a linear function of the storage duration. Under a variety of assumptions, researchers like Babangida and Baraya (2019a), Dutta and Kumar (2015), Selvaraju and Ghuru (2018), Baraya and Sani (2011), and others create an inventory model with time-varying holding costs.

The traditional inventory model forbids shortages. Nevertheless, there are instances when the provider cannot meet the client's needs from the available inventory; this is referred to as a stock-out or shortage scenario. Roy (2008) created an EOQ model for immediately decaying items with a price-dependent demand rate. In this model, shortages are permitted, the backlog is fully accounted for, and the deterioration rate and holding cost are considered as linearly increasing functions of time. An inventory model for non-instantaneous deteriorating items with a stock-dependent demand rate, a time-varying holding cost, and fully backlogged shortages was created by Choudhury et al. (2013).

Under the trade credit policy, Babangida and Baraya (2019b) created an inventory model for non-instantaneous deteriorating goods with two-phase demand and shortages. Complete backlogs and shortages are acceptable. In order to minimize the overall variable cost, the best period with positive inventory, cycle length, and order amount are identified.

For the majority of commodities, including electronics, fashion, cars and their spare parts, photographic film, seasonal goods, and so on, the amount of time it takes for the next replenishment will determine whether or not the backlog is acceptable. As a result, the backlog rate ought to fluctuate based on how long it takes for the next replenishment. In other words, the backlog rate will decrease while the waiting time increases and vice versa. An EOQ-based model for non-instantaneous deteriorating items with a constant demand rate under allowable payment delays was created by Geetha and Uthayakumar (2010). Partial backlogs and shortages are permitted; the pace of backlogs varies based on the time it takes for the next replenishment. An inventory model for degrading items with a stock-dependent demand rate and time-varying deterioration was created by Sarkar and Sarkar (2013). Partial backlogs and shortages are permitted; the pace of backlogs is determined by the time it takes for the next replenishment. In accordance with the trade credit policy, Babangida and Baraya (2022) created an EOQ model for non-instantaneous deteriorating items with two-phase demand rates, linear holding costs, and time-dependent partial backlog rates.

This study looked at an EOQ model for non-instantaneous deteriorating items with linear holding costs, two-phase demand rates, time-dependent partial backlog rates, and two-level pricing strategies under trade credit policy. The adequate and necessary requirements for the ideal solution have been identified. In order to optimize the overall profit per unit time, the optimal time with positive inventory, cycle length, and order quantity will be determined. A few numerical examples have been given to illustrate the models' theoretical results. Through sensitivity analysis, the effects of changing a number of the proposed models' parameters on the decision factors were investigated, and suggestions for maximizing total profit were also offered.

2. NOTATION AND ASSUMPTION.

2.1 Notation:

The following notations are used in the development of the inventory system.

A The fixed cost of each order

C The purchasing cost per unit time

S_1 Unit selling price during the interval $[0,\ t_d]$

S_2 Unit selling price during the open interval$[t_d, T]$, where $S_1 > S_2 > C$

C_b Shortage cost per unit time

I_c The interest charged in stock by the supplier

I_e The interest earned

M The trade credit period (in year for settling account)

θ The constant deterioration rate function

t_d The length of time in which the product exhibit more deterioration

t_1 Length of time in which the inventory has no shortage

T The length of replenishment cycle time

Q_m The maximum inventory level

B_m The backorder level during the shortage period

Q The order quantity during the cycle length i.e. $Q = Q_m + B_m$

C_π Unit cost of lost sales per unit

2.2 Assumptions

In addition to assumptions 8 and 9, which are not taken into consideration in Babangida and Baraya (2021), this model develops under the following assumptions, which have been adapted from the aforementioned research.

1. The replenishment rate is infinite, i.e., the replenishment rate is instantaneous, and the lead time is zero.
2. During the fixed period, t_d, there is no deterioration, and at the end of this period, the inventory item deteriorates at the rate θ.
3. There is no replacement or repair for deteriorating items.
4. Demand rate before deterioration begins is assumed to be continuous time-dependent quadratic and is given by $a + bt + t^2$, where $a \geq 0, b \neq 0, c \neq 0c \neq 0$. Here a is the initial demand rate, b is the rate at which the demand rate changes and c is the accelerated change in the demand rate.
5. Demand rate after deterioration sets in is assumed to be constant and is given by d,$d > 0$.
6. During the trade credit period $M(0 < M < 1)$, e account is not settled; generated sales revenue is deposited in an interest-bearing account. At the end of the period, the retailer pays off all units bought and starts to pay the capital opportunity cost for the items in stock. No interest is earned after the trade credit period.
7. The unit selling price is not the same as the unit purchasing cost. It is assumed that the unit selling price before deterioration sets in is greater than that after deterioration sets n $(S_1 > S_2 > C)$.
8. Shortages are allowed and partially backlogged during the stock-out period; the backlogging rate is variable and depends on the waiting time for the next replenishment, i.e. the longer the waiting time is, the smaller the backlogging rate will be. The backlogging rate for negative inventory is given by $B(t) = \frac{1}{1+\delta(T-t)}$, δ is backlogging parameter ($0 < \delta < 1$)and $(T-t)$ is waiting time $(t_1 \leq t \leq T)$, $1 - B(t)$ is the remaining fraction lost.
9. Holding cost $C_1(t)$per unit time is linear time-pendent and is assumed to be $C_1(t) = h_1 + h_2 t$; where $h_1 > 0$ and $h_2 > 0$.

3. MODELLING

At the start of the cycle, Q_m units are ordered (i.e., at time $t = 0$). The inventory level gradually depletes due to market demand alone during the interval $[0, t_d]$, and the demand rate is assumed to be time-dependent quadratic. At the time interval $[t_d, t_1]$, the inventory level depletes due to the combined effects of customer demand and deterioration, and the demand rate decreases to d. At time $t = t_1$, the inventory level depletes to zero. Shortages occur at the time interval $[t_1, T]$ and are partially backlogged; the backlogging rate is variable and depends on the waiting time for the next replenishment. Figure 1 below illustrates the behavior of the inventory system.

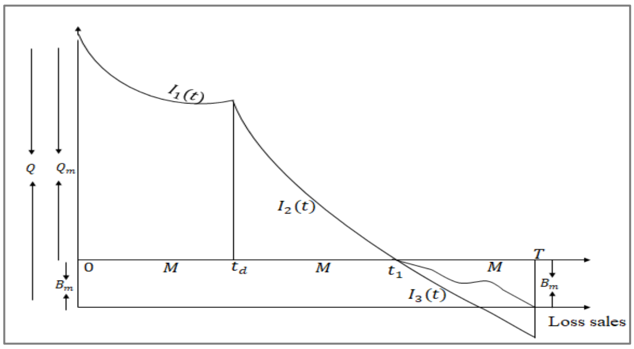

Figure 1: Graphical representation of the inventory system

During the time interval $[0, t_d]$, the change of inventory at any time t is represented by the following differential equation

$$\frac{dI_1(t)}{dt} = -(a + bt + ct^2), \quad 0 \leq t \leq t_d \tag{1}$$

with boundary conditions $I_1(0) = Q_m$ and $I_1(t_d) = Q_d$.

$$\frac{dI_2(t)}{dt} + \theta I_2(t) = -d, \quad t_d \leq t \leq t_1 \tag{2}$$

With the boundary condition $I_2(t_1) = 0$ at $t = t_1$ and $I_2(t_d) = Q_d$ at $t = t_d$

$$\frac{dI_3(t)}{dt} = -\frac{d}{1 + \delta(T-t)}, \quad t_1 \leq t \leq T \tag{3}$$

with boundary condition $I_3(t_1) = 0$ at $t = t_1$.

The solution of equations (1), (2) and (3) are respectively given by

$$I_1(t) = \frac{d}{\theta}\left(e^{\theta(t_1-t_d)} - 1\right) + a(t_d - t) + \frac{b}{2}(t_d^2 - t^2) + \frac{c}{3}(t_d^3 - t^3) \quad 0 \leq t \leq t_d \tag{4}$$

$$I_2(t) = \frac{d}{\theta}\left(e^{\theta(t_1-t)} - 1\right), \quad t_d \leq t \leq t_1 \tag{5}$$

$$I_3(t) = -\frac{d}{\delta}[\ln[1 + \delta(T - t_1)) - \ln[1 + \delta(T-t)], \quad t_1 \leq t \leq T \tag{6}$$

From Fig.1, using the condition $I_1(0) = Q_m$ in equation (4), the maximum stock level is given by

$$Q_m = \frac{d}{\theta}\left(e^{\theta(t_1-t_d)} - 1\right) + \left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3}\right) \tag{7}$$

Similarly, the value of Q_d can be derived at $t = t_d$, then it follows from equation (5) that

$$Q_d = \frac{d}{\theta}\left(e^{\theta(t_1-t_d)} - 1\right) \tag{8}$$

The maximum backordered inventory B_m is obtained at $t = T$, and then from equation (6), it follows that

$$B_m = \frac{d}{\delta}\left[ln[1 + \delta(T - t_1)]\right] \tag{9}$$

Consequently, the maximum inventory level Q_m and the maximum backordered inventory B_m are added to determine the order size Q for the period $[0,\ T]$.

$$Q = \frac{d}{\theta}\left(e^{\theta(t_1-t_d)} - 1\right) + \left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3}\right) + \frac{d}{\delta}\left[ln[1+\delta(T-t_1)]\right] \quad (10)$$

(i) *The total demand during the period* $[t_d,\ t_1]$ *is given by*

$$D_M = \int_{t_d}^{t_1} d\, dt = d(t_1 - t_d) \quad (11)$$

(ii) *The total number of deteriorated items per cycle is given by*

$$D_P = Q_d - D_M = \frac{d}{\theta}\left[e^{\theta(t_1-t_d)} - 1 - \theta(t_1 - t_d)\right] \quad (12)$$

(iii) *Total number of items sold*

$$SN = Q - D_P = \left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3}\right) + d(t_1 - t_d) + \frac{d}{\delta}\left[ln[1+\delta(T-t_1)]\right] \quad (13)$$

(iv) Sale Revenue (SR)

$$SR = S_1\left[\int_0^{t_d}(a + bt + ct^2)dt\right] + S_2\left[\int_{t_d}^{t_1} d dt + \int_{t_1}^{T}\frac{d}{1-\delta(T-t)}dt\right]$$
$$= S_1\left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3}\right) + S_2 d(t_1 - t_d) + S_2\frac{d}{\delta}\left[ln[1 + \delta(T - t_1)]\right] \quad (14)$$

(v) Purchasing cost *(PC)*

$$PC = CQ = C\left[\frac{d}{\theta}\left(e^{\theta(t_1-t_d)} - 1\right) + \left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3}\right) + \frac{d}{\delta}\left[ln[1+\delta(T-t_1)]\right]\right] \quad (15)$$

(iv) *The fixed ordering cost per order is given by* A

(v) *The inventory holding cost for the entire cycle is given by*

$$C_H = \int_0^{t_d}(h_1 + h_2 t)I_1(t)dt + \int_{t_d}^{t_1}(h_1 + h_2 t)I_2(t)dt \quad (16)$$

Substituting the values of equations (4) and (5) in equation (16)

$$C_H = h_1\left[\frac{dt_d}{\theta}e^{\theta(t_1-t_d)} + \frac{a}{2}t_d^2 + \frac{b}{3}t_d^3 + \frac{c}{4}t_d^4 + \frac{d}{\theta^2}e^{\theta(t_1-t_d)} - \frac{d}{\theta^2} - \frac{dt_1}{\theta}\right] + h_2\left[\frac{dt_d^2}{2\theta}e^{\theta(t_1-t_d)} + \frac{a}{6}t_d^3 + \frac{b}{8}t_d^4 + \frac{c}{10}t_d^5 + \frac{dt_d}{\theta^2}e^{\theta(t_1-t_d)} - \frac{dt_1}{\theta^2} - \frac{d}{\theta^3} + \frac{d}{\theta^3}e^{\theta(t_1-t_d)} - \frac{dt_1^2}{2\theta}\right] \quad (17)$$

(vi) *The backordered cost per cycle is given by*

$$SC = C_b\int_{t_1}^{T} -I_3(t)dt = \frac{C_b d}{\delta}\left((T - t_1) - \frac{ln\left(1 + \delta(T - t_1)\right)}{\delta}\right) \quad (18)$$

(vii) The opportunity cost per cycle due to lost sales is given by

$$LC = C_\pi d\int_{t_1}^{T}\left(1 - \frac{d}{1 + \delta(T-t)}\right)dt = C_\pi d\left[(T - t_1) - \frac{ln\left(1+\delta(T-t_1)\right)}{\delta}\right] \quad (19)$$

(vii) The total profit per unit time for a replenishment cycle (denoted by $TP(t_1, T)$ is given by

$$TP(t_1, T) = \begin{cases} TP_1(t_1, T) & 0 < M \le t_d \\ TP_2(t_1, T) & t_d < M \le t_1 \\ TP_3(t_1, T) & M > t_1 \end{cases} \quad (20)$$

where $TP_1(t_1, T)$, $TP_2(t_1, T)$, and $TP_3(t_1, T)$ are discussed for three different cases follows.

Case 1: ($0 < M \le t_d$)

The interest payable

The interest payable is listed below since this is the time before degradation occurs and payment for goods is resolved using the capital opportunity cost rate I_c for the items in stock.

$$I_{P1} = CI_c\left[\int_M^{t_d} I_1(t)dt + \int_{t_d}^{t_1} I_2(t)dt\right]$$
$$= CI_c\left[\frac{d(t_d - M)}{\theta}\left(e^{\theta(t_1-t_d)} - 1\right) + \frac{a}{2}(t_d - M)^2 + \frac{b}{6}(2t_d + M)(t_d - M)^2 + \frac{c}{12}(3t_d^2 + 2t_d M + M^2)(t_d - M)^2 + \frac{d}{\theta^2}\left(e^{\theta(t_1-t_d)} - 1 - \theta(t_1 - t_d)\right)\right] \quad (21)$$

The interest earned

While the retailer must settle the accounts at period M, he must arrange funds at a certain interest rate to finance his remaining stocks for the period M to t_d. In this scenario, the retailer can earn interest on sales revenue up to the trade credit period M. The interest earned is

$$I_{E1} = S_1 I_e \left[\int_0^M (a + bt + ct^2)tdt\right] = S_1 I_e \left(a\frac{M^2}{2} + b\frac{M^3}{3} + c\frac{M^4}{4}\right) \quad (22)$$

The total profit per unit time for case 1 $(0 < M \leq t_d)$ is

$TP_1(t_1, T) = \frac{1}{T}${Sales Revenue - Purchasing cost - Ordering cost - inventory holding cost - backordered cost - lost sales cost- interest payable during the permissible delay period $+$ interest earned during the cycle}

$$\begin{aligned}
&= \frac{1}{T}\Bigg\{(S_1 - C)\left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3}\right) + S_2 d(t_1 - t_d) + (S_2 - C)\frac{d}{\delta}\left[ln[1 + \delta(T - t_1)]\right] - C\left[\frac{d}{\theta}\left(e^{\theta(t_1 - t_d)} - 1\right)\right] - A \\
&\quad - h_1\left[\frac{dt_d}{\theta}e^{\theta(t_1 - t_d)} + \frac{a}{2}t_d^2 + \frac{b}{3}t_d^3 + \frac{c}{4}t_d^4 + \frac{d}{\theta^2}e^{\theta(t_1 - t_d)} - \frac{d}{\theta^2} - \frac{dt_1}{\theta}\right] \\
&\quad - h_2\left[\frac{dt_d^2}{2\theta}e^{\theta(t_1 - t_d)} + \frac{a}{6}t_d^3 + \frac{b}{8}t_d^4 + \frac{c}{10}t_d^5 + \frac{dt_d}{\theta^2}e^{\theta(t_1 - t_d)} - \frac{dt_1}{\theta^2} - \frac{d}{\theta^3} + \frac{d}{\theta^3}e^{\theta(t_1 - t_d)} - \frac{dt_1^2}{2\theta}\right] \\
&\quad - \left(C_\pi d + \frac{dC_b}{\delta}\right)\left[(T - t_1) - \frac{ln\big(1 + \delta(T - t_1)\big)}{\delta}\right] \\
&\quad - cI_c\Bigg[\frac{d(t_d - M)}{\theta}\left(e^{\theta(t_1 - t_d)} - 1\right) + \frac{a}{2}(t_d - M)^2 + \frac{b}{6}(2t_d + M)(t_d - M)^2 \\
&\quad + \frac{c}{12}(3t_d^2 + 2t_d M + M^2)(t_d - M)^2 + \frac{d}{\theta^2}\left(e^{\theta(t_1 - t_d)} - 1 - \theta(t_1 - t_d)\right)\Bigg] \\
&\quad + S_1 I_e\left(a\frac{M^2}{2} + b\frac{M^3}{3} + c\frac{M^4}{4}\right)\Bigg\} \qquad (23)
\end{aligned}$$

Case 2: $(t_d < M \leq t_1)$

The interest payable

The interest payable is higher when the credit period's endpoint is longer than the period without deterioration but less than or equal to the period with a positive inventory stock of the products.

$$I_{P2} = cI_c\left[\int_M^{t_1} I_2(t)dt\right] = cI_c\left[\frac{d}{\theta^2}\left(e^{\theta(t_1 - M)} - 1 - \theta(t_1 - M)\right)\right] \quad (24)$$

The interest earned

The retailer can earn interest on sales revenue up to the trade credit period M in this scenario. However, in order to settle the accounts at period M, he must arrange funds at a specific interest rate to finance his remaining stocks for period M to t_d. The interest earned is

$$I_{E2} = S_1 I_e\left[\int_0^{t_d}(a + bt + ct^2)tdt\right] + S_2 I_e\left[\int_{t_d}^M dtdt\right]$$

$$= S_1 I_e\left(a\frac{t_d^2}{2} + b\frac{t_d^3}{3} + c\frac{t_d^4}{4}\right) + S_2 I_e\left(\frac{dM^2}{2} - \frac{dt_d^2}{2}\right) \quad (25)$$

The total profit per unit time for case 2$(t_d < M \leq t_1)$ is

$TP_2(t_1, T) = \frac{1}{T}${Sales Revenue - Purchasing cost - Ordering cost - inventory holding cost - backordered cost - lost sales cost- interest payable during the permissible delay period $+$ interest earned during the cycle}

$$= \frac{1}{T}\Big\{(S_1 - C)\left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3}\right) + S_2 d(t_1 - t_d) + (S_2 - C)\frac{d}{\delta}[ln[1 + \delta(T - t_1)]] - C\left[\frac{d}{\theta}\left(e^{\theta(t_1 - t_d)} - 1\right)\right]$$
$$- A - h_1\left[\frac{dt_d}{\theta}e^{\theta(t_1 - t_d)} + \frac{a}{2}t_d^2 + \frac{b}{3}t_d^3 + \frac{c}{4}t_d^4 + \frac{d}{\theta^2}e^{\theta(t_1 - t_d)} - \frac{d}{\theta^2} - \frac{dt_1}{\theta}\right]$$
$$- h_2\left[\frac{dt_d^2}{2\theta}e^{\theta(t_1 - t_d)} + \frac{a}{6}t_d^3 + \frac{b}{8}t_d^4 + \frac{c}{10}t_d^5 + \frac{dt_d}{\theta^2}e^{\theta(t_1 - t_d)} - \frac{dt_1}{\theta^2} - \frac{d}{\theta^3} + \frac{d}{\theta^3}e^{\theta(t_1 - t_d)} - \frac{dt_1^2}{2\theta}\right]$$
$$- \left(C_\pi d + \frac{dC_b}{\delta}\right)\left[(T - t_1) - \frac{ln\big(1 + \delta(T - t_1)\big)}{\delta}\right] - cI_c\left[\frac{d}{\theta^2}\left(e^{\theta(t_1 - M)} - 1 - \theta(t_1 - M)\right)\right]$$
$$+ S_1 I_e\left(a\frac{t_d^2}{2} + b\frac{t_d^3}{3} + c\frac{t_d^4}{4}\right) + S_2 I_e\left(\frac{dM^2}{2} - \frac{dt_d^2}{2}\right)\Big\} \quad (26)$$

Case 3: ($M > t_1$)

The interest payable

$I_{P3} = 0,$ in this instance, since the retailer pays no interest and the payment delay period is longer than the period with a positive inventory.

The interest earned

In this case, the period of delay in payment (M) is greater than the period with positive inventory(t_1). In this case, the retailer earns interest on the sales revenue up to the permissible delay period, and no interest is payable during the period for the item kept in stock. Interest earned for the time period $[0, T]$

$$I_{E3} = S_1 I_e\left[\int_0^{t_d}(a + bt + ct^2)tdt + (M - t_1)\int_0^{t_d}(a + bt + ct^2)dt\right] + S_1 I_e\left[\int_{t_d}^{t_1} dtdt + (M - t_1)\int_{t_d}^{t_1} ddt\right]$$
$$= S_1 I_e\left[\left(a\frac{t_d^2}{2} + b\frac{t_d^3}{3} + c\frac{t_d^4}{4}\right) + (M - t_1)\left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3}\right)\right] + S_2 I_e\left[-\frac{d}{2}(t_1 - t_d)^2 + Md(t_1 - t_d)\right] \quad (27)$$

The total profit per unit time for case 3 $(M > t_1)$ is

$TP_3(t_1, T) = \frac{1}{T}${Sales Revenue - Purchasing cost - Ordering cost - inventory holding cost - backordered cost - lost sales cost + interest earned during the cycle}

$$= \frac{1}{T}\Big\{(S_1 - C)\left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3}\right) + S_2 d(t_1 - t_d) + (S_2 - C)\frac{d}{\delta}[ln[1 + \delta(T - t_1)]] - C\left[\frac{d}{\theta}\left(e^{\theta(t_1 - t_d)} - 1\right)\right] - A$$
$$- h_1\left[\frac{dt_d}{\theta}e^{\theta(t_1 - t_d)} + \frac{a}{2}t_d^2 + \frac{b}{3}t_d^3 + \frac{c}{4}t_d^4 + \frac{d}{\theta^2}e^{\theta(t_1 - t_d)} - \frac{d}{\theta^2} - \frac{dt_1}{\theta}\right]$$
$$- h_2\left[\frac{dt_d^2}{2\theta}e^{\theta(t_1 - t_d)} + \frac{a}{6}t_d^3 + \frac{b}{8}t_d^4 + \frac{c}{10}t_d^5 + \frac{dt_d}{\theta^2}e^{\theta(t_1 - t_d)} - \frac{dt_1}{\theta^2} - \frac{d}{\theta^3} + \frac{d}{\theta^3}e^{\theta(t_1 - t_d)} - \frac{dt_1^2}{2\theta}\right]$$
$$- \left(C_\pi d + \frac{dC_b}{\delta}\right)\left[(T - t_1) - \frac{ln\big(1 + \delta(T - t_1)\big)}{\delta}\right]$$
$$+ S_1 I_e\left[\left(a\frac{t_d^2}{2} + b\frac{t_d^3}{3} + c\frac{t_d^4}{4}\right) + (M - t_1)\left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3}\right)\right]$$
$$+ S_2 I_e\left[-\frac{d}{2}(t_1 - t_d)^2 + Md(t_1 - t_d)\right]\Big\} \quad (28)$$

Using the well-known approximations $e^x = 1 + x + \frac{x^2}{2} + \cdots$ and $ln(1 + x) = x - \frac{x^2}{2} + \cdots$ when $-1 < x < 1$ in equations(23), (26) and (28) yields

$$TP_1(t_1, T) = \frac{d}{T}\Big\{-\frac{1}{2}\mathrm{P}_1 t_1^2 + \mathrm{Q}_1 t_1 - \mathrm{R}_1 - \frac{1}{2}(C_\pi\delta + C_b)T^2 - (S_2 - C)\frac{\delta}{2}T^2 + (C_\pi\delta + C_b)Tt_1 + (S_2 - C)\delta Tt_1$$
$$+ (S_2 - C)T\Big\} \quad (29)$$

Where

$$\mathrm{P}_1 = \left[h_1(t_d\theta + 1) + h_2\left(\frac{t_d\theta}{2} + 1\right)t_d + C\theta + (C_\pi\delta + C_b) + cI_c(\theta(t_d - M) + 1) + (S_2 - C)\delta\right],$$

$$\mathrm{Q}_1 = \left[h_1 t_d^2\theta + \frac{h_2}{2}(1 + t_d\theta)t_d^2 + Ct_d\theta + cI_c(M + (t_d - M)\theta t_d)\right]$$

and

$$\begin{aligned}\mathrm{R}_1 = -\frac{1}{d}\Bigg[&(S_1 - C)\left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3}\right) - (S_2 - C)dt_d - \frac{Cd\theta t_d^2}{2} - A\\ &- h_1\left(\frac{a}{2}t_d^2 + \frac{b}{3}t_d^3 + \frac{c}{4}t_d^4 - \frac{dt_d^2}{2} + \frac{dt_d^3\theta}{2}\right) - h_2\left(\frac{a}{6}t_d^3 + \frac{b}{8}t_d^4 + \frac{c}{10}t_d^5 + \frac{dt_d^4\theta}{4}\right)\\ &- CI_c\Bigg(\frac{a}{2}(t_d - M)^2 + \frac{b}{6}(2t_d + M)(t_d - M)^2 + \frac{c}{12}(3t_d^2 + 2t_dM + M^2)(t_d - M)^2\\ &+ dMt_d - \frac{dt_d^2}{2} + \frac{d}{2}(t_d - M)\theta t_d^2\Bigg) + S_1 I_e\left(a\frac{M^2}{2} + b\frac{M^3}{3} + c\frac{M^4}{4}\right)\Bigg]\end{aligned}$$

Similarly,

$$\begin{aligned}TP_2(t_1, T) = \frac{d}{T}\Big\{&-\frac{1}{2}\mathrm{P}_2 t_1^2 + \mathrm{Q}_2 t_1 - \mathrm{R}_2 - \frac{1}{2}(C_\pi\delta + C_b)T^2 - (S_2 - C)\frac{\delta}{2}T^2 + (C_\pi\delta + C_b)Tt_1 + (S_2 - C)\delta Tt_1\\ &+ (S_2 - C)T\Big\}\end{aligned} \quad (30)$$

Where

$$\mathrm{P}_2 = \left[h_1(t_d\theta + 1) + h_2\left(\frac{t_d\theta}{2} + 1\right)t_d + C\theta + (C_\pi\delta + C_b) + CI_c + (S_2 - C)\delta\right],$$

$$\mathrm{Q}_2 = \left[h_1 t_d^2\theta + \frac{h_2}{2}(1 + t_d\theta)t_d^2 + Ct_d\theta + cI_cM\right]$$

and

$$\begin{aligned}\mathrm{R}_2 = -\frac{1}{d}\Bigg[&(S_1 - C)\left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3}\right) - (S_2 - C)dt_d - \frac{Cd\theta t_d^2}{2} - A\\ &- h_1\left(\frac{a}{2}t_d^2 + \frac{b}{3}t_d^3 + \frac{c}{4}t_d^4 - \frac{dt_d^2}{2} + \frac{dt_d^3\theta}{2}\right) - h_2\left(\frac{a}{6}t_d^3 + \frac{b}{8}t_d^4 + \frac{c}{10}t_d^5 + \frac{dt_d^4\theta}{4}\right)\\ &- CI_c\frac{d}{2}M^2 + S_1 I_e\left(a\frac{t_d^2}{2} + b\frac{t_d^3}{3} + c\frac{t_d^4}{4}\right) + S_2 I_e\left(\frac{dM^2}{2} - \frac{dt_d^2}{2}\right)\Bigg]\end{aligned}$$

and

$$\begin{aligned}TP_3(t_1, T) = \frac{d}{T}\Big\{&-\frac{1}{2}\mathrm{P}_3 t_1^2 + \mathrm{Q}_3 t_1 - \mathrm{R}_3 - \frac{d}{2}(C_\pi\delta + C_b)T^2 - (S_2 - C)d\frac{\delta}{2}T^2 + d(C_\pi\delta + C_b)Tt_1\\ &+ (S_2 - C)d\delta Tt_1 + (S_2 - C)dT\Big\}\end{aligned} \quad (31)$$

Where

$$\mathrm{P}_3 = \left[h_1(t_d\theta + 1) + h_2\left(\frac{t_d\theta}{2} + 1\right)t_d + C\theta + (C_\pi\delta + C_b) + S_2 I_e + (S_2 - C)\delta\right],$$

$$\mathrm{Q}_3 = \left[h_1 t_d^2\theta + \frac{h_2}{2}(1 + t_d\theta)t_d^2 + Ct_d\theta - \frac{1}{d}\left\{S_1 I_e\left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3}\right)\right\} + S_2 I_e t_d + S_2 I_e M\right]$$

and

$$\begin{aligned}\mathrm{R}_3 = -\frac{1}{d}\Bigg[&(S_1 - C)\left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3}\right) - (S_2 - C)dt_d - \frac{Cd\theta t_d^2}{2} - A\\ &- h_1\left(\frac{a}{2}t_d^2 + \frac{b}{3}t_d^3 + \frac{c}{4}t_d^4 - \frac{dt_d^2}{2} + \frac{dt_d^3\theta}{2}\right) - h_2\left(\frac{a}{6}t_d^3 + \frac{b}{8}t_d^4 + \frac{c}{10}t_d^5 + \frac{dt_d^4\theta}{4}\right)\\ &+ S_1 I_e\left[\left(a\frac{t_d^2}{2} + b\frac{t_d^3}{3} + c\frac{t_d^4}{4}\right) + \left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3}\right)M\right] - S_2 I_e\frac{d}{2}t_d^2 - S_2 I_e Mdt_d\Bigg]\end{aligned}$$

4. Optimal Decision

The best ordering practices that maximize the overall profit per unit of time are identified in this section. It has been determined what the necessary and sufficient conditions are for optimal solutions to exist and be unique. The necessary conditions for the total profit per unit time $TP_i(t_1,T)$ to be maximum are $\frac{\partial TP_i(t_1,T)}{\partial t_1}=0$ and $\frac{\partial TP_i(t_1,T)}{\partial T}=0$ for $i=1,2,3$. The value of (t_1,T) obtained from $\frac{\partial TP_i(t_1,T)}{\partial t_1}=0$ and $\frac{\partial TP_i(t_1,T)}{\partial T}=0$ and for which the sufficient condition $\left\{\left(\frac{\partial^2 TP_i(t_1,T)}{\partial t_1^2}\right)\left(\frac{\partial^2 TP_i(t_1,T)}{\partial T^2}\right)-\left(\frac{\partial^2 TP_i(t_1,T)}{\partial t_1\,\partial T}\right)^2\right\}>0$ is satisfied gives a maximum value for the total profit per unit time $TP_i(t_1,T)$.

For case 1 $(0<M\le t_d)$

The necessary condition for the total profit $TP_1(t_1,T)$ in equation (38) to be the maximum are $\frac{\partial TP_1(t_1,T)}{\partial t_1}=0$ and $\frac{\partial TP_1(t_1,T)}{\partial T}=0$, which give

$$\frac{\partial TP_1(t_1,T)}{\partial t_1}=\frac{d}{T}\{-\mathrm{P}_1t_1+\mathrm{Q}_1+(C_\pi\delta+C_b)T+(S_2-C)\delta T\}$$

Setting $\frac{\partial TP_1(t_1,T)}{\partial t_1}=0$ gives

$$\{\mathrm{P}_1t_1-\mathrm{Q}_1+(C_\pi\delta+C_b)T+(S_2-C)\delta T\}=0 \tag{32}$$

and

$$T=\frac{1}{\{(C_\pi\delta+C_b)+(S_2-C)\delta\}}(\mathrm{P}_1t_1-\mathrm{Q}_1) \tag{33}$$

Since $(t_d-M)\ge 0,(t_1-t_d)>0,(t_1-M)>0$, it should be noted that

$$\begin{aligned}(\mathrm{P}_1t_1-\mathrm{Q}_1)=\Big[&(S_2-C)\delta+h_1(t_d\theta(t_1-t_d)+t_1)+h_2\left(t_1-\frac{t_d}{2}\right)t_d+\frac{h_2t_d\theta}{2}(t_1-t_d)t_d+C\theta(t_1-t_d)\\&+C_\pi(\delta-1)+(C_\pi\delta+C_b)t_1+cI_c\big((t_1-M)+\theta(t_d-M)(t_1-t_d)\big)\Big]>0\end{aligned}$$

Similarly,

$$\frac{\partial TP_1(t_1,T)}{\partial T}=-\frac{d}{T^2}\left\{-\frac{1}{2}\mathrm{P}_1t_1^2+\mathrm{Q}_1t_1-\mathrm{R}_1+\frac{[(C_{\pi Z}\delta+C_b)+(S_2-C)\delta]T^2}{2}\right\} \tag{34}$$

Setting $\frac{\partial TP_1(t_1,T)}{\partial T}=0$ to obtain

$$-\frac{d}{T^2}\left\{-\frac{1}{2}\mathrm{P}_1t_1^2+\mathrm{Q}_1t_1-\mathrm{R}_1+\frac{[(C_\pi\delta+C_b)+(S_2-C)\delta]T^2}{2}\right\}=0 \tag{35}$$

Substituting T from equation (33) into equation (35) yields

$$\begin{aligned}\{\mathrm{P}_1([(C_\pi\delta+C_b)+(S_2-C)\delta]-\mathrm{P}_1)t_1^2-2\mathrm{Q}_1([(C_\pi\delta+C_b)+(S_2-C)\delta]-\mathrm{P}_1)t_1\\-(\mathrm{Q}_1^2-2[(C_\pi\delta+C_b)+(S_2-C)\delta]\mathrm{R}_1)\}=0\end{aligned} \tag{36}$$

Let

$$\begin{aligned}\Delta_1=\{\mathrm{P}_1([(C_\pi\delta+C_b)+(S_2-C)\delta]-\mathrm{P}_1)t_d^2-2\mathrm{Q}_1([(C_\pi\delta+C_b)+(S_2-C)\delta]-\mathrm{P}_1)t_d\\-(\mathrm{Q}_1^2-2[(C_\pi\delta+C_b)+(S_2-C)\delta]\mathrm{R}_1)\}-(2[(C_\pi\delta+C_b)+(S_2-C)\delta]\mathrm{R}_1+\mathrm{Q}_1^2),\end{aligned}$$

then the following result is obtained.

Lemma 1

(i) If $\Delta_1\ge 0$, then the solution of $t_1\in[t_d,\infty)$ (say t_{11}^*) which satisfies equation (36) not only exists but also is unique.

See the proof in Appendix 1a

(ii) If $\Delta_1 < 0$, then the solution of $t_1 \in [t_d, \infty)$ which satisfies equation (36) does not exist.

See the proof in Appendix 1b

Therefore, the value of t_1 (denoted by t_{11}^*) can be found from equation (36) and is given by

$$t_{11}^* = \frac{\mathrm{Q}_1}{\mathrm{P}_1} + \frac{1}{\mathrm{P}_1}\sqrt{\frac{[(C_\pi\delta + C_b) + (S_2 - C)\delta](2\mathrm{P}_1\mathrm{R}_1 - \mathrm{Q}_1^2)}{(\mathrm{P}_1 - [(C_\pi\delta + C_b) + (S_2 - C)\delta])}} \tag{37}$$

Once the value of t_{11}^* is obtained, then the value of T (denoted by T_1^*) can be found from (33) and is given by

$$T_1^* = \frac{1}{[(C_\pi\delta + C_b) + (S_2 - C)\delta]}(\mathrm{P}_1 t_{11}^* - \mathrm{Q}_1) \tag{38}$$

Equations (37) and (38) give the optimal values of t_{11}^* and T_1^* for the profit function in equation (23) only if Q_1 satisfies the inequality given in equation (39)

$$2\mathrm{P}_1\mathrm{R}_1 > \mathrm{Q}_1^2 \tag{39}$$

Theorem 1

(i) If $\Delta_1 \geq 0$, then the total profit $TP_1(t_1, T)$ is concave and reaches its global maximum at the point (t_{11}^*, T_1^*), where (t_{11}^*, T_1^*) is the point which satisfies equations (36) and (32), if all principal minors are negative definite i.e., if

$$\left(\frac{\partial^2 TP_1(t_1,\ T)}{\partial t_1^2}\bigg|_{(t_{11}^*,\ T_1^*)}\right) < 0, \left(\frac{\partial^2 TP_1(t_1,\ T)}{\partial T^2}\bigg|_{(t_{11}^*,\ T_1^*)}\right) < 0$$

and

$$\begin{vmatrix} \frac{\partial^2 TP_1(t_1,\ T)}{\partial t_1^2}\bigg|_{(t_{11}^*,\ T_1^*)} & \frac{\partial^2 TP_1(t_1,\ T)}{\partial t_1 \partial T}\bigg|_{(t_{11}^*,\ T_1^*)} \\ \frac{\partial^2 TP_1(t_1,\ T)}{\partial t_1 \partial T}\bigg|_{(t_{11}^*,\ T_1^*)} & \left(\frac{\partial^2 TP_1(t_1,\ T)}{\partial T^2}\bigg|_{(t_{11}^*,\ T_1^*)}\right) \end{vmatrix} > 0.$$

See the proof in Appendix 1c

(ii) If $\Delta_1 < 0$, then the total profit $TP_1(t_1, T)$ has a maximum value at the point (t_{11}^*, T_1^*) where $t_{11}^* = t_d$ and $T_1^* = \frac{1}{[(C_\pi\delta + C_b) + (S_2 - C)\delta]}(\mathrm{P}_1 t_d - \mathrm{Q}_1)$

See the proof in Appendix 1d

Proof of part (ii). When $\Delta_1 < 0$, then $F_1(t_1) < 0$ for all $t_1 \in [t_d,\ \infty)$. Therefore, $\frac{\partial TP_1(t_1,\ T)}{\partial T} = \frac{F_1(t_1)}{T^2} < 0$ for all $t_1 \in [t_d, \infty)$ which implies $TP_1(t_1,\ T)$ is a strictly decreasing function of t_1. Therefore, $TP_1(t_1,\ T)$ has a maximum value when t_1 is minimum. Therefore, $TP_1(t_1,\ T)$ has a maximum value at the point $(t_{11}^*,\ T_1^*)$ where $t_{11}^* = t_d$ and $T_1^* = \frac{1}{\{(C_\pi\delta + C_b) + (S_2 - C)\delta\}}(\mathrm{P}_1 t_d - \mathrm{Q}_1)$. This completes the proof.

For case 2 ($t_d < M \leq t_1$)

The necessary condition for the total profit $TP_1(t_1, T)$ in equation (23) to be the maximum are $\frac{\partial TP_2(t_1,T)}{\partial t_1} = 0$ and $\frac{\partial TP_2(t_1,T)}{\partial T} = 0$, which give

$$\frac{\partial TP_2(t_1,T)}{\partial t_1} = \frac{d}{T}\{-\mathrm{P}_2 t_1 + \mathrm{Q}_2 + (C_\pi\delta + C_b)T + (S_2 - C)\delta T\}$$

Setting $\frac{\partial TP_2(t_1,T)}{\partial t_1} = 0$ gives

$$\{\mathrm{P}_2 t_1 - \mathrm{Q}_2 + (C_\pi\delta + C_b)T + (S_2 - C)\delta T\} = 0 \tag{40}$$

and

$$T = \frac{1}{\{(C_\pi\delta + C_b) + (S_2 - C)\delta\}}(\mathrm{P}_2 t_1 - \mathrm{Q}_2) \tag{41}$$

Since $(t_1 - t_d) > 0, (t_1 - M) \geq 0$, it should be noted that

$$(\mathrm{P}_2 t_1 - \mathrm{Q}_2) = \Big[(S_2 - C)\delta t_1 + h_1(t_d\theta(t_1 - t_d) + t_1) + h_2\left(t_1 - \frac{t_d}{2}\right)t_d + \frac{h_2 t_d \theta}{2}(t_1 - t_d)t_d + C\theta(t_1 - t_d) + (C_\pi\delta + C_b)t_1 + cI_c(t_1 - M)\Big] > 0$$

Similarly,

$$\frac{\partial TP_2(t_1, T)}{\partial T} = -\frac{d}{T^2}\left\{-\frac{1}{2}\mathrm{P}_2 t_1^2 + \mathrm{Q}_2 t_1 - \mathrm{R}_2 + \frac{[(C_{\pi Z}\delta + C_b) + (S_2 - C)\delta]T^2}{2}\right\} \tag{42}$$

Setting $\frac{\partial TP_2(t_1,T)}{\partial T} = 0$ to obtain

$$-\frac{d}{T^2}\left\{-\frac{1}{2}\mathrm{P}_2 t_1^2 + \mathrm{Q}_2 t_1 - \mathrm{R}_2 + \frac{[(C_\pi\delta + C_b) + (S_2 - C)\delta]T^2}{2}\right\} = 0 \tag{43}$$

Substituting T from equation (41) into equation (43) yields

$$\{\mathrm{P}_2([(C_\pi\delta + C_b) + (S_2 - C)\delta] - \mathrm{P}_2)t_1^2 - 2\mathrm{Q}_2([(C_\pi\delta + C_b) + (S_2 - C)\delta] - \mathrm{P}_2)t_1 - (\mathrm{Q}_2^2 - 2[(C_\pi\delta + C_b) + (S_2 - C)\delta]\mathrm{R}_2)\} = 0 \tag{44}$$

Let

$$\Delta_2 = \{\mathrm{P}_2([(C_\pi\delta + C_b) + (S_2 - C)\delta] - \mathrm{P}_2)M^2 - 2\mathrm{Q}_2([(C_\pi\delta + C_b) + (S_2 - C)\delta] - \mathrm{P}_2)M - (\mathrm{Q}_2^2 - 2[(C_\pi\delta + C_b) + (S_2 - C)\delta]\mathrm{R}_2)\}$$

Lemma 2

(i) If $\Delta_2 \geq 0$, then the solution of $t_1 \in [M, \infty)$ (say t_{12}^*) which satisfies equation (44) not only exists but also is unique.

The proof is similar to Appendix 1a, hence is omitted

(ii) If $\Delta_2 < 0$, then the solution of $t_1 \in [M, \infty)$ which satisfies equation (44) does not exist.

The proof is similar to Appendix 1b, hence is omitted

Therefore,, the value of t_1 (denoted by t_{12}^*) can be found from equation (44) and is given by

$$t_{12}^* = \frac{\mathrm{Q}_2}{\mathrm{P}_2} + \frac{1}{\mathrm{P}_2}\sqrt{\frac{[(C_\pi\delta + C_b) + (S_2 - C)\delta](2\mathrm{P}_2\mathrm{R}_2 - \mathrm{Q}_2^2)}{(\mathrm{P}_2 - [(C_\pi\delta + C_b) + (S_2 - C)\delta])}} \tag{45}$$

Once the value of t_{12}^* is obtained, then the value of T (denoted byT_2^*) can be found from (41) and is given by

$$T_2^* = \frac{1}{[(C_\pi\delta + C_b) + (S_2 - C)\delta]}(\mathrm{P}_2 t_{12}^* - \mathrm{Q}_2) \tag{46}$$

Equations (45) and (46) give the optimal values of t_{12}^* and T_2^* for the profit function in equation (26) only if Q_2 satisfies the inequality given in equation (47)

$$2\mathrm{P}_2\mathrm{R}_2 > \mathrm{Q}_2^2 \tag{47}$$

Theorem 2

(i) If $\Delta_2 \geq 0$, then the total profit $TP_2(t_1, T)$ is concave and reaches its global maximum at the point (t_{12}^*, T_2^*), where (t_{12}^*, T_2^*) is the point which satisfies equations (44) and (40), if all principal minors are negative definite i.e., if

$$\left(\frac{\partial^2 TP_2(t_1,\ T)}{\partial t_1^2}\bigg|_{(t_{12}^*,\ T_2^*)}\right) < 0, \left(\frac{\partial^2 TP_2(t_1,\ T)}{\partial T^2}\bigg|_{(t_{12}^*,\ T_2^*)}\right) < 0$$

and

$$\begin{vmatrix} \frac{\partial^2 TP_2(t_1,\ T)}{\partial t_2^2}\bigg|_{(t_{12}^*,\ T_2^*)} & \frac{\partial^2 TP_2(t_1,\ T)}{\partial t_1 \partial T}\bigg|_{(t_{12}^*,\ T_2^*)} \\ \frac{\partial^2 TP_2(t_1,\ T)}{\partial t_1 \partial T}\bigg|_{(t_{12}^*,\ T_2^*)} & \left(\frac{\partial^2 TP_2(t_1,\ T)}{\partial T^2}\bigg|_{(t_{12}^*,\ T_2^*)}\right) \end{vmatrix} > 0.$$

The proof is similar to Appendix 1c, hence is omitted

(ii) If $\Delta_2 < 0$, then the total profit $TP_2(t_1, T)$ has a maximum value at the point (t_{12}^*, T_2^*) where $t_{12}^* = t_d$ and $T_2^* = \frac{1}{[(C_\pi\delta + C_b) + (S_2 - C)\delta]}(\mathrm{P}_2 t_d - \mathrm{Q}_2)$

The proof is similar to Appendix 1d, hence is omitted.

For case 3 (M> t_1)

The necessary condition for the total profit $TP_3(t_1,T)$ in equation (28) to be the maximum are $\frac{\partial TP_3(t_1,T)}{\partial t_1} = 0$ and $\frac{\partial TP_3(t_1,T)}{\partial T} = 0$, which give

$$\frac{\partial TP_3(t_1,T)}{\partial t_1} = \frac{d}{T}\{-\mathrm{P}_3 t_1 + \mathrm{Q}_3 + (C_\pi\delta + C_b)T + (S_2 - C)\delta T\}$$

Setting $\frac{\partial TP_3(t_1,T)}{\partial t_1} = 0$ gives

$$\{\mathrm{P}_3 t_1 - \mathrm{Q}_3 + (C_\pi\delta + C_b)T + (S_2 - C)\delta T\} = 0 \tag{48}$$

and

$$T = \frac{1}{\{(C_\pi\delta + C_b) + (S_2 - C)\delta\}}(\mathrm{P}_3 t_1 - \mathrm{Q}_3) \tag{49}$$

Since$(t_1 - t_d) > 0$, it should be noted that

$$(\mathrm{P}_3 t_1 - \mathrm{Q}_3) = \Bigg[(S_2 - C)\delta t_1 + h_1(t_d\theta(t_1 - t_d) + t_1) + h_2\left(t_1 - \frac{t_d}{2}\right)t_d + \frac{h_2 t_d\theta}{2}(t_1 - t_d)t_d + C\theta(t_1 - t_d) + +(C_\pi\delta + C_b)t_1 + \frac{1}{d}\left\{S_1 I_e\left(at_d + b\frac{t_d^2}{2} + c\frac{t_d^3}{3}\right)\right\} + S_2 I_e t_1 - S_2 I_e(t_d + M)\Bigg] > 0$$

Similarly,

$$\frac{\partial TP_3(t_1,T)}{\partial T} = -\frac{d}{T^2}\left\{-\frac{1}{2}\mathrm{P}_3 t_1^2 + \mathrm{Q}_3 t_1 - \mathrm{R}_3 + \frac{[(C_{\pi Z}\delta + C_b) + (S_2 - C)\delta]T^2}{2}\right\} \tag{50}$$

Setting $\frac{\partial TP_3(t_1,T)}{\partial T} = 0$ to obtain

$$-\frac{d}{T^2}\left\{-\frac{1}{2}\mathrm{P}_3 t_1^2 + \mathrm{Q}_3 t_1 - \mathrm{R}_3 + \frac{[(C_\pi\delta + C_b) + (S_2 - C)\delta]T^2}{2}\right\} = 0 \tag{51}$$

Substituting T from equation (49) into equation (51) yields

$$\{\mathrm{P}_3([(C_\pi\delta + C_b) + (S_2 - C)\delta] - \mathrm{P}_3)t_1^2 - 2\mathrm{Q}_3([(C_\pi\delta + C_b) + (S_2 - C)\delta] - \mathrm{P}_3)t_1 - (\mathrm{Q}_3^2 - 2[(C_\pi\delta + C_b) + (S_2 - C)\delta]\mathrm{R}_3)\} = 0 \tag{52}$$

Let

$$\Delta_{3a} = \{P_3([C_\pi\delta + C_b) + (S_2 - C)\delta] - P_3)t_d^2 - 2Q_3([(C_\pi\delta + C_b) + (S_2 - C)\delta] - P_3)t_d - (Q_3^2 - 2[(C_\pi\delta + C_b) + (S_2 - C)\delta]R_3)\} > 0$$

$$\Delta_{3b} = \{\mathrm{P}_3([(C_\pi\delta + C_b) + (S_2 - C)\delta] - \mathrm{P}_3)M^2 - 2\mathrm{Q}_3([(C_\pi\delta + C_b) + (S_2 - C)\delta] - \mathrm{P}_3)M - (\mathrm{Q}_3^2 - 2[(C_\pi\delta + C_b) + (S_2 - C)\delta]\mathrm{R}_3)\} < 0$$

Lemma 3

(i) If $\Delta_{3b} \le 0 \le \Delta_{3a}$, then the solution of $t_1 \in [t_d, M]$ (say t_{13}^*) which satisfies equation (52) not only exists but also is unique.
The proof is similar to Appendix 1a, hence is omitted.

(ii) If $\Delta_{3a} < 0$, then the solution of $t_1 \in [t_d, M]$ which satisfies equation (52) does not exist.
The proof is similar to Appendix 1b, hence is omitted

Therefore,, the value of t_1 (denotedt_{13}^* by) can be found from equation (52) and is given by

$$t_{13}^* = \frac{\mathrm{Q}_3}{\mathrm{P}_3} + \frac{1}{\mathrm{P}_3}\sqrt{\frac{[(C_\pi\delta + C_b) + (S_2 - C)\delta](\mathrm{Q}_3^2 - 2\mathrm{P}_3\mathrm{R}_3)}{([(C_\pi\delta + C_b) + (S_2 - C)\delta] - \mathrm{P}_3)}} \tag{53}$$

Once the value of t_{13}^* is obtained, then the value of T (denoted byT_3^*) can be found from (62) and is given by

$$T_3^* = \frac{1}{[(C_\pi\delta + C_b) + (S_2 - C)\delta]}(\mathrm{P}_3 t_{13}^* - \mathrm{Q}_3) \quad (54)$$

Equations (53) and (54) give the optimal values of t_{13}^* and T_3^* for the profit function in equation (28) only if Q_3 satisfies the inequality given in equation (55)

$$2\mathrm{P}_3\mathrm{R}_3 > \mathrm{Q}_3^2 \quad (55)$$

Theorem 3

(i) If $\Delta_{3a} \geq 0$, then the total profit $TP_3(t_1, T)$ is concave and reaches its global maximum at the point (t_{13}^*, T_3^*), where (t_{13}^*, T_3^*) is the point which satisfies equations (52) and (48), if all principal minors are negative definite i.e., if

$$\left(\left.\frac{\partial^2 TP_3(t_1,\ T)}{\partial t_1^2}\right|_{(t_{13}^*,\ T_3^*)}\right) < 0, \left(\left.\frac{\partial^2 TP_3(t_1,\ T)}{\partial T^2}\right|_{(t_{13}^*,\ T_3^*)}\right) < 0$$

and

$$\begin{vmatrix} \left.\frac{\partial^2 TP_3(t_1,\ T)}{\partial t_3^2}\right|_{(t_{13}^*,\ T_3^*)} & \left.\frac{\partial^2 TP_3(t_1,\ T)}{\partial t_1 \partial T}\right|_{(t_{13}^*,\ T_3^*)} \\ \left.\frac{\partial^2 TP_3(t_1,\ T)}{\partial t_1 \partial T}\right|_{(t_{13}^*,\ T_3^*)} & \left(\left.\frac{\partial^2 TP_3(t_1,\ T)}{\partial T^2}\right|_{(t_{13}^*,\ T_3^*)}\right) \end{vmatrix} > 0.$$

The proof is similar to Appendix 1c, hence is omitted

(ii) If $\Delta_{3a} < 0$, then the total profit $TP_3(t_1, T)$ has a maximum value at the point (t_{13}^*, T_3^*) where $t_{13}^* = M$ and $T_3^* = \frac{1}{\{(C_\pi\delta + C_b) + (S_2 - C)\delta\}}(P_3 M - Q_3)$.

The proof is similar to Appendix 1d, hence is omitted

(iii) If $\Delta_{3b} > 0$, then the total profit $TP_3(t_1, T)$ has a maximum value at the point (t_{13}^*, T_3^*) where $t_{13}^* = t_d$ and $T_3^* = \frac{1}{\{(C_\pi\delta + C_b) + (S_2 - C)\delta\}}(P_3 t_d - Q_3)$

The proof is similar to Appendix 1d, hence is omitted

5. NUMERICAL RESULTS

Example 5.1 ($M \leq t_d$)

The following parameters were adopted from Babangida and Baraya (2021) in addition to h_1, δ, C_π and C_b which are not considered in their study. The parameters and their values are as follows:

Table 1: parameters and their Values

Parameter(s)	Value(s)
A	\$250/order
h_1	\$2 unit/year
h_2	\$15 unit/year
θ	0.01 unit/year
a	180 unit
b	30 unit
c	15 unit
d	120 unit
t_d	0.1354 year
M	0.0888 year
I_c	0.1
I_e	0.08
C_b	\$30
δ	0.85
C_π	1

It is seen that $M \leq t_d, \Delta_1$=46.7063 $> 0, 2P_1R_1$ =58.7894,$Q_1^2 = 0.0851$ and $2P_1R_1 > Q_1^2$. Substituting the above values in equation (39), (37), (23) and (56).

The result is obtained in the table below.

Table 2: Optimal Solutions for case 1

Parameters	Values
t_{11}^*	0.4739 (172 days)
T_1^*	0.5424 (197 days)
$TP_1(t_{11}^*, T_1^*)$	\$311.6589
EOQ_1^*	73.3331 unit.

Example 5.2 ($M > t_d$)

The values of the parameters are same as in example 5.1 [as in Babangida and Baraya (2021)] except that $M = 0.1523.$ It is seen that $M > t_d$, $\Delta_2 = 45.0853 > 0$, $2P_2R_2 = 58.0326, = Q_2^2 = 0.1496$ and $2P_2R_2 > Q_2^2$. Substituting the above values in equation (46), (45), (26) and (56). The result is obtained in the table below

Table 3: Optimal Solutions for case 2

Parameters	Values
t_{12}^*	0.4730 (172 days)
T_2^*	0.5386 (196 days)
$TP_2(t_{12}^*, T_2^*)$	\$323.7361
EOQ_2^*	72.8984 unit.

Example 5.3 ($M > t_1$)

The values of the parameters are same as in example 5.1 except that $M = 0.36.$ It is seen that $M > t_d$, $\Delta_{3a} = 22.8756 > 0$, $\Delta_{3b} = -2.0383 < 0$ $2P_3R_3 = 31.0511, Q_3^2 = 0.2916$ and $2P_3R_3 > Q_3^2$. Substituting the above values in equation (54), (53), (28) and (56). The result is obtained in the table below.

Table 4: Optimal Solutions for case 3

Parameters	Values
t_{13}^*	0.3473 (126 days)
T_3^*	0.3892 (142 days)
$TP_3(t_{13}^*, T_3^*)$	\$423.6718
EOQ_3^*	55.0559 unit.

It is readily visible from the table above that average total profit per unit for case 1 and case 2 of the suggested model is greater than that of Babangida and Baraya (2021). Thus, the optimal result is case 3.

6. SENSITIVITY ANALYSIS

The sensitivity analysis of some model parameters of the optimal result is performed by changing each of the parameters from $-6\%, -4\%, -2, +2, +4$ $to + 6\%$ taking one parameter at a time and keeping the remaining parameters unchanged. The effects of these changes of parameters on cycle length, optimal time with positive inventory total profit and economic order quantity per cycle are discussed and summarised in Table 6 below:

Table 5: Comparison table

Comparison of our model with Babangida and Baraya (2021)			
Models	Average total profit per unit for case 1	Average total profit per unit for case 2	Average total profit per unit for case 3
Babangida and Baraya (2021)	\$4.1341	\$4.3176	-
Proposed Model	\$4.2410	\$4.3460	\$7.1653

7. DISCUSSION ON SENSITIVITY ANALYSIS

Based on the results shown in Table 6, the following managerial insights are obtained.

(i) From Table 6, it is obviously seen that the higher the rate of deterioration (θ) the lower the optimal time with positive inventory (t_1^*), cycle length (T^*), order quantity (EOQ^*) and the total profit $TP(T^*)$ and vice versa. This implies that the retailer needs to take all the necessary measures to avoid or reduce deterioration in order to maximize higher profit.

(ii) From Table 6, it is apparently seen that as the unit selling price before deterioration sets in (S_1) increases, the optimal time with positive inventory (t_1^*), cycle length (T^*) and order quantity (EOQ^*) decrease while the total profit $TP(T^*)$ increases and vice versa. This implies that as the selling price increases the retailer will order less quantity to enjoy the benefits of trade credit more frequently.

(iii) From Table 6, it is evidently seen that as the unit selling price after deterioration sets in (S_2) increases, the optimal time with positive inventory (t_1^*), cycle length (T^*), order quantity (EOQ^*) and the total profit $TP(T^*)$ increase and vice versa. This implies that as the selling price is increasing the retailer maximizes higher profit.

(iv) From table 6, it evidently seen that as the unit cost of lost sales per unit (C_π) increases the optimal time with positive inventory (t_1^*) also increases while cycle length (T^*), order quantity (EOQ^*) and the total profit $TP(T^*)$ decrease.This implies that the retailer should order less quantity when the unit cost of lost sales is high.

Table 6. Effect of some model parameters from -6% to +6% on decision variables.

Parameter	% change in Parameter	% change in t_1^*	% change in T^*	% change in EOQ^*	% change in $TP(t_1^*,$
θ	-6%	0.0711	0.0595	0.0478	0.0151
	-4%	0.0473	0.0397	0.0318	0.0101
	-2%	0.0237	0.0198	0.0159	0.0050
	2%	-0.0236	-0.0198	-0.0159	-0.0050
	4%	-0.0473	-0.0400	-0.0318	-0.0101
	6%	-0.0709	-0.0593	-0.0476	-0.0151
S_1	-6%	36.1743	37.4237	31.6385	-19.8899
	-4%	25.3085	26.1886	22.1435	-13.9385
	-2%	13.3811	13.8499	11.7126	-7.3830
	2%	-15.5517	-16.1063	-13.6266	8.6183
	4%	-34.8953	-36.1544	-30.5975	19.3948
	6%	-64.2589	-66.6184	-56.4069	35.8740
S_2	-6%	-24.2549	-24.8304	-21.0158	-20.8158
	-4%	-15.2895	-15.6297	-13.2277	-14.3218
	-2%	-7.2839	-7.4361	-6.2930	-7.3411
	2%	6.7147	6.8385	5.7870	7.6210
	4%	12.965	13.1893	11.16116	15.4670
	6%	18.8260	19.1318	16.1897	23.4987
C_π	-6%	-0.0057	0.0097	0.0078	0.0033
	-4%	-0.0038	0.0065	0.0052	0.0022
	-2%	-0.0019	0.0032	0.0026	0.0011
	2%	0.0019	-0.0032	-0.0026	-0.0011
	4%	0.0038	-0.0065	-0.0052	-0.0022
	6%	0.0057	-0.0097	-0.0078	-0.0032

8. CONCLUSION

This research developed an economic order quantity model for non-instantaneous deteriorating items with two phase demand rates, linear holding cost, time dependent partial backlogging rate and two-level pricing strategies under trade credit policy. The purpose of the model is to determine the optimal time with positive inventory, cycle length and order quantity such that the total profit of the inventory system has a maximum value. Some numerical examples have been given to illustrate the theoretical result of the model. Sensitivity analysis of some model parameters on the decision variables has been carried out, and suggestions towards maximising the total profit were also given. The retailer can maximize the total profit by ordering less quantity and shorten the cycle length if the rate of deterioration, unit purchasing cost, and interest charged, ordering cost and shortage cost increase and unit selling price before deterioration start, unit selling price after deterioration start and interested earned decrease. The model can be used in inventory control and management of items such as food items (e.g. beans, maize, corns, millet), electronics (e.g. mobile phones, computers), automobiles, fashionable items, etc. The proposed model can be extended by considering factors such as variable deterioration, inflation and time value of money, quantity discount, and order size dependent trade credit.

REFERENCES

Ahmad, Y., Hudu, D., & Adam, K. M. (2019). Ordering policy for ameliorating inventory with linear demand rate and unconstrained retailers capital. *Abacus (Mathematics Science Series), 44*(1).

Ahmed, Y., & Musa, A. (2016). Economic order quantity model for delayed deteriorating items with time dependent exponential declining demand and shortages. *ABACUS: Journal of the Mathematical Association of Nigeria, 43*(2), 14–24.

Babangida, B., & Baraya, Y. M. (2019a). An inventory model for non-instantaneous deteriorating item with time dependent quadratic demand and linear holding cost under trade credit policy. *ABACUS: Journal of the Mathematical Association of Nigeria, 46*(1), 191–217. [Crossref]

Babangida, B., & Baraya, Y. M. (2019b). An inventory model for non-instantaneous deteriorating item with time dependent quadratic demand and complete backlogging under trade credit policy.

ABACUS: Journal of the Mathematical Association of Nigeria, 46(1), 488–505. [Crossref]

Babangida, B., & Baraya, Y. M. (2020). An inventory model for non-instantaneous deteriorating items with time dependent quadratic demand, two storage facilities and shortages under trade credit policy. *International Journal of Modelling in Operations Management, 8*(1), 1–44. [Crossref]

Babangida, B., & Baraya, Y. M. (2021). An EOQ model for non-instantaneous deteriorating items with two-phase demand rates and two-level pricing strategies under trade credit policy. *The Transactions of the Nigerian Association of Mathematical Physics, 17*, 117–130.

Babangida, B., & Baraya, Y. M. (2022). An EOQ model for non-instantaneous deteriorating items with two-phase demand rates, linear holding cost and time dependent partial backlogging rate under trade credit policy. *ABACUS: Journal of the Mathematical Association of Nigeria, 49*(2), 91–125.

Baraya, Y. M., & Sani, B. (2011). An economic production quantity (EPQ) model for delayed deteriorating items with stock-dependent demand rate and linear time dependent holding cost. *Journal of the Nigerian Association of Mathematical Physics, 19*(5), 123–130.

Baraya, Y. M., & Sani, B. (2016). A deteriorating inventory model for items with constant demand rate under warehouse capacity constraint. *ABACUS: Journal of the Mathematical Association of Nigeria, 43*(2), 180–192.

Chakrabarti, T., Giri, B. C., & Chaudhuri, K. S. (1998). A heuristic for replenishment of deteriorating items with time-varying demand and shortages in all cycles. *International Journal of Systems Science, 29*(6), 551–555. [Crossref]

Choudhury, K. D., Karmakar, B., Das, M., & Datta, T. K. (2013). An inventory model for deteriorating items with stock dependent demand, time-varying holding cost and shortages. *Journal of the Operational Research Society, 23*(1), 137–142.

Dash, B. P., Singh, T., & Pattnayak, H. (2014). An inventory model for deteriorating items with exponential declining demand and time-varying holding cost. *American Journal of Operations Research, 4*(1), 1–7. [Crossref]

Dutta, D., & Kumar, P. (2015). A partial backlogging inventory model for deteriorating items with time-varying demand and holding cost. *International Journal of Mathematics in Operational Research, 7*(3), 281–296. [Crossref]

Geetha, K. V., & Uthayakumar, R. (2010). Economic design of an inventory policy for non-instantaneous deteriorating items under permissible delay in payments. *Journal of Computational and Applied Mathematics, 233*(10), 2492–2505. [Crossref]

Giri, B. C., Chakraborty, T., & Chaudhuri, K. S. (2000). A note on lot-sizing heuristic for deteriorating items with time-varying demands and shortages. *Computers and Operations Research, 27*(6), 495–505. [Crossref]

Haley, C. W., & Higgins, R. C. (1973). Inventory policy and trade credit financing. *Management Science, 20*(4), 464–471. [Crossref]

Harris, F. (1913). How many parts to make at once, factory. *The Magazine of Management, 10*(2), 135–136.

Jaggi, C. K., Gupta, M., & Tiwari, S. (2019). Credit financing in economic ordering policies for deteriorating items with stochastic demand and promotional efforts in two-warehouse environment. *International Journal of Operational Research, 35*(4), 529–550. [Crossref]

Kar, S., Bhunia, A. K., & Maiti, M. (2001). Deterministic inventory model with two-levels of storage, a linear trend in demand and a fixed time horizon. *Computers and Operations Research, 28*(13), 1315–1331. [Crossref]

Khanra, S., Ghosh, S. K., & Chaudhuri, K. S. (2011). An EOQ model for a deteriorating items with time dependent quadratic demand under permissible delay in payment. *Applied Mathematics and Computation, 218*(1), 1–9. [Crossref]

Mandal, D., & Venkataraman, S. V. (2019). A dynamic programming model for perishable inventory management. *International Journal of Operational Research, 35*(2), 147–177. [Crossref]

Musa, A., & Sani, B. (2012). Inventory ordering policies of delayed deteriorating items under permissible delay in payments. *International Journal of Production Economics, 136*(1), 75–83. [Crossref]

Priya, R. K., & Senbagam, K. (2018). An EOQ inventory model for two parameter Weibull deterioration with quadratic time dependent demand and shortages. *International Journal of Pure and Applied Mathematics, 119*(7), 467–478.

Roy, A. (2008). An inventory model for deteriorating items with price dependent demand and time-varying holding cost. *Advance Modelling and Optimization, 10*(1), 25–37.

Sakar, B., & Sakar, S. (2013). An improved inventory model with partial backlogging, time-varying deterioration and stock-dependent demand. *Economic Modelling, 30*(1), 924–932. [Crossref]

Selvaraju, P., & Ghuru, S. K. (2018). EOQ models for deteriorative items with constant, linear and quadratic holding cost and shortages – a comparative study. *International Journal of Operational Research, 33*(4), 462–480. [Crossref]

Shaikh, A. A., Tiwari, S., & Cárdenas-Barrón, L. E. (2018). Closed-form solutions for the EPQ-based inventory model for exponentially deteriorating items under retailer partial trade credit policy in supply chain. *International Journal of Applied and Computational Mathematics, 4*(2), 1–9. [Crossref]

Uthayakumar, R., & Karuppasamy, S. K. (2017). An inventory model for variable deteriorating pharmaceutical items with time dependent demand and time dependent holding cost under trade credit in healthcare industries. *Communications in Applied Analysis, 21*(4), 533–549. [Crossref]

Whitin, T. M. (1957). *Theory of inventory management.* Princeton University Press

APPENDIX1a: Proof of lemma 1(i)

From equation (36), a new function $F_1(t_1)$ is defined as follows

$$F_1(t_1) = \{\mathrm{P}_1([(C_\pi\delta + C_b) + (S_2 - C)\delta] - \mathrm{P}_1)t_1^2 - 2\mathrm{Q}_1([(C_\pi\delta + C_b) + (S_2 - C)\delta] - \mathrm{P}_1)t_1 - (\mathrm{Q}_1^2 - 2[(C_\pi\delta + C_b) + (S_2 - C)\delta]\mathrm{R}_1)\}, \qquad t_1 \in [t_d, \infty) \quad (57)$$

Taking the first-order derivative of $F_1(t_1)$ with respect to $t_1 \in [t_d, \infty)$, it follows that

$$\frac{F_1(t_1)}{dt_1} = 2\{[(C_\pi\delta + C_b) + (S_2 - C)\delta] - \mathrm{P}_1\}(\mathrm{P}_1 t_1 - \mathrm{Q}_1) < 0$$

Because $(\mathrm{P}_1 t_1 - \mathrm{Q}_1) > 0$

And

$$\{[(C_\pi\delta + C_b) + (S_2 - C)\delta] - \mathrm{P}_1\} = -\left[h_1(t_d\theta + 1) + h_2\left(\frac{t_d\theta}{2} + 1\right)t_d + C\theta + cI_c(\theta(t_d - M) + 1)\right] < 0$$

Hence $F_1(t_1)$ is a strictly decreasing function of t_1 in the interval $[t_d, \infty)$. Moreover, $\lim_{t_1\to\infty} F_1(t_1) = -\infty$ and $F_1(t_d) = \Delta_1 \geq 0$. Therefore, by applying intermediate value theorem, there exists a unique t_1 say $t_{11}^* \in [t_d, \infty)$ such that $F_1(t_{11}^*) = 0$. Hence t_{11}^* is the unique solution of equation (36).

APPENDIX 1b: proof of lemma 1(ii)

If $\Delta_1 < 0$, then from equation (37), $F_1(t_1) < 0$. Since $F_1(t_1)$ is a strictly decreasing function of $t_1 \in [t_d, \infty)$ and $F_1(t_1) < 0$ for all $T \in [t_d, \infty)$. Therefore,, a value of $T \in [t_d, \infty)$ such that $F_1(t_1) = 0$ cannot found. This completes the proof.

APPENDIX 1c: proof of Theorem 1(i)

When $\Delta_1 \geq 0$, it is seen that t_{11}^* and T_1^* are the unique solutions of equations (36) and (32) respectively from Lemma l(i). Taking the second derivative of $TP_1(t_1, T)$ with respect to t_1 and T, and then finding the values of these functions at the point (t_{11}^*, T_1^*), it follows that

$$\left.\frac{\partial^2 TP_1(t_1, T)}{\partial t_1^2}\right|_{(t_{11}^*, T_1^*)} = -\frac{d}{T_1^*}\mathrm{P}_1 < 0$$

$$\left.\frac{\partial^2 TP_1(t_1, T)}{\partial t_1 \partial T}\right|_{(t_{11}^*, T_1^*)} = \frac{d}{T_1^*}\{(C_\pi\delta + C_b) + (S_2 - C)\delta\}$$

$$\left.\frac{\partial^2 TP_1(t_1, T)}{\partial T^2}\right|_{(t_{11}^*, T_1^*)} = -\frac{d}{T_1^*}\{(C_\pi\delta + C_b) + (S_2 - C)\delta\} < 0$$

and

$$\left(\left.\frac{\partial^2 TP_1(t_1, T)}{\partial t_1^2}\right|_{(t_{11}^*, T_1^*)}\right)\left(\left.\frac{\partial^2 TP_1(t_1, T)}{\partial T^2}\right|_{(t_{11}^*, T_1^*)}\right) - \left(\left.\frac{\partial^2 TP_1(t_1, T)}{\partial t_1 \partial T}\right|_{(t_{11}^*, T_1^*)}\right)^2 = \frac{d^2 C_b\delta}{T_1^{*2}}\left(2C_b\delta + \left[h_1(t_d\theta + 1) + h_2\left(\frac{t_d\theta}{2} + 1\right)t_d + C\theta + cI_c(\theta(t_d - M) + 1)\right]\right) > 0 (58)$$

It is therefore conclude from (58) and Lemma 1 that $TP_1(t_{11}^*, T_1^*)$ is concave and (t_{11}^*, T_1^*) is the global maximum point of $TP_1(t_1, T)$. Hence the values of t_1 and T in (37) and (38) are optimal.

APPENDIX 1d: proof of Theorem 1(ii)

When $\Delta_1 < 0$, then $F_1(t_1) < 0$ for all $t_1 \in [t_d,\ \infty)$. Therefore, $\frac{\partial TP_1(t_1,\ T)}{\partial T} = \frac{F_1(t_1)}{T^2} < 0$ for all $t_1 \in [t_d, \infty)$ which implies $TP_1(t_1,\ T)$ is a strictly decreasing function of t_1. Therefore, $TP_1(t_1,\ T)$ has a maximum value when t_1 is minimum. Therefore, $TP_1(t_1,\ T)$ has a maximum value at the point $(t_{11}^*,\ T_1^*)$ where $t_{11}^* = t_d$ and $T_1^* = \frac{1}{\{(C_\pi\delta + C_b) + (S_2 - C)\delta\}}(\mathrm{P}_1 t_d - \mathrm{Q}_1)$. This completes the proof.

.