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New Analytical Solutions for the Fractional Modified Korteweg de Vries-Zakharov-Kuznetsov Equation with Beta Derivative

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ABSTRACT
This paper employs the sine-cosine method to derive new exact solutions for the modified fractional (3+1)-dimensional Korteweg-de Vries–Zakharov-Kuznetsov equation using the Beta fractional derivative approach. The effectiveness of the sine-cosine method is demonstrated through the successful construction of exact solutions, which are further illustrated with 3D and 2D graphical representations. The results highlight the potential of this method as a powerful and efficient tool for solving nonlinear fractional partial differential equations, providing insights into the behavior and structure of complex physical phenomena described by such equations.

INTRODUCTION

Fractional calculus is a specialized branch of applied mathematics that extends classical calculus to incorporate derivatives and integrals of non-integer order. It provides powerful tools for describing complex and irregular phenomena Akinyemi et al. (2022), Acey et al. (2019), Ahmed et al. (2020), Akinyemi and Huseen(2020); Benjamin and Mahony(1992); Gardner et al.: (1967), Helal (2009), Hirota (1971), Wakil (2006). The origins of fractional calculus date back to the works of Leibniz and Euler in the 17th and 18th centuries Kadomtsev et al.:(1970), Kalil et al. (2014). Over time, it has gained significant attention in fluid mechanics, plasma physics, engineering, and many other scientific disciplines. Differential equations involving fractional derivatives are known as non-classical differential equations. Leibniz was among the first to explore the generalization of differentiation and integration to non-integer orders. In the 19th and early 20th centuries, mathematicians like Augustin-Louis Cauchy and Karl Weierstrass contributed to the theoretical development of calculus. However, fractional calculus became more formally structured in the late 19th century, with significant contributions from Liouville and Riemann Koca and Atangans(2017). In the 20th century, mathematicians such as Caputo, Miller, and Ross advanced the field by developing a systematic framework for fractional calculus. Today, fractional calculus is widely applied in various scientific and engineering domains, including physics, biology, and control theory. Its principles have been extensively studied in recent years due to their importance in modeling complex phenomena in applied physical sciences Korteweg and Vries (1895). Non-integer-order models have been particularly useful in representing processes in areas such as signal processing, fluid dynamics, acoustics, electromagnetism, analytical chemistry, and multiple engineering disciplines Kumar et al. (2018); Machado et al.:(2011), Miller and Ross (2003). Additionally, methods like the sine–cosine and tan techniques have been employed in solving fractional differential equations (Oldham and Spanier 1974).

In recent times, there has been a growing interest in obtaining exact analytical solutions for nonlinear wave equations using appropriate methods Podlubny(1999), Samko et al. (1993), Sousa (2018), Tariq and Seadawy(2019), Wazwaz (2008). The investigation of exact traveling wave solutions for nonlinear partial differential equations (NPDEs) plays a vital role in comprehending nonlinear physical phenomena Xu (2005), Yusuf et al. (2019). These solutions offer valuable insights into the underlying mechanisms governing complex physical processes and dynamical behaviors described by nonlinear evolution equations.

Among the various nonlinear evolution equations, the (3+1)-dimensional spacetime fractional modified Korteweg-de Vries Zakharov-Kuznetsov (KdV-ZK) equation is a significant model. This equation provides a fundamental framework for exploring nonlinear dynamics in higher-dimensional settings.

$$U_t + \theta_1 U^2 U_x + \theta_2 U_{xxx} + \theta_3 (U_{yy} + U_{zz}))_x = 0. \qquad (1)$$

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In the equation (1), the nonlinear term U^2U_x is responsible for causing the waveform to steepen, while the dispersion term U_{xxx} counteracts this by causing the waveform to spread out. The interplay between these two effects results in the formation of solitons. In the lowest-order approximation, the dispersion effects are sufficiently weak that they can be neglected, allowing the nonlinear term to dominate and facilitate the creation of soliton structures.

2 METHODOLOGY

In this section, we briefly introduced the notation of the Beta derivative.

2.1 Beta Derivative

The beta derivative can be stated by Yusuf et al. (2019)

1. ${}_0^AT_\eta^\alpha(F(\eta)) = \lim_{\epsilon\to 0}\frac{F(\eta+\eta\epsilon(\eta+\frac{1}{\Gamma(\alpha)}))-F(\eta)}{\epsilon}$

   along with the properties as comes next

2. ${}_0^AT_\eta^\alpha(aF(\eta)+bG(\eta)) = a{}_0^AT_\eta^\alpha F(\eta) + b{}_0^AF_\eta^\alpha G(\eta){}_0^A$

3. $T_\eta^\alpha(c) = 0$, for any$c$depicting a constant,

4. ${}_0^AT_\eta^\alpha(F(\eta).G(\eta)) = G(\eta){}_0^AT_\eta^\alpha F(\eta) + F(\eta){}_0^AT_\eta^\alpha G(\eta)$

5. ${}_0^AT_\eta^\alpha\left(\frac{F(\eta)}{G(\eta)}\right) = \frac{G(\eta){}_0^AT_\eta^\alpha F(\eta)-F(\eta){}_0^AT_\eta^\alpha G(\eta)}{G^2(\eta)}$.

$$considering \quad \epsilon = (\eta+\frac{1}{\Gamma(\alpha)})^{1-\alpha}h, h$$
$$0 \quad when \quad \epsilon\to 0,$$

therefore we have

6. ${}_0^AT_\eta^\alpha F(\eta) = (\eta+\frac{1}{\Gamma(\alpha)})^{1-\alpha}\frac{dF(\eta)}{d\eta}, with \quad \zeta = \frac{l}{\alpha}(\eta+\frac{1}{\Gamma(\alpha)})^\alpha$,

   where l is a constant.

7. ${}_0^AT_\eta^\alpha(\frac{F(\tau)}{G(\eta)}) = l\frac{dF(\tau)}{d(\tau)}$.

The β-fractional derivative of the modified Korteweg –de Vries–Zakharov–Kuznetsov equation is expressed as:

$$D_t^\alpha U + \theta_1 U^2 D_x^\alpha U + \theta_2 D_{xxx}^{3\alpha}U + \theta_3(D_{yyx}^{3\alpha}U + D_{zzx}^{3\alpha})U = 0. \quad (2)$$

where the coefficients θ_j for $j = 1,2,3,4,5$ are nonzero constants.

3 ANALYSIS OF THE SINE-COSINE METHOD

Here, we outline the key steps of the sine-cosine method, as introduced by Wazwaz (2008), which will be applied to solve nonlinear partial differential equations.

Let assume

$$P(U, U_t, U_x, U_{xx}, U_{yy}, U_{zz}, \cdots) = 0. \quad (3)$$

This variable characterizes the dynamic wave solution $U(x,t)$. It is beneficial to outline the key steps of the method.

Step 1. In order to determine the traveling wave solution of equation (3), we introduce the wave transformation variable.

$$U(x,y,z,t) = P(\zeta), \;\; \zeta = x+y+z-ct \quad (4)$$

Step 3. The Sine-Cosine method used the following changes:

$$\frac{\partial u}{\partial t} = -c\frac{d}{d\zeta}, \cdots \frac{\partial^2}{\partial t^2} = c^2\frac{d^2}{d\zeta^2}, \qquad \frac{\partial}{\partial x} = \frac{d}{d\zeta}, \frac{\partial^2}{\partial x^2} = \frac{d^2}{d\zeta^2} \quad (5)$$

and so on for the rest of the derivatives. The transformation (5) converts the PDE (3) to an ODE.

$$Q(U, U_\zeta, U_{\zeta\zeta}, U_{\zeta\zeta\zeta}, \cdots) = 0, \quad (6)$$

whereU$_\zeta$ denotes $\frac{dU}{d\zeta}$.

Step 4. Next, we integrate the obtained ODE as many times as possible, assuming the constants of integration to be zero.

Step .5 We may set the solution in the form of

$$U(x,y,z,t) = \lambda\cos^\beta(\mu\zeta), \quad (7)$$

or

$$U(x,y,z,t) = \lambda\sin^\beta(\mu\zeta), \quad (8)$$

where β, λ, and μ are parameters that will be calculated.

Step6. The derivatives of equation(7) and (8) gives

$$U = \lambda cos^\beta(\mu\zeta) \quad (9)$$

$$U^n = \lambda^n cos^{n\beta}(\mu\zeta) \quad (10)$$

$$(U^n)_\zeta = -n\mu\beta\lambda^n cos_{(\mu\zeta)}^{n\beta-1} sin(\mu\zeta) \quad (11)$$

$$(U^n)_{\zeta\zeta} = -n^2\mu^2\beta^2\lambda^n cos_{(\mu\zeta)}^{n\beta} + n\mu^2\lambda^n\beta(n\beta - 1)cos^{n\beta-2}(\mu\zeta) \quad (12)$$

and

$$U = \lambda sin^\beta(\mu\zeta) \quad (13)$$

$$U^n = \lambda^n sin^{n\beta}(\mu\zeta) \quad (14)$$

$$(U^n)_\zeta = -n\mu\beta\lambda^n Sin^{n\beta-1}(\mu\zeta)Cos(\mu\zeta) \quad (15)$$

$$(U^n)_{\zeta\zeta} = -n^2\mu^2\beta^2\lambda^n sin^{n\beta}(\mu\zeta) + n\mu^2\lambda^n\beta(n\beta - 1)sin^{n\beta-2}(\mu\zeta) \quad (16)$$

and so on for the other derivatives.

Step 7. By substituting equations (9) to (12) into the derived ODE (6), or alternatively equations (13) to (16) into the same equation (6), we obtain a trigonometric equation involving either the cosine or sine function depending on the chosen approach. The parameters λ, β, and μ are then computed by first balancing the exponent of each pair of sine or cosine. We then collect the coefficients of the same power in $cos(\mu\zeta)$ or $sin(\mu\zeta)$ where these coefficients have to vanish and solve the resulting system of algebraic equations by using the computerized symbolic calculations to obtain the possible values of the unknown variables λ, β, and μ

4 APPLICATION OF THE METHOD

In this section, we demonstrate the application of the sine-cosine method for solving the Korteweg-de-Vries Benjami-Bona-Mahony equation.

Application 1 *Consider the(3+1)-dimensional beta fractional modified Korteweg-de Vries Zakarov-Kuznetsov equation*

$$D_t^\alpha U + \theta_1 U^2 D_x^\alpha U + \theta_2 D_{xxx}^{3\alpha} U + \theta_3(D_{yyx}^{3\alpha} + D_{zzx}^{3\alpha}) = 0 \quad (17)$$

Using the wave transformation

$$U(x, y, z, t) = U(\zeta), \quad (18)$$

$$\zeta = \frac{a}{\alpha}\left(x + \frac{1}{\Gamma(\alpha)}\right)^\alpha + \frac{b}{\alpha}\left(y + \frac{1}{\Gamma(\alpha)}\right)^\alpha + \frac{c}{\alpha}\left(z + \frac{1}{\Gamma(\alpha)}\right)^\alpha + \frac{d}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^\alpha \quad (19)$$

Differentiating $U(\zeta)$ with respect to x, y, z and t we have

$$dU' + \theta_1 U^2 U' + (a^3\theta_2 + ab^2\theta_3 + ac^2\theta_3)U''' = 0 \quad (20)$$

Integrating equation (20) once, we get

$$dU + \theta_1 \frac{U^3}{3} + (a^3\theta_2 + ab^2\theta_3 + ac^2\theta_3)U'' = 0 \quad (21)$$

Using the proposed method, we set

$$U(x, y, z, t) = \lambda sin^\beta(\mu\zeta),$$

$$or \quad U(x, y, z, t) = \lambda cos^\beta(\mu\zeta). \quad (22)$$

Let the solution be

$$U = \lambda cos^\beta(\mu\zeta) \quad (23)$$

$$U'' = -\lambda\mu^2\beta^2 cos^\beta(\mu\zeta) + \lambda\mu^2\beta(\beta - 1)cos^{\beta-2}(\mu\zeta) \quad (24)$$

Inserting U and U'' in equation (21), we

$$d\lambda cos^\beta(\mu\zeta) + \frac{\theta_1}{3}\lambda^3 cos^{3\beta}(\mu\zeta) - (a^3\theta_2 + ab^2\theta_3 + ac^2\theta_3)$$

$$\lambda\mu^2\beta^2 sin^\beta(\mu\zeta) + (a^3\theta_2 + ab^2\theta_3 + ac^2\theta_3)\lambda\mu^2\beta(\beta - 1)cos^\beta(\mu\zeta) = 0 \quad (25)$$

Then equation (25) is satisfied if the following algebraic system of equations holds:

$$(\beta - 1) \neq 0, \quad (26)$$

$$3\beta = \beta - 2, \quad (27)$$

$$d\lambda = (a^3\theta_2 + ab^2\theta_3 + ac^2\theta_3)\lambda\mu^2\beta^2, \quad (28)$$

$$\theta_1\frac{\lambda^2}{3} = -(a^3\theta_2 + ab^2\theta_3 + ac^2\theta_3)\mu^2\beta(\beta - 1). \quad (29)$$

Solving the above systems, we obtained the following results

$$\beta = -1,$$

$$\mu = -\sqrt{\frac{d}{a^3\theta_2 + a^3\theta_2 + ac^2\theta_3 + ab^2\theta_3}},$$

$$\lambda = -\sqrt{\frac{6d}{a\theta_1}}.$$

Substituting the obtained results in (21), we get

$$U(x, y, z, t) = -\sqrt{\frac{6d}{\theta_1}} sec\left(\sqrt{\frac{d}{a^3\theta_2 + a^3\theta_2 + ac^2\theta_3 + ab^2\theta_3}}(\zeta)\right) \quad d > 0, \quad (30)$$

$$U(x, y, z, t) = \sqrt{\frac{6d}{\theta_1}} csc\left(\sqrt{\frac{d}{a^3\theta_2 + a^3\theta_2 + ac^2\theta_3 + ab^2\theta_3}}(\zeta)\right) \quad d > 0. \quad (31)$$

For $d < 0$, we get the following solutions

$$U(x, y, z, t) = -\sqrt{\frac{6d}{\theta_1}} sech\left(\sqrt{\frac{d}{a^3\theta_2 + a^3\theta_2 + ac^2\theta_3 + ab^2\theta_3}}(\zeta)\right), \quad (32)$$

$$U(x, y, z, t) = \sqrt{\frac{6d}{\theta_1}} csch\left(\sqrt{\frac{d}{a^3\theta_2 + a^3\theta_2 + ac^2\theta_3 + ab^2\theta_3}}(\zeta)\right). \quad (33)$$

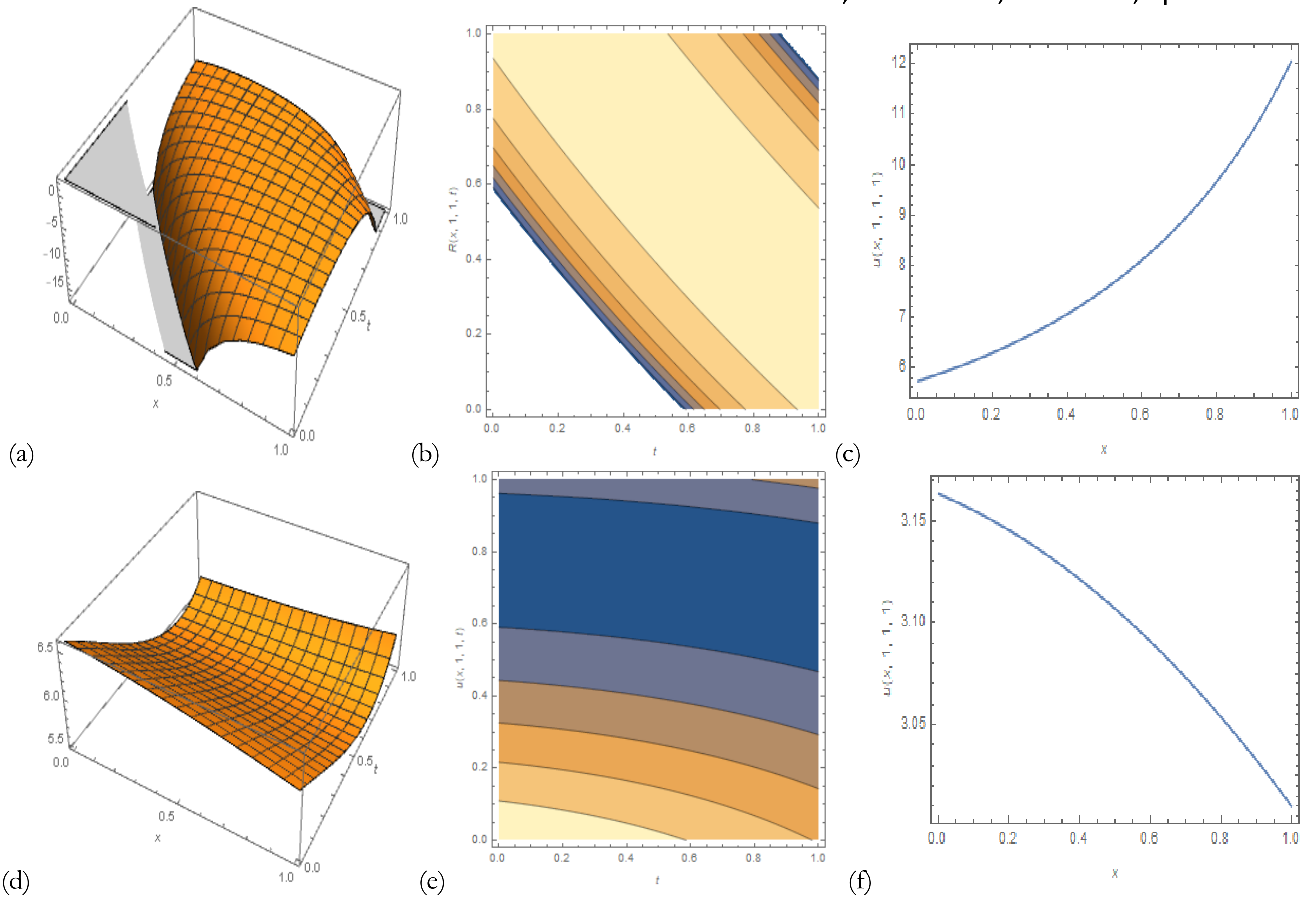

Figure 1: Analytical comparison of the fractional $(3+1)$ dimensional modified Korteweg-de Vries Zakharov-Kuznetsov (mKdV-ZK) equation using the Beta fractional derivative for various α, values time intervals.

5 RESULTS AND DISCUSSION

In this section, we discuss the behavior of analytical solutions for the fractional (3+1)-dimensional modified Korteweg-de Vries–Zakharov-Kuznetsov (mKdV-ZK) equation by employing the Beta fractional derivative and the sine-cosine method. Using computational tools like Mathematica, we illustrate 2D and 3D graphical solutions for various values of α, θ_1, θ_2, θ_3, and different parameters a, b, c, d, across different time intervals.

Figures 1(a) to 1(f) showcase the 2D and 3D graphical representations of Eq. (21), derived using the sine-cosine method. Specifically, for $\alpha = 4.5, a = b = c = d = y = z = t = 1$, and $\theta_1 = 2, \theta_2 = 4, \theta_3 = 5$, the 3D surface and 2D graphical solutions over various time intervals are displayed in Figures 1(a), 1(b), and 1(c). Similarly, Figures 1(d), 1(e), and 1(f) depict the 3D and 2D solutions under the same parameter settings and time intervals. The results reveal that increasing α amplifies wave activity, whereas decreasing α reduces wave intensity.

6 CONCLUSION

This study focused on applying the sine-cosine method to derive new exact solutions for the modified fractional (3+1)-dimensional Korteweg-de Vries–Zakharov-Kuznetsov equation using the Beta fractional derivative approach. The objective was successfully achieved, and the research also includes 3D and 2D graphical representations of the solutions. The results highlight the sine-cosine method as a reliable and effective approach for solving nonlinear partial differential.

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