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ORIGINAL RESEARCH ARTICLE

Comparing the Methods of Estimators of the Modified Inverted Kumaraswamy Distribution Using Inverse Power Function

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ABSTRACT
This article considers the problem of estimating additional parameters of the modified inverted Kumaraswamy (MIK) distribution using the inverse power function based on the general Kumaraswamy distribution. The parameters' maximum likelihood (MLE) estimators are obtained, while the Bayesian estimates are obtained using the maximum product spacing (MPS). We obtained a new model for generalizing the existing ones to make them more flexible and to aid their application in various fields. An expression for reliability measures, order statistics, and some other important properties are derived. The maximum likelihood estimation method is used to estimate the proposed model's unknown parameters. Finally, a simulation study was reported concerning different sample sizes and method schemes. The practical utility of the proposed distribution is demonstrated using two real-life datasets: (i) survival times (in months) of 101 patients diagnosed with advanced acute myelogenous leukaemia, and (ii) strengths of 63 samples of 1.5 cm glass fibres, originally obtained by workers at the UK National Physical Laboratory. The results highlight the robustness and flexibility of the proposed model in reliability and survival analysis contexts.

INTRODUCTION

Abd Al-Fattah et al. (2017) introduced the inverted Kumaraswamy (IKum) distribution and studied its properties. The IKum distribution can be used in long-term reliability predictions, producing optimistic predictions of rare events occurring in the distribution's right tail compared with other distributions.

Bayesian prediction is an important topic in statistical inference, where we try to use the previous data to predict future observations inside the same population with a specified probability. When the unobserved failures belong to the same sample, then the prediction is called one-sample Bayesian prediction, while it is called two-sample Bayesian prediction when we want to predict by a new sample using an old sample. The Bayesian prediction was discussed by many authors based on different distributions with different types of censored samples. El-Din and Shafay (2013) studied Bayesian prediction intervals based on progressively Type-II censored data. Shafay and Balakrishnan (2012) studied the Bayesian prediction intervals based on the Type-I hybrid censored data. Bayesian prediction intervals of generalized order statistics based on multiple Type-II censored data were discussed by Mohie El-Din *et al.* (2012); they also studied the Bayesian prediction for order statistics from a general class of distributions based on left Type-II censored data, see (2011). Latest El-Din *et al.* (2017) studied the one-sample Bayesian prediction intervals based on Type-II progressively hybrid censored samples. All parametric statistical techniques, such as inference, modelling, survival analysis, and reliability, are based on statistical distributions. Fitting the data to a statistical model is critical when analyzing lifetime data. For this reason, several lifespan distributions have been established in the literature. The majority of lifespan models have a limited set of behaviours. Such distributions are unable to provide a better fit for all real scenarios. As a result, a variety of distribution classes have been created by expanding common continuous distributions. The family generated from continuous distributions is a new enhancement for developing and expanding classic distributions. The newly generated distributions have been extensively researched in a variety of fields, and they provide greater application flexibility.

One of the most well-known lifetime distributions is the inverted Kumaraswamy distribution by Abd AL-Fattah *et al.* (2017). The distribution has a wide range of

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applications in problems related to econometrics, biological sciences, survey sampling, engineering sciences, medical research, and life testing problems. In addition, it is employed in financial literature, environmental studies, and survival and reliability theory. Many researchers focused on the inverted distribution and its applications; for example, Calabria and Pulcini (1990) studied the inverse Weibull distribution, AL-Dayian (1999) introduced the inverted Burr Type XII distribution, Abd EL-Kader *et al.* (2003) also described the inverted Pareto Type I distribution, AL-Dayian (2004) discussed the inverted Pareto Type II distribution, and Aljuaid (2013) presented the exponentiated inverted Weibull distribution. Kumaraswamy (1980) presented a distribution with many similarities to the beta distribution. This distribution applies to many natural phenomena whose outcomes have lower and upper bounds, such as the height of individuals, scores obtained on a test, atmospheric temperatures, and hydrological data such as daily rainfall and daily stream flow (see Kumaraswamy, 1980; Jones, 2009; Yakubu and Doguwa, 2017; and Jamal *et al.,* 2021).

The current study contributes significantly to the growing literature on advanced lifetime distribution models and robust estimation methods. Habu *et al.* (2024) investigated the extension of the Topp-Leone distribution using Maximum Product Spacing (MPS) and Maximum Likelihood Estimation (MLE) techniques. Their findings emphasized the superior performance of MPS over MLE in terms of lower bias and RMSE, especially for smaller sample sizes, an observation consistent with the results of this study. Similarly, Obafemi et al. (2024) proposed a New Extension of the Topp-Leone Distribution (NETD) using a generalized logarithmic function, showcasing improved modelling flexibility for survival and reliability data.

In parallel, Sadiq *et al.* (2023c) developed the New Generalized Odd Fréchet-Odd Exponential-G family, combining the strengths of two generalized structures to produce a highly adaptable distribution with superior performance in various data modelling contexts. Their approach resonates with the methodology adopted in the current study, particularly in extending existing models to better capture the characteristics of lifetime data.

Earlier contributions by Sadiq *et al.* (2023a, 2023b) and Sadiq *et al.* (2022) introduced related flexible families such as the NGOF-G, NGOF-Exponentiated-G, and New Odd Fréchet-G distributions, all of which emphasized statistical properties and practical applications. Most recently, Sadiq *et al.* (2024) presented the Odd Rayleigh-G family, further advancing the field by providing comparative insights into distributional behaviour and estimation performance.

Taken together, these studies support the methodological choices in the present research, validating the efficacy of estimation methods and affirming the scientific value of developing extended distributions like the Modified Inverted Kumaraswamy (MIK). The current study adds to this literature by deriving a new distribution and empirically demonstrating its practical utility using real-life survival and reliability datasets.

Despite the growing interest in the Modified Inverted Kumaraswamy (MIK) distribution and its variants, most existing studies have focused primarily on traditional parameter estimation techniques such as Maximum Likelihood Estimation (MLE), often overlooking alternative methods that may offer improved accuracy and efficiency, especially with small or moderate sample sizes. Furthermore, previous models based on the MIK distribution have limited flexibility in capturing diverse data behaviours due to a lack of structural generalization. Notably, little to no attention has been given to incorporating the inverse power function to enhance the flexibility of the MIK distribution. Additionally, comparative assessments of estimation techniques, particularly between MLE and Maximum Product Spacing (MPS) have been inadequately explored within this context. This study bridges these gaps by (i) introducing a novel generalization of the MIK distribution using the inverse power function, (ii) deriving key statistical and reliability properties of the new model, and (iii) rigorously comparing MLE and MPS based on bias and RMSE through extensive simulation studies and real-life applications. By addressing these gaps, this research contributes significantly to improving parameter estimation accuracy and expanding the applicability of generalized Kumaraswamy-type models in survival and reliability analysis.

"Unlike classical lifetime distributions such as Weibull, Burr, and the standard Kumaraswamy, the proposed generalized MIK model augmented by the inverse power function offers enhanced flexibility in capturing a wider range of data behaviours, making it a robust and versatile alternative for reliability and survival analysis."

Abd AL-Fattah *et al.* (2017) propose the inverted Kumaraswamy distribution, while the Modified Inverted Kumaraswamy (MIK) distribution using the inverse power function was generated from the inverted Kumaraswamy distributions, and by comparing their consistency and performance of the estimated parameters using two different methods of estimation (MLE and MPS) is the aim of this study. The cumulative distribution function (cdf) and the probability density function (pdf) of our proposed distribution, called the modified inverted Kumaraswamy model, are given as:

$$F(x;\alpha,\beta,\lambda) = (1-(1+x^{\frac{1}{\lambda}})^{-\alpha})^{\beta}; 0 < x < \infty, \alpha,\beta,\lambda > 0 \quad (1)$$

$$f(x;\alpha,\beta,\lambda) = \frac{\alpha\beta}{\lambda}x^{\frac{1}{\lambda}-1}(1+x^{\frac{1}{\lambda}})^{-(\alpha+1)}(1-(1+x^{\frac{1}{\lambda}})^{-\alpha})^{\beta-1}; 0 < x < \infty, \alpha,\beta,\lambda > 0 \quad (2)$$

This research aims to compare the estimators methods and see which best fits the proposed distribution called the modified inverted Kumaraswamy distribution. By incorporating both theoretical advancements and practical applications, the proposed model not only generalizes the

MIK distribution but also presents a more powerful alternative to classical lifetime models, particularly in the domains of reliability engineering and biomedical survival analysis. This paper is organized as follows: Section 1 presents the introduction and background of the study; we obtained some important representations for the MIK distribution in Section 2. The parameters were estimated using the maximum likelihood estimation (MLE) and maximum product of spacing (MPS) approach in Section 3. The simulation study was conducted to show that the estimated parameters are efficient and consistent using MLE and MPS in Section 4. Finally, Section 5 concludes the paper.

IMPORTANT REPRESENTATION

In this section, we derived a useful representation for the MIK cdf and pdf using some standard generalized binomial series expansion for negative and positive power. However, equation (1) can be expressed by related to the mentioned expansion as;

$$F(x) = \sum_{i=1}^{\infty}(-1)^i \binom{\beta}{i}(1 + x^{1/\lambda})^{-\alpha i} \tag{3}$$

The simplest form of the probability density function (pdf) given in equation (2) can also be expressed by related to the mentioned expansion as;

$$f(x) = \frac{\alpha\beta}{\lambda} x^{\frac{1}{\lambda}-1} \sum_{j=0}^{\infty}(-1)^j \binom{\beta-1}{j}(1 + x^{\frac{1}{\lambda}})^{-\alpha(1+j)-1} \tag{4}$$

Using the generalized binomial theorem given as,

$$(1+z)^{-b} = \sum_{k=0}^{\infty}\binom{-b}{k} z^k \quad for\ |b| < 1 \tag{5}$$

Then, the last term of equation (4) reduces to;

$$(1 + x^{\frac{1}{\lambda}})^{-\alpha(1+j)-1} = \sum_{k=0}^{\infty}(-1)^k \left(-\alpha(1+j) - 1_k\right)(x^{\frac{1}{\lambda}})^k \tag{6}$$

Substituting equation (6) into equation (4) and simplifying further, we have;

$$f(x) = \frac{\alpha\beta}{\lambda} \sum_{j,k=0}^{\infty}(-1)^{j+k} \binom{\beta-1}{j}\binom{-\alpha(1+j)-1}{k} x^{\frac{1}{\lambda}(1+k)-1} \tag{7}$$

Therefore, equation (7) is comfortably reduced to;

$$f(x) = \sum_{k=0}^{\infty} \psi_j\, x^{\frac{1}{\lambda}(1+k)-1} \tag{8}$$

where, $\psi_j = \frac{\alpha\beta}{\lambda} \sum_{j=0}^{\infty}(-1)^{j+k} \binom{\beta-1}{j}\binom{-\alpha(1+j)-1}{k}$

PARAMETER ESTIMATION METHODS

In this section, the method of Maximum Likelihood Estimation (MLE) and Maximum Product Spacing (MPS) will be extensively used to estimate the parameter of the proposed distribution (MIK).

MAXIMUM LIKELIHOOD ESTIMATION

The maximum likelihood method is the technique for estimating parameters in a continuous probability model. Suppose that X is a random variable with a probability density function $f(x;\theta)$ where θ is a single unknown parameter. Let $X_1, X_2, \ldots, X_n$ be the observed values in a random sample of size n. In such a case, the log-likelihood function of the MIK distribution is expressed as

$$L(\theta) = n\,log(\alpha) + n\,log(\beta) - n\,log(\lambda) + \left(\frac{1}{\lambda} - 1\right)\sum_{i=1}^{n} log\, x_i - (\alpha+1)\sum_{i=1}^{n} log\left(1 + x_i^{\frac{1}{\lambda}}\right) + (\beta - 1)\sum_{i=1}^{n} log\left[1 - \left(1 + x_i^{\frac{1}{\lambda}}\right)^{\alpha}\right] \tag{9}$$

Differentiating equation (9) with respect to α, β, and λ

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} log\left(1 + x_i^{\frac{1}{\lambda}}\right) - (\beta - 1)\sum_{i=1}^{n} \frac{\left(1+x_i^{\frac{1}{\lambda}}\right)^{\alpha} log\left(1+x_i^{\frac{1}{\lambda}}\right)}{\left[1-\left(1+x_i^{\frac{1}{\lambda}}\right)^{\alpha}\right]} = 0 \tag{10}$$

$$\frac{\partial L}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} log\left[1 - \left(1 + x_i^{\frac{1}{\lambda}}\right)^{\alpha}\right] = 0 \tag{11}$$

$$\frac{\partial L}{\partial \lambda} = -\frac{n}{\lambda} - \frac{1}{\lambda^2}\sum_{i=1}^{n} log\, x_i - (\alpha+1)\sum_{i=1}^{n} \frac{x_i^{\frac{1}{\lambda}} log\, x_i}{\lambda^2\left(1+x_i^{\frac{1}{\lambda}}\right)} - (\beta + 1)\sum_{i=1}^{n} \frac{\alpha\left(1+x_i^{\frac{1}{\lambda}}\right)^{\alpha-1} x_i^{\frac{1}{\lambda}} log\, x_i}{\lambda^2\left[1-\left(1+x_i^{\frac{1}{\lambda}}\right)^{\alpha}\right]} = 0 \tag{12}$$

Therefore, equations (10), (11), and (12) are non-linear, and cannot be solved analytically, necessitating the use of analytical tools to solve them in numerical.

MAXIMUM OF PRODUCT SPACING (MPS)

Let $x_1, x_2, \ldots x_n$ be a random sample from the MIK distribution have CDF $F(x;\lambda,\alpha,\beta)$ presented in equation (1) and $x_1, x_2, \ldots x_n$represents the corresponding ordered samples. The spacing.

$$\Psi = F\left(x_{(i)}\right) - F\left(x_{(i-1)}\right);\ \forall i = 1,2,\ldots,n+1 \tag{13}$$

where $F\left(x_{(0)}\right) = 0$ and $F\left(x_{(n+1)}\right) = 1$

Therefore,

$$F\left(x_{(i)}; x; \lambda, \alpha, \beta\right) = \left[1 - \left[1 + x_{(i)}^{\frac{1}{\lambda}}\right]^{-\alpha}\right]^{\beta} \tag{14}$$

and

$$F\left(x_{(i-1)}; x; \lambda, \alpha, \beta\right) = \left[1 - \left[1 + x_{(i-1)}^{\frac{1}{\lambda}}\right]^{-\alpha}\right]^{\beta} \tag{15}$$

Thus,

$$\psi = \left[1-\left[1+x_{(i)}^{\frac{1}{\lambda}}\right]^{-\alpha}\right]^{\beta} - \left[1-\left[1+x_{(i-1)}^{\frac{1}{\lambda}}\right]^{-\alpha}\right]^{\beta} \quad (16)$$

The parameter estimates are obtained by maximizing the function presented as;

$$\Omega(x;\lambda,\alpha,\beta) = \frac{1}{n+1}\sum_{i=1}^{n+1\Sigma} log \quad (17)$$

The expression in equation (17) reduces to;

$$\Omega(x;\lambda,\alpha,\beta) = \frac{1}{n+1}\sum_{i=1}^{n+1} log\left[\left[1-\left[1+x_{(i)}^{\frac{1}{\lambda}}\right]^{-\alpha}\right]^{\beta} - \left[1-\left[1+x_{(i-1)}^{\frac{1}{\lambda}}\right]^{-\alpha}\right]^{\beta}\right] \quad (18)$$

Differentiating equation (18) with respect to individual parameters yields the parameter estimates of $\lambda_{MPS}, \alpha_{MPS}, \beta_{MPS}$and solving the nonlinear equations.

$$\frac{\partial\Omega(x;\lambda,\alpha,\beta)}{\partial\alpha} = \frac{1}{n+1}\sum_{i=1}^{n+1} log\left[A_{1(i)} - A_{2(i-1)}\right] \quad (19)$$

where

$$A_{(1)}(x_{(i)};\lambda,\alpha,\beta) = \frac{\beta\left[1-\left[1+x_{(i)}^{\frac{1}{\lambda}}\right]^{-\alpha}\right]^{\beta-1}\left[1+x_{(i)}^{\frac{1}{\lambda}}\right]^{-\alpha} log\left[1+x_{(i)}^{\frac{1}{\lambda}}\right]}{\left[1-\left[1+x_{(i)}^{\frac{1}{\lambda}}\right]^{-\alpha}\right]^{\beta}} \quad (20)$$

and

$$A_{(2)}(x_{(i-1)};\lambda,\alpha,\beta) = \frac{\beta\left[1-\left[1+x_{(i-1)}^{\frac{1}{\lambda}}\right]^{-\alpha}\right]^{\beta-1}\left[1+x_{(i-1)}^{\frac{1}{\lambda}}\right]^{-\alpha} log\left[1+x_{(i-1)}^{\frac{1}{\lambda}}\right]}{\left[1-\left[1+x_{(i-1)}^{\frac{1}{\lambda}}\right]^{-\alpha}\right]^{\beta}} \quad (21)$$

$$\frac{\partial\Omega(x;\lambda,\alpha,\beta)}{\partial\beta} = \frac{1}{n+1}\sum_{i=1}^{n+1} log\left[B_{1(i)} - B_{2(i-1)}\right] \quad (22)$$

where

$$B_{(1)}(x_{(i)};\lambda,\alpha,\beta) = \frac{\left[1-\left[1+x_{(i)}^{\frac{1}{\lambda}}\right]^{-\alpha}\right]^{\beta} log\left[1-\left[1+x_{(i)}^{\frac{1}{\lambda}}\right]^{-\alpha}\right]}{\left[1-\left[1+x_{(i)}^{\frac{1}{\lambda}}\right]^{-\alpha}\right]^{\beta}} \quad (23)$$

and

$$B_{(2)}(x_{(i-1)};\lambda,\alpha,\beta) = \frac{\left[1-\left[1+x_{(i-1)}^{\frac{1}{\lambda}}\right]^{-\alpha}\right]^{\beta} log\left[1-\left[1+x_{(i-1)}^{\frac{1}{\lambda}}\right]^{-\alpha}\right]}{\left[1-\left[1+x_{(i-1)}^{\frac{1}{\lambda}}\right]^{-\alpha}\right]^{\beta}} \quad (24)$$

$$\frac{\partial\Omega(x;\lambda,\alpha,\beta)}{\partial\lambda} = \frac{1}{n+1}\sum_{i=1}^{n+1} log\left[K_{1(i)} - K_{2(i-1)}\right] \quad (25)$$

where

$$K_{(1)}(x_{(i)};\lambda,\alpha,\beta) = \frac{\beta\left[1-\left[1+x_{(i)}^{\frac{1}{\lambda}}\right]^{-\alpha}\right]^{\beta-1}\alpha\left[1+x_{(i)}^{\frac{1}{\lambda}}\right]^{-(\alpha+1)} x_{(i)}^{\frac{1}{\lambda}} log\, x_{(i)}}{\lambda^2\left[1-\left[1+x_{(i)}^{\frac{1}{\lambda}}\right]^{-\alpha}\right]^{\beta}} \quad (26)$$

and

$$K_{(2)}(x_{(i-1)};\lambda,\alpha,\beta) = \frac{\beta\left[1-\left[1+x_{(i-1)}^{\frac{1}{\lambda}}\right]^{-\alpha}\right]^{\beta-1}\alpha\left[1+x_{(i-1)}^{\frac{1}{\lambda}}\right]^{-(\alpha+1)} x_{(i-1)}^{\frac{1}{\lambda}} log\, x_{(i-1)}}{\lambda^2\left[1-\left[1+x_{(i-1)}^{\frac{1}{\lambda}}\right]^{-\alpha}\right]^{\beta}} \quad (27)$$

The MPS is obtained by setting equations (19), (22), and (25) to zero and solving these questions simultaneously. Thus, these cannot be solved analytically, necessitating the use of analytical tools to solve them numerically.

SIMULATION STUDY

In this section, a numerical analysis will be conducted to evaluate the performance of MLE and MPS for MIK Distribution.

Table 1 presents the result of the simulation study comparing the Maximum Likelihood Estimates (MLE) and Maximum Product Spacing (MPS) estimates for the parameters $(\alpha = 1, \beta = 0.5,\ and\ \lambda = 0.4)$ across different sample sizes (n). The bias for the estimated parameter (α) by MLE decreases as the sample size increases, starting from 0.2926 at $n = 25$ to 0.0332 at $n = 1000$. The bias is consistently lower for MPS, starting from -0.0466 at $n = 25$to -0.0134 at $n = 1000$. The RMSE for MLE decreases from 0.6797 at $n = 25$ to 0.1842 at $n = 1000$. For MPS, the RMSE is consistently lower, starting from 0.5574 at $n = 25$ to 0.1776 at $n = 1000$.

Similarly, the bias for the estimated parameter (β) using MLE decreases from 0.1570 at $n = 25$ to 0.0185 at $n = 1000$. For MPS, the bias is consistently lower, starting from 0.0040 at $n = 25$ to -0.0033 at $n = 1000$. The RMSE for MLE decreases from 0.4129 at $n = 25$ to 0.0942 at $n = 1000$. The RMSE is consistently lower for MPS, starting from 0.2918 at $n = 25$ to 0.0877 at $n = 1000$. Furthermore, the bias for the estimated parameter (λ) MLE decreases from 0.0460 at $n = 25$ to 0.0089 at $n = 1000$. For MPS, the bias is consistently lower, starting from 0.0094 at $n = 25$to -0.0017 at $n = 1000$. The RMSE for MLE decreases from 0.1846 at $n = 25$ to 0.0542 at $n = 1000$. The RMSE is consistently lower for MPS, starting from 0.1742 at $n = 25$ to 0.0528 at $n = 1000$.

The MPS consistently shows lower bias compared to MLE across all parameters and sample sizes. The MPS consistently shows lower RMSE compared to MLE across all parameters and sample sizes. This suggests that the MPS method generally provides more accurate and reliable estimates compared to the MLE method, especially for smaller sample sizes.

Table 1 Maximum Likelihood Estimates (MLE) and Maximum Product Spacing (MPS) Estimates of (α =1, β =0.5, λ =0.4)

n	Parameters	M.L.E			MPS		
		Mean	Bias	RMSE	Mean	Bias	RMSE
25	$\alpha = 1$	1.2936	0.2926	0.6797	0.9534	-0.0466	0.5574
	β =0.5	1.2936	0.1570	0.4129	0.5040	0.0040	0.2918
	λ =0.4	1.2936	0.0460	0.1846	0.4094	0.0094	0.1742
50	$\alpha = 1$	1.2197	0.2197	0.5633	0.9606	-0.0394	0.4809
	$\beta = 0.5$	0.6222	0.1222	0.3285	0.5025	0.0025	0.2585
	λ =0.4	0.4447	0.0447	0.1614	0.4041	0.0041	0.1438
100	$\alpha = 1$	1.1389	0.1389	0.4264	0.9536	-0.0053	0.3702
	β =0.5	0.5826	0.0826	0.2471	0.4948	-0.0464	0.1890
	λ =0.4	0.4321	0.0321	0.1225	0.3987	-0.0013	0.1159
150	$\alpha = 1$	1.0944	0.0944	0.3727	0.9422	-0.0598	0.3368
	β =0.5	0.5578	0.0578	0.2094	0.4870	-0.0130	0.1702
	λ =0.4	0.4182	0.0182	0.1067	0.3973	-0.0070	0.1015
200	$\alpha = 1$	1.0779	0.0779	0.3338	0.9658	-0.0342	0.3000
	β =0.5	0.5469	0.0469	0.1830	0.4942	-0.0058	0.1508
	λ =0.4	0.4182	0.0182	0.0965	0.3973	-0.0027	0.0896
250	$\alpha = 1$	1.0601	0.0601	0.3026	0.9542	-0.0458	0.2783
	β =0.5	0.5382	0.0382	0.1628	0.4895	-0.0105	0.1430
	λ =0.4	0.4144	0.0144	0.0892	0.3935	-0.0065	0.0834
500	$\alpha = 1$	1.0489	0.0489	0.2465	0.9741	-0.0259	0.2301
	β =0.5	0.5300	0.0300	0.1328	0.4953	-0.0047	0.1159
	λ =0.4	0.4132	0.0132	0.0725	0.3970	-0.0030	0.0696
1000	$\alpha = 1$	1.0332	0.0332	0.1842	0.9866	-0.0134	0.1776
	β =0.5	0.5185	0.0185	0.0942	0.4967	-0.0033	0.0877
	λ =0.4	0.4089	0.0089	0.0542	0.3983	-0.0017	0.0528

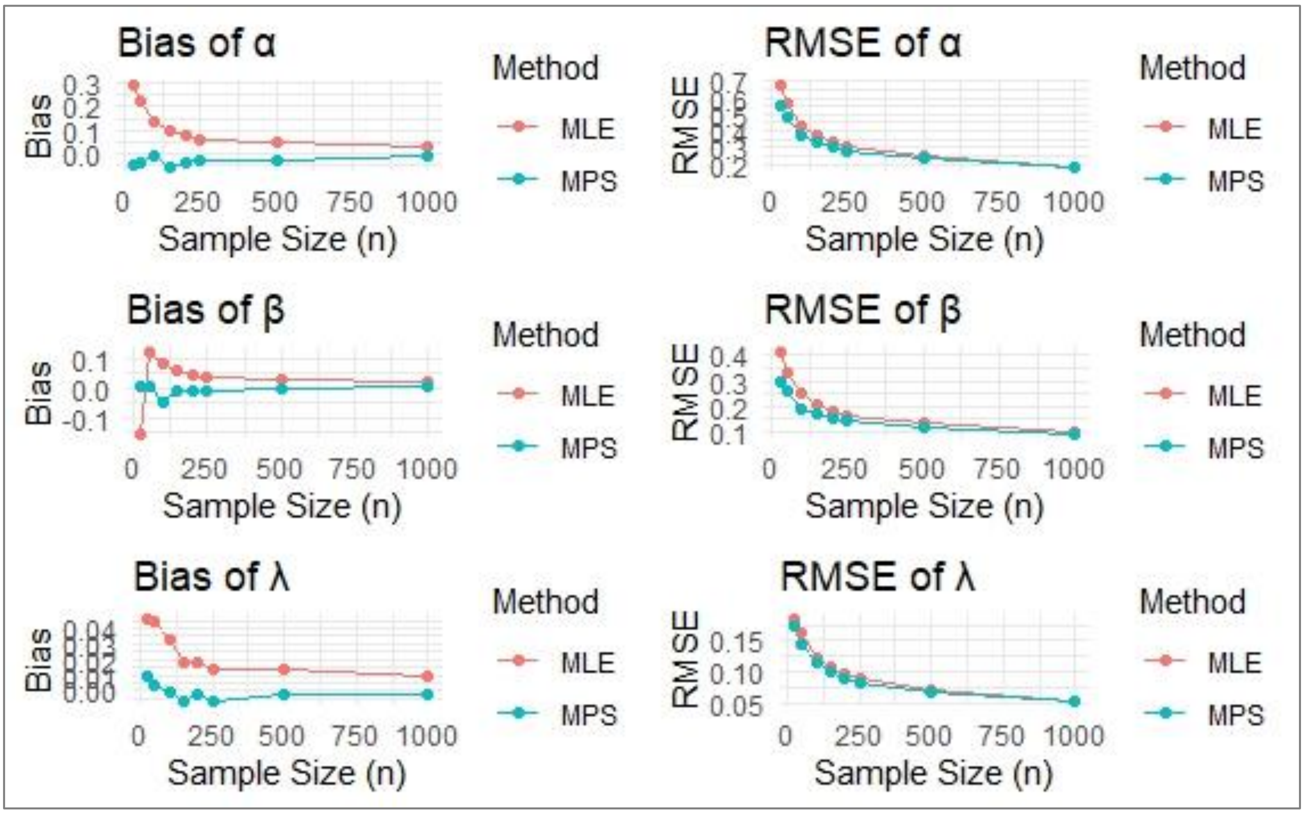

Figure 1: Bias and RMSE of MLE and MPS for each parameter estimates

Figure 1 the plots comparing the bias and RMSE (Root Mean Square Error) of the MLE (Maximum Likelihood Estimation) and MPS (Maximum Product Spacing) methods for each parameter $(\alpha, \beta, \text{and } \lambda)$.

The bias for the estimated parameter (α) decreases as the sample size increases for both methods. However, the MPS shows a slightly lower bias than MLE for larger sample sizes. The RMSE for the estimated parameter (α) decreases as the sample size increases, with MPS having a lower RMSE compared to MLE, especially for smaller sample sizes.

The bias for the estimated parameter (β) is relatively small and stable across different sample sizes, with MPS showing a slightly lower bias than MLE. The RMSE for the estimated parameter (β) is small and decreases slightly as the sample size increases, with MPS having a marginally lower RMSE compared to MLE.

The bias for the estimated parameter (λ) decreases as the sample size increases, with MPS showing a lower bias than MLE for larger sample sizes. The RMSE for the estimated parameter (λ) decreases as the sample size increases, with MPS having a lower RMSE compared to MLE, especially for smaller sample sizes. In conclusion, the MPS method generally performs better than the MLE method in terms of both bias and RMSE, particularly for smaller sample sizes. This suggests that MPS may be a more reliable method for parameter estimation in scenarios with limited data for the proposed MIK distribution.

Table 2 presents the simulation study that compares the performance of the Maximum Likelihood Estimation (MLE) and Maximum Product Spacing (MPS) methods in estimating parameters (α =1, β =0.5, λ =1.4) for various sample sizes (n). The performance measures used include **bias** and **Root Mean Square Error (RMSE).** MLE and MPS methods improve estimation accuracy as the sample size increases. While the Bias and RMSE decrease consistently across all parameters. For larger sample sizes ($n = 500, n = 1000$), both methods yield estimates closer to the true parameter values, indicating consistency of the estimators.

Figure 2 shows the plots comparing the bias and RMSE (Root Mean Square Error) of the MLE (Maximum Likelihood Estimation) and MPS (Maximum Product Spacing) methods for each parameter $(\alpha, \beta, \text{and } \lambda)$.

The bias for the estimated parameter (α) decreases as the sample size increases for both methods. However, the MPS shows a slightly lower bias than MLE for larger sample sizes. The RMSE for the estimated parameter (α) decreases as the sample size increases, with MPS having a lower RMSE compared to MLE, especially for smaller sample sizes.

The bias for the estimated parameter (β) is relatively small and stable across different sample sizes, with MPS showing a slightly lower bias than MLE. The RMSE for the estimated parameter (β) is small and decreases slightly as the sample size increases, with MPS having a marginally lower RMSE compared to MLE.

Table 2 Maximum Likelihood Estimates (MLE) And Maximum Product Spacing (MPS) Estimates of (α =1, β =0.5, λ =1.4)

n	Parameter	M.L.E			MPS		
		Mean	Bias	RMSE	Mean	Bias	RMSE
25	$\alpha = 1$	1.3371	0.3371	0.7556	0.9562	-0.0438	0.5764
	β =0.5	0.6767	0.1767	0.4202	0.5039	0.0039	0.2851
	λ =1.4	1.5833	0.1833	0.6581	1.4285	0.0285	0.5705
50	$\alpha = 1$	1.2403	0.2403	0.6177	0.9552	-0.0448	0.5057
	β =0.5	0.6382	0.1382	0.3652	0.5035	0.0035	0.2654
	λ =1.4	1.5697	0.1697	0.5900	1.4111	0.0111	0.5283
100	$\alpha = 1$	1.1526	0.1526	0.4561	0.9515	-0.0485	0.3971
	β =0.5	0.5916	0.0916	0.2624	0.4959	-0.0041	0.1987
	λ =1.4	1.5244	0.1244	0.4616	1.3924	-0.0076	0.4166
150	$\alpha = 1$	1.1093	0.1093	0.3916	0.9389	-0.0611	0.3697
	β =0.5	0.5661	0.0661	0.2134	1.3713	-0.0115	0.1884
	λ =1.4	1.4924	0.0924	0.3970	0.4885	-0.0287	0.3865
200	$\alpha = 1$	1.0955	0.0694	0.3565	0.9536	-0.0464	0.3299
	β =0.5	0.5568	0.0435	0.1958	0.4914	-0.0086	0.1669
	λ =1.4	1.4815	0.0602	0.3633	1.3783	-0.0217	0.3458
250	$\alpha = 1$	1.0694	0.0694	0.3118	0.9547	-0.0453	0.2981
	β =0.5	0.5435	0.0435	0.3210	0.4909	-0.0091	0.1520
	λ =1.4	1.4602	0.0602	0.0892	1.3761	-0.0239	0.3136
500	$\alpha = 1$	1.0487	0.0487	0.2465	0.9712	-0.0288	0.2353
	β =0.5	0.5304	0.0304	0.1326	0.4941	-0.0059	0.1180
	λ =1.4	1.4456	0.0456	0.2539	1.3862	-0.0138	0.2439
1000	$\alpha = 1$	1.0366	0.0366	0.1879	0.9855	-0.0145	0.1826
	β =0.5	0.5204	0.0204	0.0974	0.4968	-0.0032	0.0905
	λ =1.4	1.4345	0.0345	0.1922	1.3933	-0.0067	0.1877

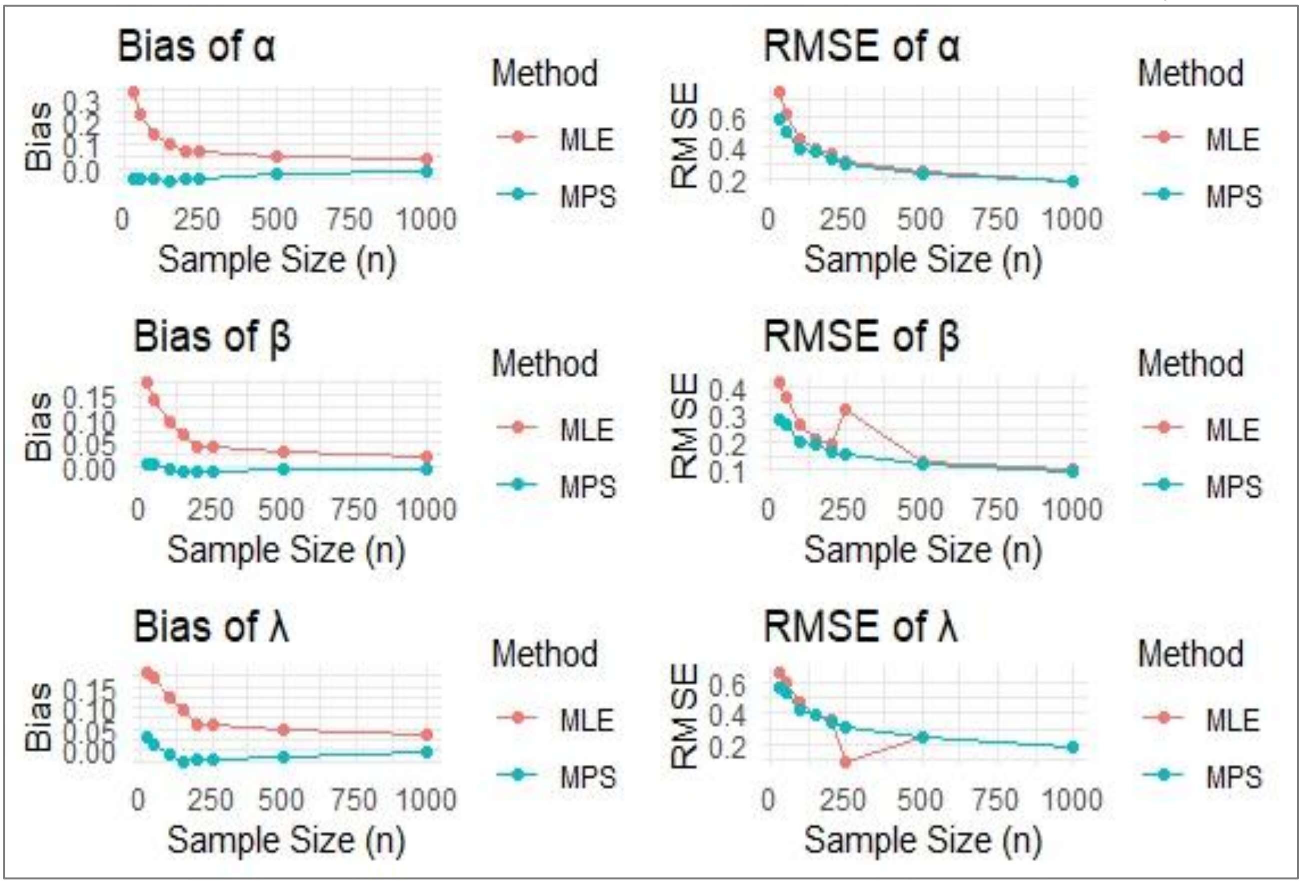

Figure 2: Bias and RMSE of MLE and MPS for each parameter estimates

Table 3 Maximum Likelihood Estimates (MLE) And Maximum Product Spacing (MPS) Estimates of (α =1, β =0.5, λ =2.4)

n	Parameter		M.L.E			MPS	
		Mean	Bias	RMSE	Mean	Bias	RMSE
	$\alpha = 1$	1.3421	0.3421	0.7746	0.9733	-0.0267	0.6097
25	$\beta = 0.5$	0.6892	0.1892	0.4586	0.5162	0.0162	0.3123
	$\lambda = 2.4$	2.7172	0.3172	1.1303	2.4741	0.0741	0.9900
	$\alpha = 1$	1.2548	0.2548	0.6260	0.9587	-0.0413	0.4041
50	$\beta = 0.5$	0.6450	0.1450	0.3676	0.5004	0.0004	0.2063
	$\lambda = 2.4$	2.7069	0.3069	0.9891	2.3968	-0.0032	0.7329
	$\alpha = 1$	1.1541	0.1541	0.4611	0.9587	-0.0278	0.5129
100	$\beta = 0.5$	0.5915	0.0915	0.2598	0.5004	0.0114	0.2649
	$\lambda = 2.4$	2.6141	0.2141	0.8007	2.3968	0.0433	0.8945
	$\alpha = 1$	1.1155	0.1155	0.4020	0.9598	-0.0402	0.3796
150	$\beta = 0.5$	0.5707	0.0707	0.2284	0.5002	0.0002	0.1986
	$\lambda = 2.4$	2.5675	0.1675	0.6962	2.3877	-0.0123	0.6716
	$\alpha = 1$	1.0938	0.0938	0.3644	0.9679	-0.0321	0.3350
200	$\beta = 0.5$	0.5578	0.0578	0.2060	0.4983	-0.0017	0.1735
	$\lambda = 2.4$	2.5364	0.1364	0.6337	2.3862	-0.0138	0.6014
	$\alpha = 1$	1.0776	0.0776	0.3222	0.9570	-0.0430	0.3027
250	$\beta = 0.5$	0.5487	0.0487	0.1780	0.4919	-0.0081	0.1534
	$\lambda = 2.4$	2.5168	0.1168	0.5661	2.3637	-0.0363	0.5415
	$\alpha = 1$	1.0552	0.0552	0.2541	0.9718	-0.0282	0.2456
500	$\beta = 0.5$	0.5337	0.0337	0.1359	0.4951	-0.0049	0.1254
	$\lambda = 2.4$	2.4888	0.0888	0.4429	2.3770	-0.0230	0.4358
	$\alpha = 1$	1.0359	0.0359	0.1875	0.9886	-0.0114	0.1866
1000	$\beta = 0.5$	0.5199	0.0199	0.0962	0.4983	-0.0017	0.0921
	$\lambda = 2.4$	2.4578	0.0578	0.3266	2.3942	-0.0058	0.3308

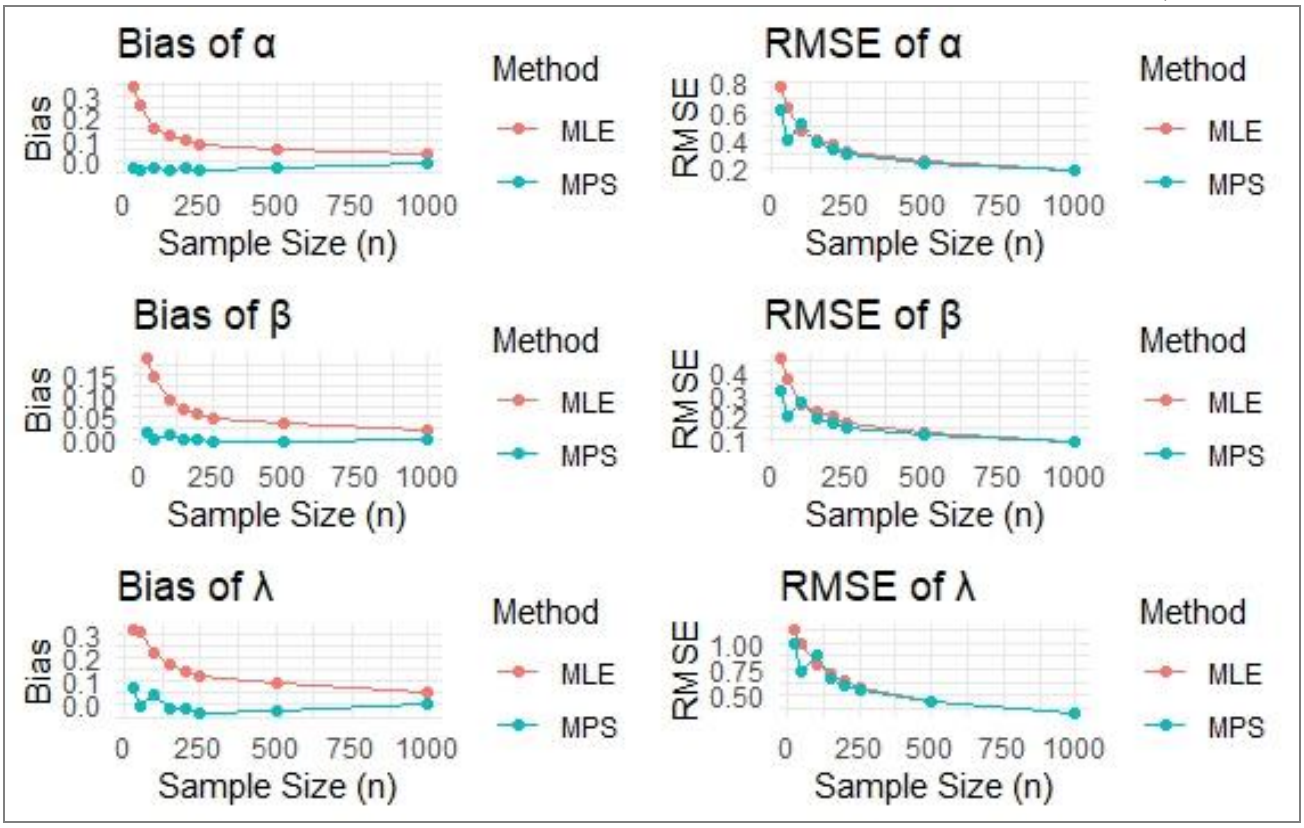

Figure 3: Bias and RMSE of MLE and MPS for each parameter estimates

Table 4: Literature Review Matrix and Comparative Findings of MLE and MPS Estimation Techniques

Study	Distribution Studied	Sample Sizes Tested	Findings on MLE vs. MPS
Bakoban & Al-Shehri 2021	A New Generalization of the Generalized Inverse Rayleigh Distribution with Applications	n = 25-200	MPS showed lower bias & RMSE; was more robust under small samples.
Gabanakgosi & Oluyede (2024)	The Topp-Leone-Gompertz-G Power Series Class of Distributions with Applications	n = 25-400	MLE provided faster convergence and better accuracy in parameter estimation.
El Fotouh *et al.* (2022)	Bayesian and Non-Bayesian Estimation of Extended Exponential Distribution under Type-I Progressive Hybrid Censoring	n = 20–150	MLE outperformed MPS in reliability and survival modelling.
Present Study (2025)	Modified Inverted Kumaraswamy (MIK)	n = 25-1000	MPS showed consistent superiority in bias and RMSE; more accurate even in large samples

The bias for the estimated parameter (λ) decreases as the sample size increases, with MPS showing a lower bias than MLE for larger sample sizes. The RMSE for the estimated parameter (λ) decreases as the sample size increases, with MPS having a lower RMSE compared to MLE, especially for smaller sample sizes. In conclusion, the MPS method generally performs better than the MLE method in terms of both bias and RMSE, particularly for smaller sample sizes. This suggests that MPS may be a more reliable method for parameter estimation in scenarios with limited data for the proposed MIK distribution.

Table 3 presents the simulation study that compares the performance of the Maximum Likelihood Estimation (MLE) and Maximum Product Spacing (MPS) methods in estimating parameters (α =1, β =0.5, λ =2.4) for various sample sizes (n). The study demonstrates that MPS is a robust and efficient estimation method compared to MLE for the given parameter values and distribution. It outperforms MLE in terms of bias and RMSE, especially for smaller sample sizes. This makes MPS an excellent choice for practical applications, particularly when data is limited or when high precision is critical.

Figure 3 shows the plots comparing the bias and RMSE (Root Mean Square Error) of the MLE (Maximum Likelihood Estimation) and MPS (Maximum Product Spacing) methods for each parameter (α, β, and λ).

The bias for the estimated parameter (α) decreases as the sample size increases for both methods. However, the MPS shows a slightly lower bias than MLE for larger sample sizes. The RMSE for the estimated parameter (α) decreases as the sample size increases, with MPS having a lower RMSE compared to MLE, especially for smaller sample sizes.

The bias for the estimated parameter (β) is relatively small and stable across different sample sizes, with MPS showing a slightly lower bias than MLE. The RMSE for the estimated parameter (β) is small and decreases slightly as the sample size increases, with MPS having a marginally lower RMSE compared to MLE.

The bias for the estimated parameter (λ) decreases as the sample size increases, with MPS showing a lower bias than MLE for larger sample sizes. The RMSE for the estimated parameter (λ) decreases as the sample size increases, with MPS having a lower RMSE compared to MLE, especially for smaller sample sizes. In conclusion, the MPS method generally performs better than the MLE method in terms of both bias and RMSE, particularly for smaller sample sizes. This suggests that MPS may be a more reliable method for parameter estimation in scenarios with limited data for the proposed MIK distribution.

COMPARING THE FINDINGS WITH EXISTING STUDIES ON MLE VS. MPS ESTIMATORS

In recent years, there has been a growing interest in exploring alternative estimation methods to the classical Maximum Likelihood Estimation (MLE), especially in the context of generalized and flexible lifetime distributions. One such method, the Maximum Product Spacing (MPS), has proven effective in several studies due to its robustness in parameter estimation, particularly for distributions with complex shapes or small sample sizes.

Bakoban & Al-Shehri 2021 demonstrated that for a new generalization of the generalized inverse Rayleigh distribution with Applications, MPS outperformed MLE in terms of reduced bias and mean squared error (MSE), especially in smaller datasets. Similarly, Gabanakgosi & Oluyede (2024) showed that MLE estimates provided better convergence and less sensitivity to the shape of the likelihood surface, a finding echoed by El Fotouh *et al.* (2022), who applied MLE in the context of beta-exponential distributions.

Consistent with these findings, the present study shows that MPS yields lower bias and RMSE across all parameter estimates of the proposed Modified Inverted Kumaraswamy (MIK) distribution with inverse power transformation. Our simulation results support the argument that MPS is a more efficient and reliable alternative to MLE for newly derived flexible distributions, confirming and extending previous conclusions in the literature.

Furthermore, from Table 4, it is important to highlight that classical lifetime distributions such as the Weibull and Burr models have been widely studied using both MLE and MPS estimation techniques. Several studies (e.g., Gabanakgosi & Oluyede 2024; Bakoban & Al-Shehri 2021) have shown that, although MLE provides consistent estimates asymptotically, it often suffers from convergence issues and large estimation variances for small sample sizes or skewed data. In contrast, MPS estimation techniques have proven to be more robust, offering smaller biases and lower RMSE values under similar conditions. Our findings for the Modified Inverted Kumaraswamy (MIK) distribution mirror these observations. While both MLE and MPS estimators perform better as the sample size increases, MPS remains superior, particularly in representing the tail behaviour and structural flexibility of the MIK distribution, similar to what has been observed in studies involving Burr-type and Weibull-related families. This further emphasizes the practical advantage of using MPS for modelling generalized lifetime data where high precision is required under limited observations.

APPLICATION TO REAL LIFE DATASETS

First Dataset

The first dataset represents 63 observations of the strengths of 1.5cm glass fibres, originally obtained by workers at the UK National Physical Laboratory. The data sets are as follows (Wani and Shafi, 2021): "0.55, 0.74, 0.77, 0.81, 0.84, 1.24, 0.93, 1.04, 1.11, 1.13, 1.30, 1.25, 1.27, 1.28, 1.29, 1.48, 1.36, 1.39, 1.42, 1.48, 1.51, 1.49, 1.49, 1.50, 1.50, 1.55, 1.52, 1.53, 1.54, 1.55, 1.61, 1.58, 1.59, 1.60, 1.61, 1.63, 1.61, 1.61, 1.62, 1.62, 1.67, 1.64, 1.66, 1.66, 1.66, 1.70, 1.68, 1.68, 1.69, 1.70, 1.78, 1.73, 1.76, 1.76, 1.77, 1.89, 1.81, 1.82, 1.84, 1.84, 2.00, 2.01, 2.24".

Table 5: The Estimates (MPSs), Log-likelihoods and Goodness of Fits Statistics of the models based on strengths of 1.5cm glass fibres (dataset 1)

Model	λ	α	β	LL	AIC	BIC
MIK	0.3782	1.7729	6.4087	-35.7952	77.5904	84.0198
TIHLIK	0.0563	0.3066	4.8367	-619.957	1245.914	1252.343
MOKEIK	142.5074	6.1266	119.996	-215.7044	437.4088	443.8382
IK	-	5.3804	84.9139	-39.1953	82.3906	86.6769

Table 5 indicates that the MIK distribution exhibits the minimum AIC and BIC values of 77.5904 and 84.0198. Therefore, among the considered distributions, the MIK distribution provides a better fit than the other models based on the strengths of the 1.5cm glass fibres dataset.

Second Dataset

The second dataset consists of Survival times (in months) of a sample of 101 patients with Advanced Acute myelogenous leukaemia. The datasets are as follows

(Yakubu and Doguwa, 2017): “0.03, 8.882, 41.118, 6.151, 17.303, 0.493, 9.145, 45.033, 6.217, 17.664, 0.855, 11.48, 46.053, 6.447, 18.092, 1.184, 11.513, 46.941, 8.651, 18.092, 1.283, 12.105, 48.289, 8.717, 18.750, 1.48, 12.796 ,57.401, 9.441, 20.625, 1.776, 12.993, 58.322, 10.329, 23.158, 2.138, 13.849, 60.625, 11.48, 27.73, 2.5, 16.612, 0.658, 12.007, 31.184, 2.763, 17.138, 0.822, 12.007, 32.434, 2.993, 20.066, 1.414, 12.237, 35.921, 3.224, 20.329, 2.5, 12.401, 42.237, 3.421, 22.368, 3.322, 13.059, 44.638, 4.178, 26.776, 3.816, 14.474, 46.48, 4.441, 28.717, 4.737, 15, 47.467, 5.691, 28.717, 4.836, 15.461, 48.322, 5.855, 32.928, 4.934, 15.757, 56.086, 6.941, 33.783, 5.033, 16.48, 6.941, 34.211, 5.757, 16.711, 7.993, 34.77, 5.855, 17.204, 8.882, 39.539, 5.987, 17.237”.

Table 6: The Estimates (MPSs), log-likelihoods, and goodness of fit statistics of the models based on the survival time of patients with leukaemia (dataset 2)

Model	λ	α	β	LL	AIC	BIC
MIK	1.8982	1.8022	9.0267	-413.1645	832.329	840.174
TIHLIK	7.8031	0.0872	8.8932	-1565.013	3136.026	3143.871
MOKEIK	6.4589	0.9722	5.9775	-448.9365	903.873	911.718
IK	-	0.7733	3.6416	-416.5064	837.0128	842.243

Table 6 displays the Maximum Product of Spacing Estimates outcomes for parameters in the MIK distribution and three comparator distributions. The MIK distribution demonstrated the lowest AIC and BIC values at 832.329 and 840.174, indicating models based on the survival time of leukaemia patients. This finding suggests that the MIK distribution is the most suitable model among the considered distributions for accurately representing the characteristics of the dataset based on the goodness of fit statistic AIC.

CONCLUSION

In this study, a new distribution was developed by modifying the inverted Kumaraswamy distribution using the inverse power function, and its parameters were estimated using Maximum Likelihood Estimation (MLE) and Maximum Product Spacing (MPS) techniques. A comprehensive simulation study was conducted across various sample sizes (n = 25, 50, 100, 150, 200, 250, 500, and 1000) to evaluate the performance of both estimation methods. The results consistently showed that MPS outperformed MLE in terms of lower bias and root mean square error (RMSE), particularly for smaller sample sizes. As the sample size increased, both methods demonstrated improved accuracy, with reduced bias and RMSE values converging toward the true parameter values. MPS produced more accurate and precise estimates even in smaller datasets and exhibited faster convergence. From a scientific and practical perspective, MPS proved to be a more reliable and efficient estimation method, making it a preferable choice in situations where precision in parameter estimation is critical. While MLE remained a viable option for larger datasets, MPS maintained a slight edge in accurately capturing the underlying distribution characteristics.

The practical utility of the proposed distribution was demonstrated using two real-life datasets: (i) survival times (in months) of 101 patients diagnosed with advanced acute myelogenous leukaemia, and (ii) strengths of 63 samples of 1.5 cm glass fibres, originally obtained by workers at the UK National Physical Laboratory (Wani and Shafi, 2021). The results highlight the proposed model's robustness, flexibility, and applicability in both reliability and survival analysis contexts.

It is recommended that future research extend the proposed model to regression-based survival frameworks and assess its performance with larger and more complex datasets. Comparative studies should be explored with other recent distributions and estimation methods, such as Bayesian techniques. Developing software tools for easy implementation is encouraged. Additionally, the model’s applicability in various fields like medicine, finance, and engineering should be further investigated.

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