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ORIGINAL RESEARCH ARTICLE

Convergence of the Fourth Order Variable Step Size Super Class of Block Backward Differentiation Formula for Solving Stiff Initial Value Problems

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ABSTRACT

In many fields of study such as science and engineering, various real life problems are created as mathematical models before they are solved. These models often lead to special class of ordinary differential equations known as stiff ODEs. A system is regarded as 'stiff'; if the existing explicit numerical methods fail to efficiently integrate it, or when the step size is determined by the requirements of its stability, rather than the accurateness. The solution of stiff ODEs contains a component with both slowly and rapidly decaying rates due to a large difference in the time scale exhibited by the system. The stiffness property prevents the conventional explicit method from handling the problem efficiently. This nature of stiff ODEs has led to considerable research efforts in developing many implicit mathematical methods. This paper discussed the convergence and order of the current variable step size super-class of block backward differentiation formula (BBDF) for solving stiff initial value problems. The necessary conditions for the convergence of the fourth order variable step size super class of BBDF for solving stiff initial value problems, has been established in this work. It has been shown that the new method is both zero-stable and consistent, which are the requirements for the convergence of any numerical method. The order of the method is also derived to be four. It is therefore concluded that the method is convergent and has significance in solving more complex stiff initial value problems, and could be robustly applied in many fields of study.

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INTRODUCTION

Many real-life problems that arise in numerous fields of study such as science and engineering; are created as mathematical models before they are solved. These models often lead to special class of ordinary differential equations known as stiff ODEs of the following form:

$$y' = f(x,y) \qquad y(a) = y_o \qquad x \in [a,b] \qquad (1)$$

The system (1) above is regarded as 'stiff'; if the existing explicit numerical methods fail to efficiently integrate it, or when the step size is determined by the requirements of its stability, rather than the accurateness (Abasi *et. al.,* 2014; Curtiss and Hirschfelder, 1952). The solution of stiff ordinary differential equations contains a component with both slowly and rapidly decaying rates due to a large difference in the time scale exhibited by the system (Suleiman *et al.,* 2013). The stiffness property prevents the conventional explicit method from handling the problem efficiently (Ahmad *et al.,* 2004; Suleiman *et al.,* 2013). This nature of stiff ODEs has led to considerable research efforts (Abasi *et. al.,* 2014; Ahmad *et al.,* 2004; Babangida *et al.,* 2016; Lambert, 1991; Suleiman *et al.,* 2013; Musa and Bala, 2019; Musa *et al.,* 2013) in developing many implicit mathematical methods. To add to the existing chain of different methods of solving stiff system equations, the step size super-class of block backward differentiation formula is explored and, from the foregone research efforts by many authors, a fourth order variable step-size is been presented in this work, such that the complexities and complications experienced in the second and third order patterns could be well smoothened and taken care of to, among other things, explore robust and wide applications in most science and engineering systems for improved and acceptable solution to research problems which was not possible using the other orders, due to certain limitations such as inability to be applied for the solution of some linear multi-step stiff complex equations.

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This paper; therefore, discussed the convergence and order of the current variable step size super-class of block backward differentiation formula for solving stiff initial value problems; as reported by Suleiman *et al.,* (2013), which can be written as:

$$y_{n+1} = -\frac{1}{4}\frac{2r\rho + r + 1}{r^2(4r^2\rho - 3r - 2)}y_{n-2} + \frac{8r^2\rho + 4r^2 + 4r\rho + 4r + 1}{r^2(4r^3\rho + 8r^2\rho - 3r^2 - 8r - 4)}y_{n-1}$$

$$+\frac{1}{4}\frac{12r^4\rho - 4r^4 + 6r^3\rho - 12r^3 - 12r^2\rho - 13r^2 - 6r\rho - 6r - 1}{r^2(4r^2\rho - 3r - 2)}y_n$$

$$+\frac{1}{4}\frac{4r^3\rho + 4r^3 + 2r^2\rho + 8r^2 + 5r + 1}{(4r^3\rho + 8r^2\rho - 3r^2 - 8r - 4)}y_{n+2} + \frac{2r^2 + 3r + 1}{(4r^2\rho - 3r - 2)}hf_{n+1} - \frac{2r^2 + 3r + 1}{(4r^2\rho - 3r - 2)}h\rho f_n$$

$$y_{n+2} = \frac{r^2\rho + 2r^2 + 3r\rho + 8r + 2\rho + 8}{r^2(2r+1)(2r^2\rho - 6r^2 + 3r\rho - 24r + \rho - 20)}y_{n-2} - \frac{4(2r\rho + 4r + \rho + 4)}{r^2(2r^2\rho - 6r^2 + 3r\rho - 24r + \rho - 20)}y_{n-1} \quad (2)$$

$$+\frac{2r^4\rho + 2r^4 + 9r^3\rho + 12r^3 + 14r^2\rho + 26r^2 + 9r\rho + 24r + 2\rho + 8}{r^2(2r^2\rho - 6r^2 + 3r\rho - 24r + \rho - 20)}y_n$$

$$+\frac{4(4r^3\rho + 3r^2\rho + 20r^2\rho + 8r\rho + 32r + 4\rho + 16)}{(2r + 1)(2r^2\rho - 6r^2 + 3r\rho - 24r + \rho - 20)}y_{n+1} - \frac{4(r^2 + 3r + 2)}{(2r^2\rho - 6r^2 + 3r\rho - 24r + \rho - 20)}hf_{n+2}$$

$$+\frac{4(r^2 + 3r + 2)}{(2r^2\rho - 6r^2 + 3r\rho - 24r + \rho - 20)}h\rho f_{n+1} - \frac{4(r^2 + 3r + 2)}{(2r^2\rho - 6r^2 + 3r\rho - 24r + \rho - 20)}hf_{n+2}$$

$$+\frac{4(r^2 + 3r + 2)}{(2r^2\rho - 6r^2 + 3r\rho - 24r + \rho - 20)}h\rho f_{n+1}$$

For stability reasons, the value ρ is restricted within the interval $(-1, 1)$ as in Suleiman *et al.,* (2013). In this work, however, $\rho = -\frac{1}{2}$ was chosen and for different value of r in equation (2), therefore the following formulae are obtained:

For $r = 1$:

$$\left.\begin{aligned} y_{n+1} &= \frac{1}{28}y_{n-2} - \frac{1}{7}y_{n-1} + \frac{9}{7}y_n - \frac{5}{28}y_{n+2} + \frac{6}{7}hf_{n+1} + \frac{3}{7}hf_n \\ y_{n+2} &= -\frac{5}{53}y_{n-2} + \frac{26}{53}y_{n-1} - \frac{54}{53}y_n + \frac{86}{53}y_{n+1} + \frac{24}{53}hf_{n+2} + \frac{12}{53}hf_{n+1} \end{aligned}\right\} \quad (3)$$

For $r = 2$:

$$\left.\begin{aligned} y_{n+1} &= \frac{1}{256}y_{n-2} - \frac{5}{256}y_{n-1} + \frac{315}{256}y_n - \frac{55}{256}y_{n+2} + \frac{15}{16}hf_{n+1} + \frac{15}{32}hf_n \\ y_{n+2} &= -\frac{13}{995}y_{n-2} + \frac{19}{199}y_{n-1} - \frac{99}{199}y_n + \frac{1408}{995}y_{n+1} + \frac{96}{199}hf_{n+2} + \frac{48}{199}hf_{n+1} \end{aligned}\right\} \quad (4)$$

For $r = \frac{5}{8}$:

$$\left.\begin{aligned} y_{n+1} &= \frac{512}{3725}y_{n-2} - \frac{12288}{26075}y_{n-1} + \frac{22113}{14900}y_n - \frac{627}{4172}y_{n+2} + \frac{117}{149}hf_{n+1} + \frac{117}{298}hf_n \\ y_{n+2} &= -\frac{63616}{188025}y_{n-2} + \frac{88064}{62675}y_{n-1} - \frac{117117}{62675}y_n + \frac{13552}{7521}y_{n+1} + \frac{1092}{2507}hf_{n+2} + \frac{546}{2507}hf_{n+1} \end{aligned}\right\} \quad (5)$$

ORDER AND ERROR CONSTANT OF THE METHOD

This section derives the order and error constant of the method corresponding to the equations in (3), (4) and (5). We first consider method (3) which can also be written in the following form:

$$\left.\begin{aligned} -\tfrac{1}{28}y_{n-2}+\tfrac{1}{7}y_{n-1}-\tfrac{9}{7}y_n+y_{n+1}+\tfrac{5}{28}y_{n+2} &= +\tfrac{6}{7}hf_{n+1}+\tfrac{3}{7}hf_n \\ \tfrac{5}{53}y_{n-2}-\tfrac{26}{53}y_{n-1}+\tfrac{54}{53}y_n-\tfrac{86}{53}y_{n+1}+y_{n+2} &= \tfrac{24}{53}hf_{n+2}+\tfrac{12}{53}hf_{n+1} \end{aligned}\right\} \quad (6)$$

The matrix associated with equation (6) is:

$$\begin{pmatrix} 0 & -\frac{1}{28} \\ 0 & \frac{5}{53} \end{pmatrix}\begin{pmatrix} y_{n-3} \\ y_{n-2} \end{pmatrix} + \begin{pmatrix} \frac{1}{7} & -\frac{9}{7} \\ -\frac{26}{53} & \frac{54}{53} \end{pmatrix}\begin{pmatrix} y_{n-1} \\ y_n \end{pmatrix} + \begin{pmatrix} 1 & \frac{5}{28} \\ -\frac{86}{53} & 1 \end{pmatrix}\begin{pmatrix} y_{n+1} \\ y_{n+2} \end{pmatrix} = h\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} f_{n-3} \\ f_{n-2} \end{pmatrix}$$
$$+ h\begin{pmatrix} 0 & \frac{3}{7} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} f_{n-1} \\ f_n \end{pmatrix} + h\begin{pmatrix} \frac{6}{7} & 0 \\ \frac{12}{53} & \frac{24}{53} \end{pmatrix}\begin{pmatrix} f_{n+1} \\ f_{n+2} \end{pmatrix} \quad (7)$$

Therefore, Let;

$$D_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, D_1 = \begin{pmatrix} -\frac{1}{28} \\ \frac{5}{53} \end{pmatrix}, D_2 = \begin{pmatrix} \frac{1}{7} \\ -\frac{26}{53} \end{pmatrix}, D_3 = \begin{pmatrix} -\frac{9}{7} \\ \frac{54}{53} \end{pmatrix}, D_4 = \begin{pmatrix} 1 \\ -\frac{86}{53} \end{pmatrix}, D_5 = \begin{pmatrix} \frac{5}{28} \\ 1 \end{pmatrix},$$

$$G_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, G_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, G_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, G_3 = \begin{pmatrix} \frac{3}{7} \\ 0 \end{pmatrix}, G_4 = \begin{pmatrix} \frac{6}{7} \\ \frac{12}{53} \end{pmatrix}, G_5 = \begin{pmatrix} 0 \\ \frac{24}{53} \end{pmatrix},$$

Definition 1: The order and associated linear difference operator of the block method (3), as given by:

$$L[y(x);h] = \sum_{j=0}^{5}\left[D_j y(x+jh) - hG_j y'(x+jh)\right] \quad (8)$$

is the unique integer '*p*'; such that the operator in (8) and the associated block method in (3); are said to be of order '*p*' if $E_0 = E_1 = E_2 = \cdots = E_P = 0$ and $E_{P+1} \neq 0$.

Where;

$$E_0 = \sum_{j=0}^{5} D_j = D_O + D_1 + D_2 + D_3 + D_4 + D_5 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{1}{28} \\ \frac{5}{53} \end{pmatrix} + \begin{pmatrix} \frac{1}{7} \\ -\frac{26}{53} \end{pmatrix} + \begin{pmatrix} -\frac{9}{7} \\ \frac{54}{53} \end{pmatrix} + \begin{pmatrix} 1 \\ -\frac{86}{53} \end{pmatrix} + \begin{pmatrix} \frac{5}{28} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$E_1 = \sum_{j=0}^{5}\left(jD_j - G_j\right) = D_1 + 2D_2 + 3D_3 + 4D_4 + 5D_5 - (G_3 + G_4 + G_5) = \begin{pmatrix} -\frac{1}{28} \\ \frac{5}{53} \end{pmatrix} + (2)\begin{pmatrix} \frac{1}{7} \\ -\frac{26}{53} \end{pmatrix} +$$
$$(3)\begin{pmatrix} -\frac{9}{7} \\ \frac{54}{53} \end{pmatrix} + (4)\begin{pmatrix} 1 \\ -\frac{86}{53} \end{pmatrix} + (5)\begin{pmatrix} \frac{5}{28} \\ 1 \end{pmatrix} - \left(\begin{pmatrix} \frac{3}{7} \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{6}{7} \\ \frac{12}{53} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{24}{53} \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$E_2 = \sum_{j=0}^{5}\left(\frac{1}{2!}j^2 D_j - \frac{1}{1!}jG_j\right) = \frac{1}{2!}(D_1 + 2^2 D_2 + 3^2 D_3 + 4^2 D_4 + 5^2 D_5) - \frac{1}{1!}(3G_3 + 4G_4 + 5G_5) =$$

$$\frac{1}{2!}\left(\begin{pmatrix}-\frac{1}{28}\\ \frac{5}{53}\end{pmatrix} + (2)^2\begin{pmatrix}\frac{1}{7}\\ -\frac{26}{63}\end{pmatrix} + (3)^2\begin{pmatrix}-\frac{9}{7}\\ \frac{54}{54}\end{pmatrix} + (4)^2\begin{pmatrix}1\\ -\frac{86}{53}\end{pmatrix} + (5)^2\begin{pmatrix}\frac{5}{28}\\ 1\end{pmatrix}\right) - \frac{1}{1!}\left((3)\begin{pmatrix}\frac{3}{7}\\ 0\end{pmatrix} + (4)\begin{pmatrix}\frac{6}{7}\\ \frac{12}{53}\end{pmatrix} + (5)\begin{pmatrix}0\\ \frac{24}{53}\end{pmatrix}\right) = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$$E_3 = \sum\left(j^3 D_j - j^2 G_j\right) = \frac{1}{3!}(D_1 + 2^3 D_2 + 3^3 D_3 + 4^3 D_4 + 5^3 D_5) - \frac{1}{2!}((3)^2 G_3 + (4)^2 G_4 + (5)^2 G_5) =$$

$$\frac{1}{3!}\left(\begin{pmatrix}-\frac{1}{28}\\ \frac{5}{53}\end{pmatrix} + (2)^3\begin{pmatrix}\frac{1}{7}\\ -\frac{26}{63}\end{pmatrix} + (3)^3\begin{pmatrix}-\frac{9}{7}\\ \frac{54}{54}\end{pmatrix} + (4)^3\begin{pmatrix}1\\ -\frac{86}{53}\end{pmatrix} + (5)^3\begin{pmatrix}\frac{5}{28}\\ 1\end{pmatrix}\right) - \frac{1}{2!}\left((3)^2\begin{pmatrix}\frac{3}{7}\\ 0\end{pmatrix} + (4)^2\begin{pmatrix}\frac{6}{7}\\ \frac{12}{53}\end{pmatrix} + (5)^2\begin{pmatrix}0\\ \frac{24}{53}\end{pmatrix}\right) = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$$E_4 = \sum_{j=0}^{5}\left(\frac{1}{4!}j^4 D_j - \frac{1}{3!}j^3 G_j\right) = \frac{1}{4!}(D_1 + 2^4 D_2 + 3^4 D_3 + 4^4 D_4 + 5^4 D_5) - \frac{1}{3!}((3)^3 G_3 + (4)^3 G_4 + (5)^3 G_5) =$$

$$\frac{1}{4!}\left(\begin{pmatrix}-\frac{1}{28}\\ \frac{5}{53}\end{pmatrix} + (2)^4\begin{pmatrix}\frac{1}{7}\\ -\frac{26}{63}\end{pmatrix} + (3)^4\begin{pmatrix}-\frac{9}{7}\\ \frac{54}{54}\end{pmatrix} + (4)^4\begin{pmatrix}1\\ -\frac{86}{53}\end{pmatrix} + (5)^4\begin{pmatrix}\frac{5}{28}\\ 1\end{pmatrix}\right) - \frac{1}{3!}\left((3)^3\begin{pmatrix}\frac{3}{7}\\ 0\end{pmatrix} + (4)^3\begin{pmatrix}\frac{6}{7}\\ \frac{12}{53}\end{pmatrix} + (5)^3\begin{pmatrix}0\\ \frac{24}{53}\end{pmatrix}\right) = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$$E_5 = \sum_{j=0}^{5}\left(\frac{1}{5!}j^5 D_j - \frac{1}{4!}j^4 G_j\right) = \frac{1}{5!}(D_1 + 2^5 D_2 + 3^5 D_5 + 4^5 D_5 + 5^5 D_5) - \frac{1}{4!}((3)^4 G_3 + (4)^4 G_4 + (5)^4 G_5) =$$

$$\frac{1}{5!}\left(\begin{pmatrix}-\frac{1}{28}\\ \frac{5}{53}\end{pmatrix} + (2)^5\begin{pmatrix}\frac{1}{7}\\ -\frac{26}{63}\end{pmatrix} + (3)^5\begin{pmatrix}-\frac{9}{7}\\ \frac{54}{54}\end{pmatrix} + (4)^5\begin{pmatrix}1\\ -\frac{86}{53}\end{pmatrix} + (5)^5\begin{pmatrix}\frac{5}{28}\\ 1\end{pmatrix}\right) - \frac{1}{4!}\left((3)^4\begin{pmatrix}\frac{3}{7}\\ 0\end{pmatrix} + (4)^4\begin{pmatrix}\frac{6}{7}\\ \frac{12}{53}\end{pmatrix} + (5)^5\begin{pmatrix}0\\ \frac{24}{53}\end{pmatrix}\right) = \begin{pmatrix}\frac{1}{35}\\ -\frac{21}{265}\end{pmatrix} \neq \begin{pmatrix}0\\ 0\end{pmatrix}$$

It could therefore be stated that method (3) above is of order four (4) and with the below error constant expression:

$$E_5 = \begin{pmatrix}\frac{1}{35}\\ -\frac{21}{265}\end{pmatrix}$$

Hence by applying a similar procedure to methods (4) and (5), it shows that the order of the two methods is 4.

Convergence of the Method

This section represents the convergence of method (3). The necessary and sufficient conditions for any linear multistep method to be convergent are that it should be consistent and zero-stable. Consistency controls the magnitude of the local truncation error; while zero stability controls the manner in which the error is propagated at each step of calculation.

Definition 2: Linear Multistep Method (LMM)

A general k-step linear multistep method is defined as;
$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\sum_{j=0}^{k} \beta_j f_{n+j}$ **(9)**

Where α_j and β_j are constants and $\alpha_k \neq 0$. Both α_0 and β_0 cannot be zero at the same time. For any k step method, α_k is normalized to one.

Theorem 1: The necessary and sufficient conditions for the linear multistep method (9) to be convergent are that it be consistent and zero-stable.

The details of the proof of the above theorem can be found in the work of Henrici, (1962). To show that the fourth 2-point variable step size super-class of block backward differentiation formula converges, we proceed to show the consistent nature of the method.

CONSISTENCY OF THE METHOD

***Definition 3*:** A linear multistep method (9); is said be consistent if it has an order of $p \geq 1$. It also follows, as previously indicated in section (2), that method (9) is said to be consistent if, and only if the following conditions are satisfied:

$$\left.\begin{array}{r}\sum_{j=o}^{k} D_j = 0 \\ \sum_{j=o}^{k} jD_j = \sum_{j=o}^{k} G_j\end{array}\right\} \quad (10)$$

Based on these definitions, from section 2, we deduced that the order of the VSSBBDF method is 4 which is greater than 1. Hence, by definition, the method is consistent.

Let $D_0, D_1, D_2, D_3, D_4, D_5$ and $G_0, G_1, G_2, G_3, G_4, G_5$ be as previously defined. Then

$$\sum_{j=0}^{5} D_j = D_O + D_1 + D_2 + D_3 + D_4 + D_5 = \begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}-\frac{1}{28}\\ \frac{5}{53}\end{pmatrix} + \begin{pmatrix}\frac{1}{7}\\ -\frac{26}{53}\end{pmatrix} + \begin{pmatrix}-\frac{9}{7}\\ \frac{54}{53}\end{pmatrix} + \begin{pmatrix}\frac{1}{86}\\ -\frac{}{53}\end{pmatrix} + \begin{pmatrix}\frac{5}{28}\\ 1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \quad (11)$$

Therefore, the first condition is satisfied in (10). We now proceed to show the second condition

$$\sum_{j=0}^{5}(jD_j) = D_1 + 2D_2 + 3D_3 + 4D_4 + 5D_5 = \begin{pmatrix}-\frac{1}{28}\\ \frac{5}{53}\end{pmatrix} + (2)\begin{pmatrix}\frac{1}{7}\\ -\frac{26}{53}\end{pmatrix} + (3)\begin{pmatrix}-\frac{9}{7}\\ \frac{54}{53}\end{pmatrix} + (4)\begin{pmatrix}\frac{1}{86}\\ -\frac{}{53}\end{pmatrix} + (5)\begin{pmatrix}\frac{5}{28}\\ 1\end{pmatrix} = \begin{pmatrix}\frac{9}{7}\\ \frac{36}{53}\end{pmatrix} \quad (12)$$

$$\sum_{j=0}^{5}(G_j) = (G_0 + G_1 + G_2 + G_3 + G_4 + G_5) = \left(\begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}\frac{3}{7}\\0\end{pmatrix} + \begin{pmatrix}\frac{6}{7}\\ \frac{12}{53}\end{pmatrix} + \begin{pmatrix}0\\ \frac{24}{53}\end{pmatrix}\right) = \begin{pmatrix}\frac{9}{7}\\ \frac{36}{53}\end{pmatrix} \quad (13)$$

Hence $\sum_{j=0}^{5} jD_j = \sum_{j=0}^{5} G_j$ which implies that the second condition in (10), remains satisfied, too.

Therefore, the method is said to be consistent since the conditions of consistency have been met. Let's proceed to show the zero-stable nature of the method.

Zero Stability of the Method

The stability polynomial of the method (3) is given by:

$$R(t,\bar{h}) = \det(At^2 - Bt - c) = \left(\left(\begin{pmatrix}\left(1-\frac{6}{7}\bar{h}\right) & \frac{5}{28}\\ \left(-\frac{86}{53}-\frac{12}{53}\bar{h}\right) & \left(1-\frac{24}{53}\bar{h}\right)\end{pmatrix}t^2 - \begin{pmatrix}0 & \frac{1}{28}\\ 0 & -\frac{5}{53}\end{pmatrix}t - \begin{pmatrix}-\frac{1}{7} & \left(\frac{9}{7}+\frac{3}{7}\bar{h}\right)\\ \frac{26}{53} & -\frac{54}{53}\end{pmatrix}\right)\right)$$

$$= \frac{957}{742}t^4 - \frac{471}{371}t^4h - \frac{621}{742}t^3 - \frac{333}{742}t^2 + \frac{144}{371}t^4h^2 - \frac{102}{53}t^3h - \frac{111}{371}t^2h - \frac{3}{742}t - \frac{36}{371}t^3h^2 \quad (14)$$

To show the zero-stable nature of method (3), we substituted $\bar{h} = 0$ in equation (14) to obtain the first characteristics polynomial as:

$$R(t,0) = \frac{957}{742}t^4 - \frac{621}{742}t^3 - \frac{333}{742}t^2 - \frac{3}{742}t \quad (15)$$

$$\Rightarrow \frac{957}{742}t^4 - \frac{621}{742}t^3 - \frac{333}{742}t^2 - \frac{3}{742}t = 0 \quad (16)$$

By solving equation (16) for '*t*', the following roots were obtained:

$t = 0,\ t = -0.3419292101,$

$t = -0.0091679685\ t = 1$

Since no magnitude of the root is greater than one and the root $(t = 1)$ remains unique, it implies that the values of $'t'$ above indicate that the method is zero-stable. Method (3) is therefore concluded to have converged, since it is both consistent and zero-stable.

CONCLUSION

The fourth order variable step size super class of block backward differentiation formula is studied in this work. It has been proven that, the method is of order four. It is also shown that the method is both consistent and zero-stable; which are the requirements for the convergence of any numerical method. We can therefore conclude that the method is convergent and of significance in resolving stiff initial value problems.

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