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ORIGINAL RESEARCH ARTICLE

A Study on Zubair Exponentiated Kumaraswamy Distribution: Properties and Application

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ABSTRACT
In this research, a new four-parameter Zubair Exponentiated Kumaraswamy (ZEKw) distribution was introduced. The density function and cumulative distribution function of the new distribution were defined. Some of its statistical properties, including moments, moment generating function, skewness, kurtosis, and order statistics, were derived. The maximum likelihood estimation method was considered for estimating the parameters of the new model. A simulation study was then carried out to investigate the consistency and efficiency of the maximum likelihood estimators. A real-life dataset was used to illustrate the usefulness and flexibility of the new model. Based on the result, the new model was found to provide a better fit than its comparators.

INTRODUCTION
The Kumaraswamy's double bound distribution is a family of continuous probability distributions defined on the interval (0,1) suitable for physical variables. The Kumaraswamy (Kw) distribution was developed by Kumaraswamy (1980). The distribution sharess some similar characteristics with the Beta distribution but is much simpler to use, especially in simulation studies. Both distributions are defined on the same support (0,1). The Kumaraswamy and Exponentiated Kumaraswamy families suffer a lack of scale parameters (Nkrumah, 2021). Effectively, they are not flexible enough for the lack of variability control (Fernando *et al.* 2017). To achieve more flexibility and applicability, it is necessary to introduce a scale parameter to the EKw distribution. The main goal of this research is to develop a new probability distribution by adding a single scale parameter to the baseline Exponentiated Kumaraswamy distribution, which can then be used to model real-world datasets with high peakedness.

To increase the flexibility in data modeling, researchers have devised multiple methods for appending a parameter(s) to an existing probability distribution. Among them are:

Exponentiated Kumaraswamy-G of Silva *et al.* (2019), Weibull Burr X of Usman *et al.* (2019), New odd Fréchet-G family of Sadiq et al. (2022) The Inverse Lomax-G of Falgore and Doguwa (2020), Odd Beta Prime-G of Sulaiman *et al.* (2023), Generalized Odd Maxwell-G of Ishaq *et al.* (2023), New generalized odd Fréchet-G (NGOF-G) family of distribution of Sadiq *et al.* (2023a), New generalized odd Fréchet-exponentiated-G family of Sadiq *et al.* (2023b), New generalized odd Fréchet-odd exponential-G family of Sadiq *et al.* (2023c), The odd Rayleigh-G family by Sadiq *et al.* (2024) among others.

Let a random variable X follow the Exponentiated Kumaraswamy distribution with parameters a,b, and γ, defined by Lemonte *et al.* (2013), then its cumulative distribution function is given as:

$$G(x;a,b,\gamma)=(1-(1-x^a)^b)^\gamma \qquad x \in (0,1) \qquad a,b,\gamma>0 \tag{1}$$

the pdf of the EKw distribution is given as:

$$g(x;a,b,\gamma)=ab\gamma x^{a-1}(1-x^a)^{b-1}(1-(1-x^a)^b)^{\gamma-1} \tag{2}$$

Several studies have employed the Exponentiated Kumaraswamy distribution to extend existing distributions. Examples include, Huang and Oluyede (2013), who developed the Exponentiated

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Kumaraswamy-Dagum (EKD) family of distribution, and Bursa and Ozel (2017) developed the Exponentiated Kumaraswamy power function distribution.

Yahaya and Mohammed (2017) developed the Transmuted Kumaraswamy Inverse Exponential distribution. Abba *et al.* (2017) presented a study on the Exponentiated Kumaraswamy Inverse Exponential distribution. Silva *et al.* (2019) introduced the Exponentiated Kumaraswamy-G family of distribution, Mohammed (2019) presented a Theoretical Analysis of the Exponentiated Transmuted Kumaraswamy distribution, Joseph and Ravindran (2023) introduced the Transmuted Exponentiated Kumaraswamy (TEKw) distribution, and Salau *et al.* (2025) also developed the Type I half logistic Exponentiated Kumaraswamy distribution. Clearly, all the aforementioned contributions to the Kumaraswamy distribution considered exponentiating the Kw distribution, thereby adding another shape parameter. Meanwhile, in this work, we employ a generator with a single scale parameter to improve on the existing distribution

Here, we employ a novel family of lifetime distributions known as the Zubair-G family developed by Zubair (2018) to come up with a new distribution. The cdf and pdf of the Zubair-G family can be defined as:

$$F(x;\alpha,\xi) = \frac{e^{\alpha G(x;\xi)^2}-1}{e^{\alpha}-1} \qquad -\infty < x < \infty \tag{3}$$

where, $\alpha > 0$ is a scale parameter and $\boldsymbol{\xi} > 0$ is a vector of parameters for any baseline distribution

the pdf of the Zubair-G family is given by:

$$f(x;\alpha,\xi) = \frac{2\alpha g(x;\xi)G(x;\xi)e^{\alpha G(x;\xi)^2}}{e^{\alpha}-1} \tag{4}$$

THE ZUBAIR EXPONENTIATED KUMARASWAMY (ZEKw) DISTRIBUTION

We define the cdf of the new ZEKw distribution by substituting (1) in (3) as:

$$F(x;\alpha,a,b,\gamma) = \frac{e^{\alpha\left(\left(1-(1-x^a)^b\right)^\gamma\right)^2}-1}{e^{\alpha}-1} \qquad 0 < x < 1 \tag{5}$$

$\alpha > 0, a > 0, b > 0\ \gamma > 0$

α is scale parameter , a, b, γ are shape parameters

differentiating (5) we have the pdf of the ZEKw distribution as:

$$f(x;\alpha,a,b,\gamma) = \frac{2\alpha ab\gamma x^{a-1}(1-x^a)^{b-1}\left(1-(1-x^a)^b\right)^{\gamma-1}\left(1-(1-x^a)^b\right)^\gamma e^{\alpha\left(\left(1-(1-x^a)^b\right)^\gamma\right)^2}}{e^{\alpha}-1} \tag{6}$$

the survival function of the ZEKw distribution is given by:

$$S(x;\alpha,a,b,\gamma) = \frac{e^{\alpha}-e^{\alpha\left(\left(1-(1-x^a)^b\right)^\gamma\right)^2}}{e^{\alpha}-1} \tag{7}$$

the hazard rate function of the ZEKw distribution is given as:

$$h(x;\alpha,a,b,\gamma) = \frac{2\alpha ab\gamma x^{a-1}(1-x^a)^{b-1}\left(1-(1-x^a)^b\right)^{\gamma-1}\left(1-(1-x^a)^b\right)^\gamma e^{\alpha\left(\left(1-(1-x^a)^b\right)^\gamma\right)^2}}{e^{\alpha}-e^{\alpha\left(\left(1-(1-x^a)^b\right)^\gamma\right)^2}} \tag{8}$$

the reverse hazard rate of the ZEKw distribution is given by:

$$rh(x;\alpha,a,b,\gamma) = \frac{2\alpha ab\gamma x^{a-1}(1-x^a)^{b-1}\left(1-(1-x^a)^b\right)^{\gamma-1}\left(1-(1-x^a)^b\right)^\gamma e^{\alpha\left(\left(1-(1-x^a)^b\right)^\gamma\right)^2}}{e^{\alpha\left(\left(1-(1-x^a)^b\right)^\gamma\right)^2}-1} \tag{9}$$

the cumulative hazard rate of the ZEKw distribution function is given as:

$$H(x;\alpha,a,b,\gamma) = -\log\left(\frac{e^{\alpha}-e^{\alpha\left(\left(1-(1-x^a)^b\right)^\gamma\right)}}{e^{\alpha}-1}\right) \tag{10}$$

Plots of the ZEKw distributions pdf, cdf, survival, and hazard rate with various parameter values are shown in Figures 1, 2, 3, and 4, respectively.

In Figure 1, it is observed that the distribution is left-skewed, and the peakedness of the distribution increases with an increase in the value of the additional scale parameter.

STATISTICAL PROPERTIES OF THE ZEKw DISTRIBUTION

In this section, we derive some necessary statistical properties, including the quantile function, moment, moment generating function, skewness, kurtosis, and distribution of order statistics.

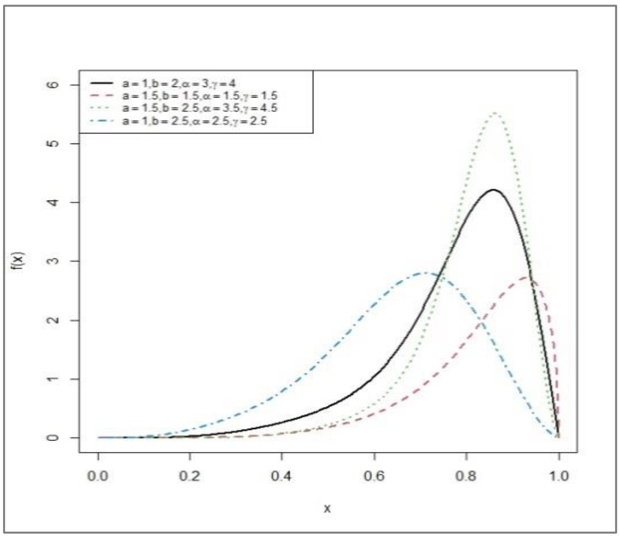

Figure 1: PDF plot of the ZEKw Distribution

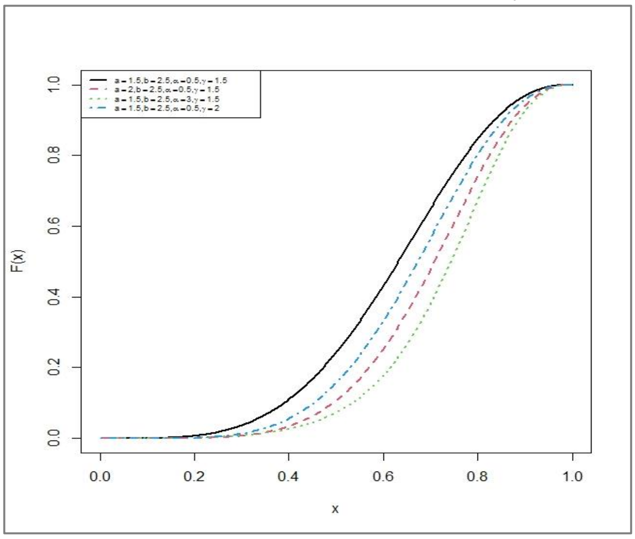

Figure 2: CDF plot of ZEKw Distribution

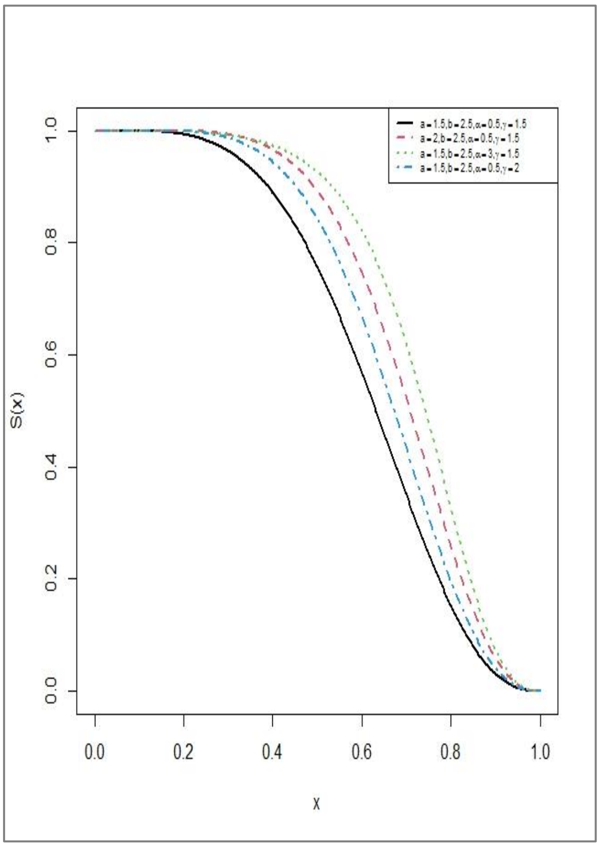

Figure 3: Survival function plot of ZEKw Distribution

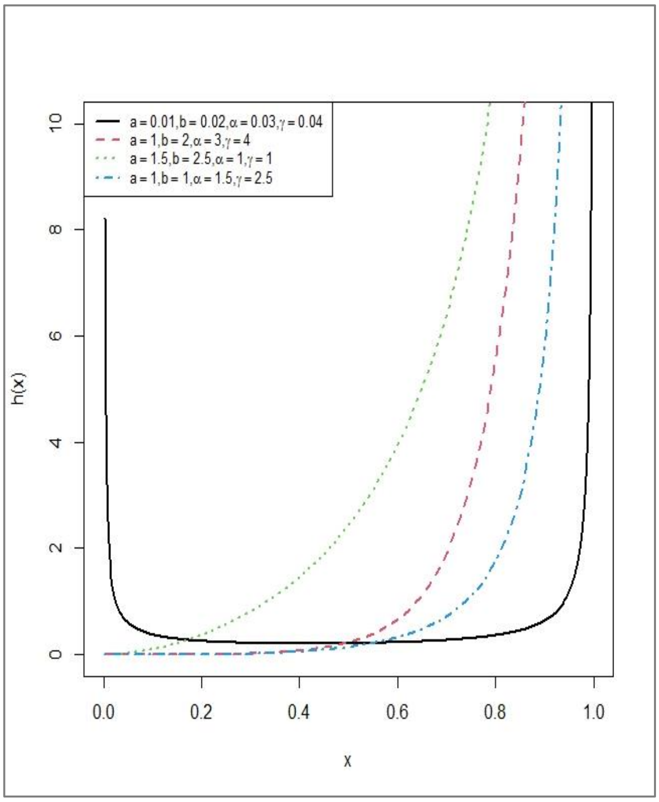

Figure 4: Hazard rate function plot of ZEKw Distribution

QUANTILE FUNCTION OF THE ZEKw DISTRIBUTION

The quantile function of X, say Q(u), is the inverse of the cdf and is given as:

$$x = F^{-1}(u) \tag{10}$$

where u is distributed uniformly throughout the [0,1] interval.

using (5) the quantile function of the ZEKw distribution can be derived as:

$$x = \left[1 - \left[1 - \left[\left[\frac{log[1+u(e^{\alpha}-1)]}{\alpha}\right]^{\frac{1}{2}}\right]^{\frac{1}{\gamma}}\right]^{\frac{1}{b}}\right]^{\frac{1}{a}} \tag{11}$$

MEDIAN OF THE ZEKw DISTRIBUTION

The median of the ZEKw distribution can be derived by substituting for u = 0.5 in equation (3.25).

$$Q(0.5) = \left[1 - \left[1 - \left[\left[\frac{\log[1+0.5(e^{\alpha}-1)]}{\alpha}\right]^{\frac{1}{2}}\right]^{\frac{1}{\gamma}}\right]^{\frac{1}{b}}\right]^{\frac{1}{a}} \quad (12)$$

MOMENT OF THE ZEKw DISTRIBUTION

Moments can be used to derive the mean, skewness, and kurtosis of a probability distribution. The r^{th} moment of a random variable X is defined as;

$$E(X^r) = \int_{-\infty}^{\infty} x^r f(x)\,dx \quad (13)$$

according to Zubair (2018), the rth moment of the Zubair-G random variable is defined as:

$$E(X^r) = 2\sum_{i=0}^{\infty} \frac{\alpha^{i+1}}{(e^{\alpha}-1)i!} \int_{-\infty}^{\infty} x^r g(x;\xi) G(x;\xi)^{2i+1}\,dx \quad (14)$$

substituting equation (1) and (2) in (14) we have:

$$E(X^r) = 2ab\gamma \sum_{i=0}^{\infty} \frac{\alpha^{i+1}}{(e^{\alpha}-1)i!} \int_0^1 x^{r+a-1}(1 - x^a)^{b-1}(1 - (1-x^a)^b)^{\gamma-1}\left((1 - (1-x^a)^b)\right)^{2i\gamma+\gamma} dx$$

$$E(X^r) = 2ab\gamma \sum_{i=0}^{\infty} \frac{\alpha^{i+1}}{(e^{\alpha}-1)i!} \int_0^1 x^{r+a-1}(1 - x^a)^{b-1}(1-(1-x^a)^b)^{2\gamma(i+1)-1}\,dx \quad (15)$$

by binomial expansion,

$$(1-z)^k = \sum_{j=0}^{\infty} (-1)^j \binom{k}{j} z^j \quad (16)$$

using (16) it follows that:

$$(1-(1-x^a)^b)^{2\gamma(i+1)-1} = \sum_{j=0}^{\infty} (-1)^j \binom{2\gamma(i+1)-1}{j} (1-x^a)^{bj} \quad (17)$$

substituting (17) into (15)

$$E(X^r) = 2ab\gamma \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^j \alpha^{i+1}}{(e^{\alpha}-1)i!} \binom{2\gamma(i+1)-1}{j} \int_0^1 x^{r+a-1}(1 - x^a)^{b(j+1)-1}\,dx$$

taking the integral part:

$$\int_0^1 x^{r+a-1}(1-x^a)^{b(j+1)-1}\,dx = B\left(\frac{r}{a}+1, b(j+1)\right) \quad (18)$$

where, $B\left(\frac{r}{a}+1, b(j+1)\right)$ is a beta function.

$$E(X^r) = 2b\gamma \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^j \alpha^{i+1}}{(e^{\alpha}-1)i!} \binom{2\gamma(i+1)-1}{j} B\left(\frac{r}{a}+1, b(j+1)\right) \quad (19)$$

equation (19) is the equation of rth moment of the ZEKw distribution

Let $2b\gamma \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^j \alpha^{i+1}}{(e^{\alpha}-1)i!} \binom{2\gamma(i+1)-1}{j} = W_{ij}$ (20)

FIRST, SECOND, THIRD, AND FOURTH MOMENTS OF THE ZEKw DISTRIBUTION

By substituting for r = 1 in (19) and substituting (20) in (19) we have the first moment as:

$$\mu_1 = W_{ij} B\left(\frac{1}{a}+1, b(j+1)\right) \quad (21)$$

The second moment for r = 2

$$\mu_2 = W_{ij} B\left(\frac{2}{a}+1, b(j+1)\right) \quad (22)$$

The third moment for r = 3

$$\mu_3 = W_{ij} B\left(\frac{3}{a}+1, b(j+1)\right) \quad (23)$$

The fourth moment for r = 4

$$\mu_4 = W_{ij} B\left(\frac{4}{a}+1, b(j+1)\right) \quad (24)$$

SKEWNESS AND KURTOSIS OF THE ZEKw DISTRIBUTION

Skewness is the measure of asymmetry of a probability distribution about it mean, mathematically defined as:

$$\beta_1 = \frac{(\mu_3)^2}{(\mu_2)^3} \quad (25)$$

the kurtosis is defined as:

$$\beta_2 = \frac{\mu_4}{(\mu_2)^2} \quad (26)$$

MOMENT GENERATING FUNCTION OF THE ZEKw DISTRIBUTION

The moment generating function of a continuous random variable *X* is defined as:

$$M_{x(t)} = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx \quad (27)$$

according to Zubair (2018) the general expression for the moment generating function (mgf) of the Zubair-G random variable X is given by:

$$M_x(t) = 2\sum_{i=0}^{\infty}\sum_{r=0}^{\infty}\frac{t^r\alpha^{i+1}}{(e^\alpha-1)r!i!}\int_{-\infty}^{\infty}x^r\, g(x;\xi)G(x;\xi)^{2i+1}\,dx \quad (28)$$

substituting (1) and (2) in (28)

$$M_x(t) = 2\sum_{i=0}^{\infty}\sum_{r=0}^{\infty}\frac{t^r\alpha^{i+1}}{(e^\alpha-1)r!i!}\int_0^1 x^r\, ab\gamma x^{a-1}((1-x^a)^{b-1})(1-(1-x^a)^b)^\gamma((1-(1-x^a)^b)^\gamma)^{2i+1}\,dx \quad (29)$$

the integral part have been previously simplified in (18) as:

$$b\gamma\sum_{j=0}^{\infty}(-1)^j\binom{2\gamma(i+1)-1}{j}B\left(\frac{r}{a}+1, b(j+1)\right) \quad (30)$$

substituting (30) in (29) the mgf of ZEKw is given as:

$$M_x(t) = 2b\gamma\sum_{i=0}^{\infty}\sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\binom{2\gamma(i+1)-1}{j}\frac{(-1)^j t^r\alpha^{i+1}}{(e^\alpha-1)r!i!}B\left(\frac{r}{a}+1, b(j+1)\right) \quad (31)$$

ORDER STATISTICS OF THE ZEKw DISTRIBUTION

Let $F_{r:k}(x)$ and $f_{r:k}(x)$ denote the CDF and PDF of the r[th] order statistics $X_{r:k}$ respectively. Then, from David H.A (1981), the density of $X_{r:k}$ for r =1, 2,…,k is given by,

$$g_{r:k}(x) = \frac{f(x)}{B(r,k-r+1)}\sum_{i,r=0}^{\infty}(-1)^i\binom{k-r}{i}(F(x))^{i+r-1} \quad (32)$$

the pdf of the Rth order statistic for the ZEKw distribution is derived by substituting equations (5) and (6) into equation (32).

$$f_{r:k}(x) = \frac{1}{B(r,k-r+1)}\left(\frac{2\alpha ab\gamma x^{a-1}(1-x^a)^{b-1}(1-(1-x^a)^b)^{\gamma-1}(1-(1-x^a)^b)^\gamma e^{\alpha((1-(1-x^a)^b)^\gamma)^2}}{e^\alpha-1}\right)\times$$
$$\sum_{i,r=0}^{\infty}(-1)^i\binom{k-r}{i}\left(\frac{e^{\alpha((1-(1-x^a)^b)^\gamma)}-1}{e^\alpha-1}\right)^{i+r-1} \quad (33)$$

by setting r = 1 and r = k in equation (33) the pdf of the minimum and maximum order statistic of the ZEKw distribution is obtained respectively as:

$$f_{1:k}(x) = \frac{1}{B(1,k)}\left(\frac{2\alpha ab\gamma x^{a-1}(1-x^a)^{b-1}(1-(1-x^a)^b)^{\gamma-1}(1-(1-x^a)^b)^\gamma e^{\alpha((1-(1-x^a)^b)^\gamma)^2}}{e^\alpha-1}\right)\times$$
$$\sum_{i=0}^{k-1}(-1)^i\binom{k-1}{r}\left(\frac{e^{\alpha((1-(1-x^a)^b)^\gamma)}-1}{e^\alpha-1}\right)^{i} \quad (34)$$

$$f_{k:k}(x) = \frac{1}{B(k,1)}\left(\frac{2\alpha ab\gamma x^{a-1}(1-x^a)^{b-1}(1-(1-x^a)^b)^{\gamma-1}(1-(1-x^a)^b)^\gamma e^{\alpha((1-(1-x^a)^b)^\gamma)^2}}{e^\alpha-1}\right)\times$$
$$\sum_{i,k=0}^{\infty}(-1)^i\left(\frac{e^{\alpha((1-(1-x^a)^b)^\gamma)}-1}{e^\alpha-1}\right)^{i+k-1} \quad (35)$$

MAXIMUM LIKELIHOOD ESTIMATION OF THE ZEKw DISTRIBUTION

Let $x_1, x_2, \dots, x_n$ be a random sample of size n from the ZEKw distribution with parametersα, a, b and γ. Using (6), the likelihood function is given by,

$$L(x_i;\alpha,a,b,\gamma) = \prod_{i=1}^{n}\left(\frac{2\alpha ab\gamma x_i{}^{a-1}(1-x_i{}^a)^{b-1}(1-(1-x_i{}^a)^b)^{\gamma-1}(1-(1-x_i{}^a)^b)^\gamma e^{\alpha\left(\left(1-(1-x_i{}^a)^b\right)^\gamma\right)}}{e^\alpha-1}\right) \quad (36)$$

taking the natural log of (36), the log-likelihood function,ℓ is given as:

$$\ell = -n\log(e^\alpha-1) + n\log 2 + n\log\alpha + n\log a + n\log b + n\log\gamma + (a-1)\sum_{i=1}^{n}\log x_i$$
$$+ (b-1)\sum_{i=1}^{n}\log(1-x^a{}_i) +$$

$$(\gamma-1)\sum_{i=1}^{n} log(1-(1-x_i{}^{a})^{b})+\gamma\sum_{i=1}^{n} log(1-(1-x_i{}^{a})^{b})+\alpha\sum_{i=1}^{n}((1-(1-x_i{}^{a})^{b})^{\gamma})^{2} \quad (37)$$

taking the partial derivative of equation (37) with respect to each parameter yield,

$$\frac{\partial\ell}{\partial\alpha}=\frac{-ne^{\alpha}}{(e^{\alpha}-1)}+\frac{n}{\alpha}+\sum_{i=1}^{n}((1-(1-x_i{}^{a})^{b})^{\gamma})^{2} \quad (38)$$

$$\frac{\partial\ell}{\partial a}=\frac{n}{a}+\sum_{i=1}^{n} log\, x_i-(b-1)\sum_{i=1}^{n}\frac{x_i{}^{a}\, log\, x_i}{(1-x_i{}^{a})}+b(\gamma-1)\sum_{i=1}^{n}\frac{(1-x_i{}^{a})^{b-1}x^{a}\, log\, x_i}{(1-(1-x_i{}^{a})^{b})}+$$
$$\gamma b\sum_{i=1}^{n}\frac{(1-x_i{}^{a})^{b-1}x_i{}^{a}\, log\, x_i}{(1-(1-x_i{}^{a})^{b})}+2\alpha b\gamma(1-(1-x_i{}^{a})^{b})^{2\gamma-1}\sum_{i=1}^{n}(1-x_i{}^{a})^{b-1}x_i{}^{a}\, log\, x_i \quad (39)$$

$$\frac{\partial\ell}{\partial b}=\frac{n}{b}+\sum_{i=1}^{n} log(1-x_i{}^{a})-2\gamma\sum_{i=1}^{n}\frac{(1-x_i{}^{a})^{b}\, log(1-x_i{}^{a})^{b}}{(1-(1-x_i{}^{a})^{b})}+\sum_{i=1}^{n}\frac{(1-x_i{}^{a})^{b}\, log(1-x_i{}^{a})^{b}}{(1-(1-x_i{}^{a})^{b})}-$$
$$2\alpha b\gamma(1-(1-x_i{}^{a})^{b})^{2\gamma-1}\sum_{i=1}^{n}(1-x_i{}^{a})^{b}\, log(1-x_i{}^{a}) \quad (40)$$

$$\frac{\partial\ell}{\partial\gamma}=\frac{n}{\gamma}+2\sum_{i=1}^{n} log(1-(1-x_i{}^{a})^{b})+2\alpha\sum_{i=1}^{n}((1-(1-x_i{}^{a})^{b})^{\gamma})^{2} \quad (41)$$

The maximum likelihood estimators of the ZEKw distribution parameters can be obtained by setting equations (38) to (41) individually to zero and solving them simultaneously. However, because the equations are nonlinear, they cannot be solved analytically; instead, they can be solved numerically using statistical software like RStudio.

SIMULATION STUDY

Here, we explore the consistency and efficiency of the maximum likelihood estimators. In this investigation, 1000 replicates were generated from the ZEKw distribution using the quantile function in equation (11). Sample sizes of n = 20, 50, 100, 150, 200, and 500 were selected. The actual values of the parameters that were selected were γ = 7.2, a = 1.5, b = 6.6, and γ = 4. From the generated replicates, parameter estimates, bias, and root mean square error (RMSE) were calculated for each of the chosen sample sizes.

From Table 1, it can be observed that the estimates are relatively good as the parameter estimates approach the true parameter values for each parameter as the sample size increases. Simultaneously, for each of the four parameters, the bias tends to zero as the sample size increases, indicating that the estimates are unbiased. Also, the RMSE approaches zero as the sample size increases. This proved the consistency of the maximum likelihood estimators obtained.

APPLICATION

This section uses a real-world dataset to illustrate the applicability of ZEKw distribution in data modelling. The performance of the ZEKw distribution is compared with that of the Kumaraswamy distributions Exponentiated Kumaraswamy, Transmuted Exponentiated Kumaraswamy, and Modified Half Logistic Exponentiated Kumaraswamy. AIC and BIC serve as the basis for the comparison. The model with the lowest values of AIC and BIC is then selected as the best performing model. We investigate a real-life dataset to highlight the effectiveness of the new model.

Table 1: Monte Carlo simulation study results of the first simulation.

n	Parameter	Estimate	Bias	RMSE
20	α	5.5670	-1.6330	1.7133
	a	12.6681	11.1681	11.1853
	b	0.8426	-5.7574	5.7735
	γ	14.0001	10.0001	10.0224
50	α	5.7729	-1.4271	1.5115
	a	12.2406	10.7406	10.7570
	b	1.1040	-5.4960	5.5184
	γ	13.6073	9.6073	9.6362
100	α	6.0272	-1.1728	1.2609
	a	11.9952	10.4952	10.5116
	b	1.3130	-5.2870	5.3092
	γ	13.2250	9.2250	9.2581
150	α	6.1178	-1.0822	1.1518
	a	11.8155	10.3155	10.3325
	b	1.4110	-5.1890	5.2112
	γ	13.0195	9.0195	9.0493
200	α	6.1354	-1.0646	1.1276
	a	11.7165	10.2165	10.2391
	b	1.5552	-5.0448	5.0685
	γ	12.8330	8.8330	8.8603
500	α	6.1447	-1.0553	1.1157
	a	11.2145	9.7145	9.7458
	b	1.7249	-4.8751	4.9022
	γ	12.4484	8.4484	8.4717

The dataset consists of 20 observations of the transformation capacity of a reservoir in California for each year from February 1991 to 2010. The data was used by Joseph and Ravindran (2023).

0.338936, 0.431915, 0.759932, 0.724626, 0.757583, 0.811556, 0.785339, 0.783660, 0.815627, 0.847413, 0.768007, 0.843485, 0.787408, 0.849868, 0.695970, 0.842316, 0.828689, 0.580194, 0.430681, 0.742000

Table 2 gives the descriptive summary of the Reservoir capacity dataset, which will be used to illustrate the usefulness of our proposed model and compare its modeling performance alongside the EKw distribution and other generalizations of the EKw distribution.

Table 2: Descriptive summary of the reservoir capacity dataset

N	Q1	Median	Mean	Q3	Min	Max
20	0.7175	0.7758	0.7214	0.8189	0.3389	0.8499

Table 3: The Maximum likelihood estimates, Log-likelihoods and Goodness-of-Fits statistics of the models based on Reservoir capacity dataset.

Model	a	b	α	γ	λ	-l	AIC	BIC
ZEKw	13.98	13.38	2.58	0.10	–	-16.84	-25.70	-21.71
EKw	9.64	6.13	–	0.54	–	-14.44	-22.88	-19.89
MHLEKw	31.42	65.89	–	0.21	–	-14.75	-23.50	-20.51
TEKw	1.63	2.37	–	3.83	-0.52	-12.61	-17.22	-13.23
Kw	6.35	4.49	–	–	–	-13.47	-22.95	-20.96

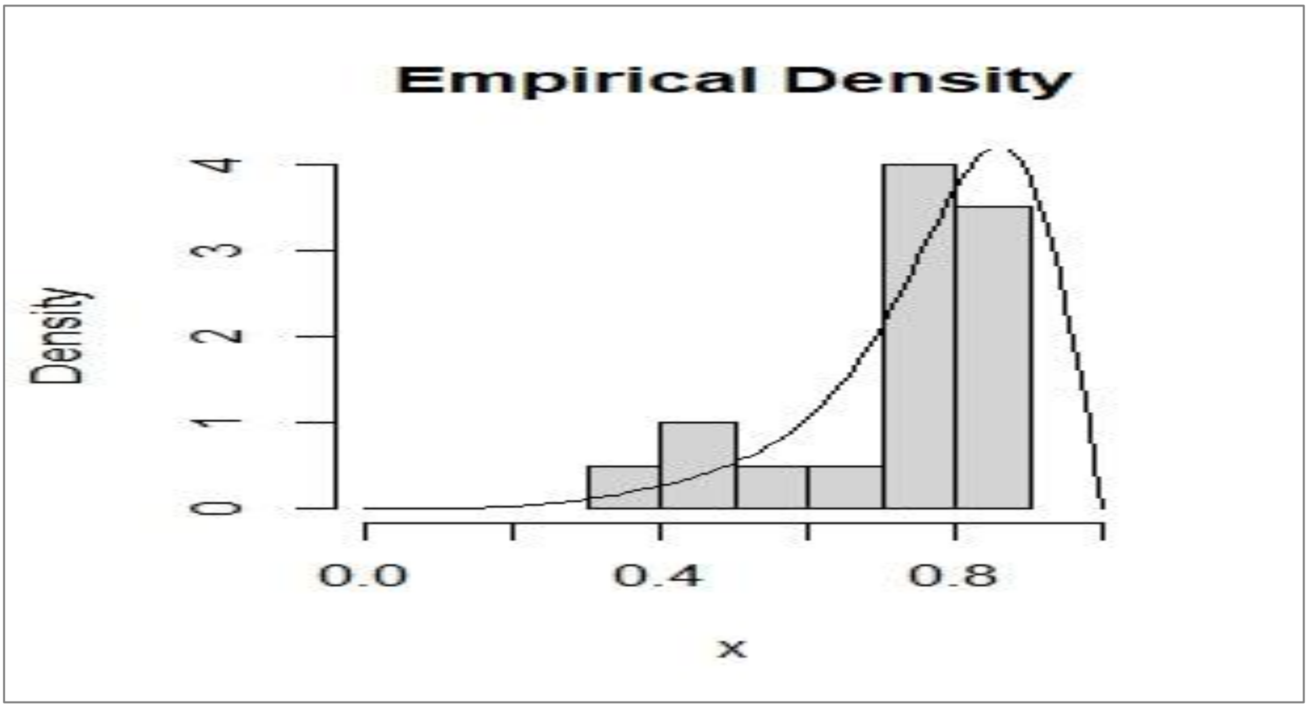

Figure 5: Histogram and density plot of the reservoir capacity dataset.

COMPETING MODELS

- Exponentiated Kumaraswamy (EKw) Distribution

According to Lemonte *et al*, (2013) pdf of the EKw distibution is given by:

$$f(x) = ab\gamma x^{a-1}(1-x^a)^{b-1}(1-(1-x^a)^{b-1})^{\gamma-1} \quad (42)$$

- Modified Half Logistic Exponentiated Kumaraswamy (MHL-EKw) Distribution

According to Xiao (2022) the pdf of the MHLEKw distribution is given by:

$$f(x) = \frac{2ab\gamma x^{a-1}(1-x^a)^{b-1}\left(1-(1-x^a)^b\right)^{-\gamma-1}}{\left(1+\left(1-(1-x^a)^b\right)^{-\gamma}\right)^2} \quad (43)$$

- Transmuted Exponentiated Kumaraswamy (TEKw) Distribution

According to Joseph and Ravindran (2023) the pdf of the T-EKw distribution is given by:

$$f(x) = ab\gamma x^{a-1}(1-x^a)^{b-1}(1-(1-x^a)^b)^{\gamma-1}((1+\lambda)-2\lambda(1-(1-x^a)^a)^\gamma) \quad (44)$$

- Kumaraswamy (Kw) Distribution

According to Kumaraswamy (1980) the pdf of the Kw distribution is given by:

$$f(x) = abx^{a-1}(1-x^a)^{b-1} \tag{45}$$

Table 3 shows the results of the MLEs, log likelihood, AIC, and BIC statistics. It is evident that the ZEKw distribution has the lowest values of AIC (-25.70) and BIC (-21.71) and therefore outperforms the baseline EKw and other competing models for modeling the Reservoir capacity dataset. By visual comparison of figure 6, it is evident that the ZEKw distribution is a better fits and shows greater superiority over its competitors in modeling in Reservoir capacity dataset.

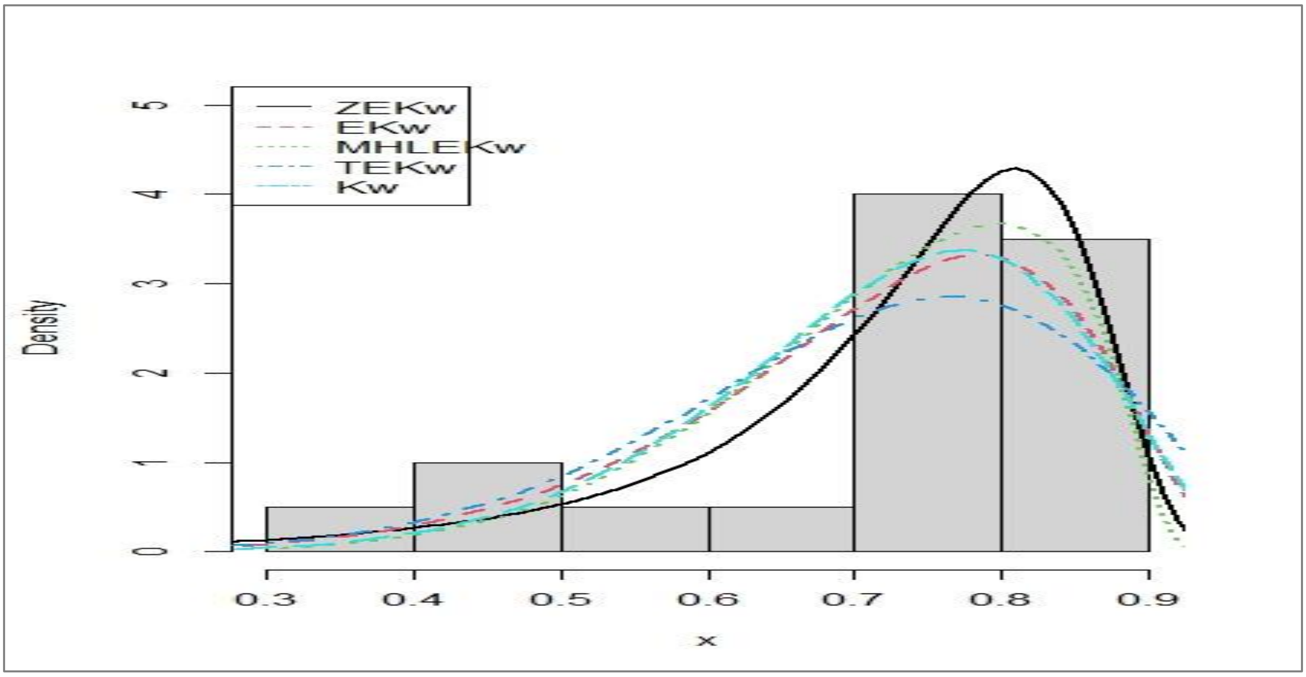

Figure 6: Histogram and estimated density plots of the ZEKw, TKw, MHLEK, EKw and Kw for the reservoir capacity dataset

CONCLUSION

This study presents the Zubair Exponentiated Kumaraswamy distribution, a novel four-parameter lifetime model. Moments, quantile function, moment generating function, and distribution of order statistics were among the statistical characteristics of the ZEKw derived. To estimate the model parameters, the maximum likelihood estimation method was considered. Additionally, a simulation was conducted, and the outcome demonstrated that the maximum likelihood estimators are consistent and efficient. Based on the graphical analysis, we can recommend that the distribution can be used to model a dataset with a negatively skewed distribution and an increasing failure rate. The new model was found to provide a better fit than the Kw distribution and several extensions of the EKw model using the Reservoir capacity dataset.

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