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ORIGINAL RESEARCH ARTICLE

An Order Inventory Model for Delayed Deteriorating Items with Two-Storage Facilities, Time-Varying Demand and Partial Backlogging Rates Under Trade-Credit Policy.

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ABSTRACT

The retailer's ideal replenishment strategy for non-instantaneous decaying goods with two-phase demand rates, two storage facilities, and shortages under a permissible payment delay has been determined in this study. While the constant demand rate is considered once deterioration has begun, the demand rate up to that point is believed to be a time-dependent quadratic function. Backlogs and shortages are also taken into consideration. Whether or not the backlog will be accepted depends on how long it will be until the next replenishment. As a result, the backlogging rate fluctuates and depends on how long it takes for the next refill. The models identify the ideal cycle length, order amount, and period at which the inventory level in the owned warehouse reaches zero in order to reduce the overall variable cost per unit of time. For the solutions to exist and be unique, both the necessary and sufficient requirements must be met. The best trade credit period is identified for each model using numerical examples, and the best model among the created models is identified using the best trade credit periods. Sensitivity analysis can offer some managerial insights.

INTRODUCTION

The foundation of many conventional inventory models is the idea of a single ware-house with infinite capacity. In most corporate setups, this assumption is, nevertheless, questionable. The retailer might buy many things at once due to stock-outs, price discounts (for bulk purchases), quantity discounts, inflation fears, demand uncertainties, and other factors. Due to their bulk, these goods might not fit in the current storage, which is known as owned ware-houses with a limited capacity. The retailer may lease a different location known as the rental ware-house. Products are transferred from the rented ware-house to the own ware-house and sold. This is because better-preserving equipment with a slower rate of deterioration will make the holding cost in the rental ware-house higher than that in the owned ware-house. Thus, it is more economical to use up the goods from rented ware-houses sooner. A two-ware-house inventory model for non-instantaneous deteriorating goods with allowable payment delays under inflationary conditions was developed by Tiwari *et al.* (2016). Because customers' patience wanes with time, shortages and partial backlogs are accepted. The model establishes the retailer's ideal replenishment procedures, optimising the optimal profit present value per unit of time. In the work of Kumar *et al.* (2017), the exponential demand rate and allowable payment delay are considered with the established two-ware-house inventory model for deteriorating goods. The shortage is not permitted, and the rate of deterioration is constant. Chandra *et al.* (2017) established ordering strategies for non-instantaneously deteriorating goods with price-dependent demand, two-storage facilities under permissible payment delays, shortages are allowed and fully backlogged, and the objective function is to maximize profit. To maximize overall profit per unit of time, Jaggi *et al.* (2017) established a two-ware-house inventory model that takes into account defective quality products, deterioration, and one level of trade credit. The model also optimizes the order quantity. Udayakumar and Geetha (2018) investigated an economic order quantity model with a constant demand rate, two-storage levels, and a permissible payment delay for non-instantaneous deteriorating items. In this model, shortages are not

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considered, and the goal is to minimize the total variable cost per unit of time. For technology products with linearly growing market sizes, Kumar and Chanda (2018) established a two-ware-house inventory model with deterioration where demand follows the innovation diffusion criterion. The approach is predicated on the idea that holding costs in a rented ware-house are higher than those in an own ware-house. According to Chakrabarty *et al.* (2018), a two-ware-house inventory model was constructed for a single item that was deteriorating under the assumptions of shortages, partial backlogs, delayed payments, the impact of inflation, and the time value of money. Similarly, in the planning for a finite horizon where backorder is envisaged, two ware-house capacities (owned and rented) have been taken into account. The system employs a two-ware-house concept when the order volume exceeds the available storage space in the own ware-house. A positive, zero, or negative ending inventory level is thus possible. An inventory model with two ware-houses, deteriorating aspect, exponentially decreasing demand rate, and limited suspension price with salvages was established by Sahoo *et al.* (2020). The model shows a rented ware-house in place of an inherent one. The intrinsic ware-house's rate of deterioration displays a linear function of time, but the rental home's rate of degradation gives a persistent function. Calculating salvage value on one's own ware-house. In a two-ware-house setting, Gupta *et al.* (2020) designed retailers' ordering procedures for time-varying deteriorating goods with partial backlogs and allowable payment delays. The model calculates the retailer's ideal ordering and backlog rules by minimising the related cost. A two-storage production inventory model with demand based on price and time was established by Datta *et al.* (2022). The selling price and the time determine the rate of demand. The rate of deterioration is constant, while the holding expense for on-site storage changes over time. By determining the most effective replenishment plan, rental storage costs can be reduced since the deterioration rate is assumed to be time-dependent while the holding cost is assumed to be constant. Babangida and Baraya (2020) devised an economic order quantity model. However, given the consumers' irritable and unpredictable character, it is impossible to know whether all customers will be willing to wait for a backorder when shortages emerge. When there are shortages, some customers who don't have immediate demands might wait for the backorders to arrive, while others might choose to purchase from alternative vendors. Due to the aforementioned, it is necessary to take into account the opportunity cost associated with lost sales when creating the model. The amount of time that must pass before the next replenishment for the majority of commodities will determine whether or not the backlog is tolerated. As a result, the backlog rate should vary and be based on when the next replenishment will arrive.

This study has considered the retailer's ideal replenishment strategy for non-instantaneous decaying goods with two-phase demand rates, two-storage facilities, and shortages under a permissible payment delay. Before the onset of deterioration, the demand rate is thought to be a time-dependent quadratic function; following this point, it is treated as a constant function until all the inventory has been consumed. Partially backlogged shortages are allowed. Backlogs are either accepted or rejected based on the length of the waiting period, hence the backlogging rate is variable and dependent on when the next replenishment will occur. This study aims to provide a mathematical model for economic order quantity that predicts the ideal time with positive inventory, the ideal cycle length, and the ideal order quantity that minimizes total variable cost per unit time and provides the optimal trade credit period. The conditions that must exist for the optimal solutions to be unique have been identified. Additionally, some numerical examples have been provided to illustrate the model's presumptive conclusions. After performing sensitivity analysis on a few model parameters to determine the best options, recommendations for reducing the overall variable cost of the inventory system have been given.

MODEL DESCRIPTION AND FORMULATION

This section provides the model notation, assumptions, and formulation.

2.1 Notations and Assumptions

2.1.1 Notations

A The ordering cost per order.

C The purchasing cost per unit per unit time ($/unit/ year).

S The selling price per unit per unit time ($/unit/ year).

C_b Shortage cost per unit per unit of time.

h_o The holding cost per unit per unit time in own ware-house ($/unit/ year).

h_r The holding cost per unit per unit time in rented ware-house ($/unit/ year).

I_c The interest charged in stock by the supplier per Dollar per year ($/unit/year).

I_e The interest earned per Dollar per year ($/unit/year) $(I_c \geq I_e)$.

M The trade credit period (in year) for settling accounts.

θ_o Constant deterioration rate in own warehouse, where $0 < \theta_o < 1$.

θ_r Constant deterioration rate in rented ware-house, where $0 < \theta_r < 1,\ \theta_r < \theta_o$

t_d The length of time in which the product exhibits no deterioration.

t_r Time at which the inventory level reaches zero in rented ware-house.

t_o Time at which the inventory level reaches zero in the owned ware-house.

T The length of the replenishment cycle time (time unit).

Q_m The maximum positive inventory level per cycle

Q_d Capacity of the owned ware-house

$(Q_m - Q_d)$ Capacity of rented ware-house

B_m The backorder level during the shortage period.

Q The order quantity during the cycle length, where $Q = (Q_m + B_m)$.

$I_O(t)$ Inventory level in the owned warehouse at any time t, where $0 \le t \le T$.

$I_r(t)$ Inventory level in the rented warehouse at any time t, where $0 \le t \le T$.

$I_s(t)$ Shortage level at any time t where $t_o \le t \le T$.

2.1.2 Assumptions

This model is established under the following assumptions.

1. The replenishment rate is instantaneous.
2. The lead time is zero.
3. A single non-instantaneous decaying item is considered.
4. The own warehouse has a fixed capacity of Q_d units; the rented ware-house has capacity of $(Q_m - Q_d)$.
5. The unit inventory holding cost per unit time in the rented ware-house is higher than that in the owned ware-house and the deterioration rate in the rented ware-house is less than that in the owned ware-house.
6. There is no replacement or repair for deteriorated goods during the period under consideration.
7. Demand before deterioration begins is a quadratic function of time t, which is more realistic because it represents both accelerated and retarded growth in demand rate of goods such as petrochemicals, aircraft, computers, seasonal products whose demand rises rapidly to a peak in the mid-season and then falls rapidly as the season wanes out and seems to be a better representation of time-varying market demand and is given by $\alpha + \beta t + \gamma t^2$, where $\alpha \ge 0, \beta \ne 0, \gamma \ne 0$.
8. Demand rate after deterioration sets in is assumed to be constant and is given by λ.
9. During the trade credit period M $(0 < M < 1)$, the account is not settled; generated sales revenue is deposited in an interest-bearing account. At the end of the period, the retailer pays off all units bought and starts to pay the capital opportunity cost for the goods in stock.
10. During the stock out phase, shortages are permitted and partially backlogged; the backlogging rate is dynamic and based on how long it takes for the next replenishment; the longer the waiting time, the smaller the backlogging rate will be. The negative inventory backlog rate is calculated as $B(t) = \frac{1}{1+\delta(T-t)}$, δ is the backlogging parameter $(0 < \delta < 1)$and $(T-t)$ is waiting time $(t_o \le t \le T)$, $1 - B(t)$ is the remaining fraction lost.

FORMULATION OF THE MODEL

The retailer's ideal replenishment strategy for non-instantaneous decaying commodities with two-phase demand rates, two-storage facilities, and shortages within a permissible payment delay has been taken into consideration in this article. Allowable payment delays encourage retailers to stock up on more goods since they boost sales, increase cash flow, lower the cost of stock holding, draw in new customers, or just retain their current ones. When the quantity exceeds the merchant's ware-house capacity, the retailer may choose to rent a ware-house to store the excess inventory. In this inventory system, Q_m units of a single product arrive at the inventory at the beginning of the cycle in which Q_d units are stored in their own ware-house and the remaining $(Q_m - Q_d)$ units in a rented ware-house. Thus, in order to find the optimal replenishment policy of the inventory system, two cases of when $t_d < t_r$ and when $t_d > t_r$ are discussed and are as follows.

3.1 Case I: when $t_d < t_r$ (Deterioration starts before the inventory level in rented ware-house becomes zero)

Figure 3.1 designates the behaviour of the inventory system. During the time interval $[0,\ t_d]$, the inventory level $I_r(t)$ in rented warehouse is depleting gradually due to market demand only and it is assumed to be a quadratic function of time t whereas in the owned ware-house inventory level remains unchanged. At time interval $[t_d,\ t_r]$ the inventory level $I_r(t)$ in the rented ware-house is depleting due to combined effects of constant market demand rate λ and deterioration while the inventory level in the owned ware-houses gets used up due to deterioration only. At time interval $[t_r,\ t_o]$, the inventory level $I_o(t)$ in the owned ware-house depletes to zero due to the combined effects of consumer demand and deterioration. Shortages occur at the time $t = t_o$ and are partially backlogged in the interval $[t_o, T]$. The whole process of the inventory system is repeated.

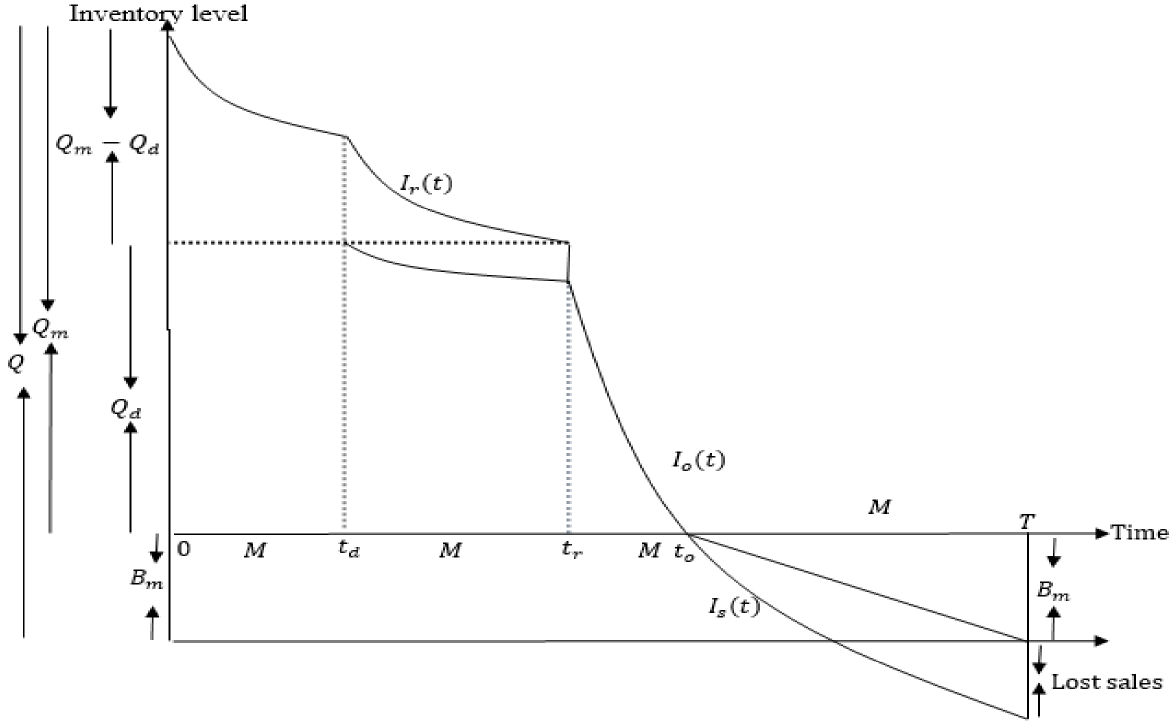

Figure 3.1: Two-ware-house inventory system when $t_d < t_r$

The differential equations that describe the inventory level in both rented ware-house and owned ware-house at any time t over the period $[0,\ T]$ are given by

$$\frac{dI_r(t)}{dt} = -(\alpha + \beta t + \gamma t^2), \qquad 0 \le t \le t_d \qquad (1)$$

$$\frac{dI_r(t)}{dt} + \theta_r I_r(t) = -\lambda, \qquad t_d \le t \le t_r \qquad (2)$$

$$\frac{dI_o(t)}{dt} + \theta_o I_o(t) = 0, \qquad t_d \le t \le t_r \qquad (3)$$

$$\frac{dI_o(t)}{dt} + \theta_o I_o(t) = -\lambda, \qquad t_r \le t \le t_o \qquad (4)$$

$$\frac{dI_s(t)}{dt} = -\frac{\lambda}{1 + \delta(T - t)}, \qquad t_o \le t \le T \qquad (5)$$

with boundary conditions $I_r(0) = Q_m - Q_d$, $I_r(t_r) = 0$, $I_o(t_d) = Q_d$, $I_o(t_o) = 0$ and $I_s(t_o) = 0$.

The solutions of equations (1), (2), (3), (4) and (5) are as follows

$$I_r(t) = Q_m - Q_d - \left(\alpha t + \beta \frac{t^2}{2} + \gamma \frac{t^3}{3}\right), \qquad 0 \le t \le t_d \qquad (6)$$

$$I_r(t) = \frac{\lambda}{\theta_r}\left(e^{\theta_r(t_r - t)} - 1\right), \qquad t_d \le t \le t_r \qquad (7)$$

$$I_o(t) = Q_d e^{\theta_o(t_d - t)}, \qquad t_d \le t \le t_r \qquad (8)$$

$$I_o(t) = \frac{\lambda}{\theta_o}\left(e^{\theta_o(t_o - t)} - 1\right), \qquad t_r \le t \le t_o \qquad (9)$$

$$I_s(t) = -\frac{\lambda}{\delta}\left[ln[1 + \delta(T - t_o)] - ln[1 + \delta(T - t)]\right], \qquad t_o \le t \le T \qquad (10)$$

Considering continuity of $I_o(t)$ at $t = t_r$, it follows from equations (8) and (9) that

$$Q_d = \frac{\lambda}{\theta_o}\left(e^{\theta_o(t_o - t_d)} - e^{\theta_o(t_r - t_d)}\right), \qquad t_o \le t \le T \qquad (11)$$

Considering continuity of $I_r(t)$ at $t = t_d$, it follows from equations (6) and (7) that

$$Q_m = \frac{\lambda}{\theta_o}\left(e^{\theta_o(t_o - t_d)} - e^{\theta_o(t_r - t_d)}\right) + \left(\alpha t_d + \beta\frac{t_d^2}{2} + \gamma\frac{t_d^3}{3}\right) + \frac{\lambda}{\theta_r}\left(e^{\theta_r(t_r - t_d)} - 1\right), \qquad t_o \le t \le T \qquad (12)$$

The maximum backordered units B_m is obtained at $t = T$, and then from equation (10), it follows that

$$B_m = -I_s(T) = \frac{\lambda}{\delta}\left[ln[1 + \delta(T - t_o)]\right] \qquad (13)$$

Consequently, the order size across the entire time period $[0,\ T]$ is

$$Q = Q_m + B_m = \frac{\lambda}{\theta_o}\left(e^{\theta_o(t_o - t_d)} - e^{\theta_o(t_r - t_d)}\right) + \left(\alpha t_d + \beta\frac{t_d^2}{2} + \gamma\frac{t_d^3}{3}\right) + \frac{\lambda}{\theta_r}\left(e^{\theta_r(t_r - t_d)} - 1\right) + \frac{\lambda}{\delta}\left[ln[1 + \delta(T - t_o)]\right] \qquad (14)$$

The total variable cost per unit time $Z(t_o, T)$ is given by

$$Z(t_o, T) = \begin{cases} Z_{11}(t_o, T), & \text{Sub} - \text{case 1.1: } 0 < M \le t_d \\ Z_{12}(t_o, T), & \text{Sub} - \text{case 1.2: } t_d < M \le t_r \\ Z_{13}(t_o, T), & \text{Sub} - \text{case 1.3: } t_r < M \le t_o \\ Z_{14}(t_o, T), & \text{Sub} - \text{case 1.4: } M > t_o \end{cases} \qquad (15)$$

where

$Z_{11}(t_o, T) = \frac{1}{T}${Ordering cost + inventory holding cost for rented ware-house+ inventory holding cost for owned ware-house + deterioration cost + backordered cost+ cost of lost sales + interest charge– interest earned}

$$\begin{aligned} &= \frac{1}{T}\Bigg\{A + h_r\left[\int_0^{t_d} I_r(t)dt + \int_{t_d}^{t_r} I_r(t)dt\right] + h_o\left[\int_0^{t_d} I_o(t)dt + \int_{t_d}^{t_r} I_o(t)dt + \int_{t_r}^{t_o} I_o(t)dt\right] \\ &\quad + C\left[\theta_r\int_{t_d}^{t_r} I_r(t)dt + \theta_o\int_{t_d}^{t_r} I_o(t)dt + \theta_o\int_{t_r}^{t_o} I_o(t)dt\right] + C_b\left[\int_{t_o}^{T} -I_s(t)dt\right] \\ &\quad + C_\pi\lambda\int_{t_o}^{T}\left(1 - \frac{\lambda}{1 + \delta(T - t)}\right)dt \\ &\quad + cI_c\left[\int_M^{t_d} I_r(t)dt + \int_{t_d}^{t_r} I_r(t)dt + \int_M^{t_d} I_o(t)dt + \int_{t_d}^{t_r} I_o(t)dt + \int_{t_r}^{t_o} I_o(t)dt\right] \\ &\quad - sI_e\left[\int_0^{M}(\alpha + \beta t + \gamma t^2)tdt\right]\Bigg\} \end{aligned}$$

$$=\frac{\lambda}{T}\left\{\frac{1}{2}W_{11}t_o^2-\mathrm{X}_{11}t_o+Y_{11}+\frac{(C_b+C_\pi\delta)}{2}T^2-(C_b+C_\pi\delta)t_oT\right\} \qquad (16)$$

but

$W_{11}=\left[h_o[\theta_o t_d+1]+C\theta_o+(C_b+C_\pi\delta)+cI_c[(t_d-M)\theta_o+1]\right], \mathrm{X}_{11}=\left[h_o\theta_o t_d^2+Ct_d\theta_o+cI_c[t_d\theta_o(t_d-M)+M]\right]$ and

$$\begin{aligned}Y_{11}=\frac{1}{\lambda}\Bigg[&A+h_r\left[\left(\alpha\frac{t_d^2}{2}+\beta\frac{t_d^3}{3}+\gamma\frac{t_d^4}{4}\right)+\frac{\lambda}{2}\{t_r^2+\theta_r(t_r-t_d)^2t_d\}\right]+h_o\left[\frac{\lambda}{2}\{\theta_o(2t_rt_d^2-t_r^2t_d)-t_r^2\}\right]\\&+C\left[\frac{\lambda}{2}\{\theta_r(t_r-t_d)^2\}+\frac{\lambda}{2}\{\theta_o(2t_rt_d-t_r^2)\}\right]\\&+cI_c\Bigg[\frac{\lambda(t_d-M)}{2}\{\theta_o(2t_rt_d-t_r^2)-2t_d+\theta_r(t_r-t_d)^2\}+\frac{\alpha}{2}(t_d-M)^2\\&+\frac{\beta}{6}(2t_d+M)(t_d-M)^2+\frac{\gamma}{12}(3t_d^2+2t_dM+M^2)(t_d-M)^2+\frac{\lambda}{2}t_d^2\Bigg]\\&-sI_e\left(\frac{\alpha}{2}M^2+\frac{\beta}{3}M^3+\frac{\gamma}{4}M^4\right)\Bigg].\end{aligned}$$

$Z_{12}(t_o,T)=\frac{1}{T}${Ordering costs plus inventory holding costs for both owned and rented ware-houses plus deterioration costs plus backordered costs plus lost sales costs plus interest charge minus interest gained}

$$\begin{aligned}=\frac{1}{T}\Bigg\{&A+h_r\left[\int_0^{t_d}I_r(t)dt+\int_{t_d}^{t_r}I_r(t)dt\right]+h_o\left[\int_0^{t_d}I_o(t)dt+\int_{t_d}^{t_r}I_o(t)dt+\int_{t_r}^{t_o}I_o(t)dt\right]\\&+C\left[\theta_r\int_{t_d}^{t_r}I_r(t)dt+\ \theta_o\int_{t_d}^{t_r}I_o(t)dt+\ \theta_o\int_{t_r}^{t_o}I_o(t)dt\right]+C_b\left[\int_{t_o}^{T}-I_s(t)dt\right]\\&+C_\pi\lambda\int_{t_o}^{T}\left(1-\frac{\lambda}{1+\delta(T-t)}\right)dt\ +cI_c\left[\int_M^{t_r}I_r(t)dt+\int_M^{t_r}I_o(t)dt+\int_{t_r}^{t_o}I_o(t)dt\right]\\&-sI_e\left[\int_0^{t_d}(\alpha+\beta t+\gamma t^2)tdt+\int_{t_d}^{M}\lambda tdt\right]\Bigg\}\end{aligned}$$

$$=\frac{\lambda}{T}\left\{\frac{1}{2}W_{12}t_o^2-\mathrm{X}_{12}t_o+Y_{12}+\frac{(C_b+C_\pi\delta)}{2}T^2-(C_b+C_\pi\delta)t_oT\right\} \qquad (17)$$

where

$W_{12}=[h_o[\theta_o t_d+1]+C\theta_o+(C_b+C_\pi\delta)+cI_c], \mathrm{X}_{12}=[h_o\theta_o t_d^2+Ct_d\theta_o+cI_cM]$ and

$$\begin{aligned}Y_{12}=\frac{1}{\lambda}\Bigg[&A+h_r\left[\left(\alpha\frac{t_d^2}{2}+\beta\frac{t_d^3}{3}+\gamma\frac{t_d^4}{4}\right)+\frac{\lambda}{2}\{t_r^2+\theta_r(t_r-t_d)^2t_d\}\right]+h_o\left[\frac{\lambda}{2}\{\theta_o(2t_rt_d^2-t_r^2t_d)-t_r^2\}\right]\\&+C\left[\frac{\lambda}{2}\{\theta_r(t_r-t_d)^2\}+\frac{\lambda}{2}\{\theta_o(2t_rt_d-t_r^2)\}\right]+cI_c\frac{\lambda}{2}M^2\\&-sI_e\left[\left(\alpha\frac{t_d^2}{2}+\beta\frac{t_d^3}{3}+\gamma\frac{t_d^4}{4}\right)+\frac{\lambda M^2}{2}-\frac{\lambda t_d^2}{2}\right]\Bigg].\end{aligned}$$

$Z_{13}(t_o,T)=\frac{1}{T}${Ordering expenses plus inventory holding expenses for rented ware-houses plus inventory holding expenses for owned ware-houses plus deterioration expenses plus backordered expenses plus the cost of lost sales plus interest charges minus interest gained}

$$= \frac{1}{T}\Bigg\{A + h_r\left[\int_0^{t_d} I_r(t)dt + \int_{t_d}^{t_r} I_r(t)dt\right] + h_o\left[\int_0^{t_d} I_o(t)dt + \int_{t_d}^{t_r} I_o(t)dt + \int_{t_r}^{t_o} I_o(t)dt\right]$$
$$+ C\left[\theta_r \int_{t_d}^{t_r} I_r(t)dt + \theta_o \int_{t_d}^{t_r} I_o(t)dt + \theta_o \int_{t_r}^{t_o} I_o(t)dt\right] + C_b\left[\int_{t_o}^{T} -I_s(t)dt\right]$$
$$+ C_\pi \lambda \int_{t_o}^{T}\left(1 - \frac{\lambda}{1+\delta(T-t)}\right)dt + cI_c\left[\int_M^{t_o} I_o(t)dt\right]$$
$$- sI_e\left[\int_0^{t_d}(\alpha + \beta t + \gamma t^2)tdt + \int_{t_d}^{t_r}\lambda t dt + \int_{t_r}^{M}\lambda t dt\right]\Bigg\}$$

$$= \frac{\lambda}{T}\left\{\frac{1}{2}W_{13}t_o^2 - X_{13}t_o + Y_{13} + \frac{(C_b + C_\pi\delta)}{2}T^2 - (C_b + C_\pi\delta)t_oT\right\} \qquad (18)$$

where

$W_{13} = [h_o[\theta_o t_d + 1] + C\theta_o + (C_b + C_\pi\delta) + cI_c], X_{13} = [h_o\theta_o t_d^2 + Ct_d\theta_o + cI_cM]$ and

$$Y_{13} = \frac{1}{\lambda}\Bigg[A + h_r\left[\left(\alpha\frac{t_d^2}{2} + \beta\frac{t_d^3}{3} + \gamma\frac{t_d^4}{4}\right) + \frac{\lambda}{2}\{t_r^2 + \theta_r(t_r - t_d)^2 t_d\}\right] + h_o\left[\frac{\lambda}{2}\{\theta_o(2t_rt_d^2 - t_r^2t_d) - t_r^2\}\right]$$
$$+ C\left[\frac{\lambda}{2}\{\theta_r(t_r - t_d)^2\} + \frac{\lambda}{2}\{\theta_o(2t_rt_d - t_r^2)\}\right] + cI_c\frac{\lambda}{2}M^2$$
$$- sI_e\left[\left(\alpha\frac{t_d^2}{2} + \beta\frac{t_d^3}{3} + \gamma\frac{t_d^4}{4}\right) + \frac{\lambda M^2}{2} - \frac{\lambda t_d^2}{2}\right]\Bigg].$$

and

$Z_{14}(t_o, T) = \frac{1}{T}${Ordering cost + inventory holding cost for rented ware-house+ inventory holding cost for owned ware-house + deterioration cost + backordered cost+ cost of lost sales – interest earned}

$$= \frac{1}{T}\Bigg\{A + h_r\left[\int_0^{t_d} I_r(t)dt + \int_{t_d}^{t_r} I_r(t)dt\right] + h_o\left[\int_0^{t_d} I_o(t)dt + \int_{t_d}^{t_r} I_o(t)dt + \int_{t_r}^{t_o} I_o(t)dt\right]$$
$$+ C\left[\theta_r \int_{t_d}^{t_r} I_r(t)dt + \theta_o \int_{t_d}^{t_r} I_o(t)dt + \theta_o \int_{t_r}^{t_o} I_o(t)dt\right] + C_b\left[\int_{t_o}^{T} -I_s(t)dt\right]$$
$$+ C_\pi \lambda \int_{t_o}^{T}\left(1 - \frac{\lambda}{1+\delta(T-t)}\right)dt$$
$$- sI_e\left[\int_0^{t_d}(\alpha + \beta t + \gamma t^2)tdt + (M - t_o)\int_0^{t_d}(\alpha + \beta t + \gamma t^2)dt + \int_{t_d}^{t_r}\lambda t dt\right.$$
$$\left. + (M - t_o)\int_{t_d}^{t_r}\lambda dt + \int_{t_r}^{t_o}\lambda t dt + (M - t_o)\int_{t_r}^{t_o}\lambda dt\right]\Bigg\}$$

$$= \frac{\lambda}{T}\left\{\frac{1}{2}W_{14}t_o^2 - X_{14}t_o + Y_{14} + \frac{(C_b + C_\pi\delta)}{2}T^2 - (C_b + C_\pi\delta)t_oT\right\} \qquad (19)$$

where

$W_{14} = [h_o[\theta_o t_d + 1] + C\theta_o + (C_b + C_\pi\delta) + sI_e], X_{14} = \left[h_o\theta_o t_d^2 + Ct_d\theta_o + sI_e\left[t_d + M - \frac{1}{\lambda}\left(\alpha t_d + \beta\frac{t_d^2}{2} + \gamma\frac{t_d^3}{3}\right)\right]\right]$ and

$$Y_{14} = \frac{1}{\lambda}\Bigg[A + h_r\left[\left(\alpha\frac{t_d^2}{2} + \beta\frac{t_d^3}{3} + \gamma\frac{t_d^4}{4}\right) + \frac{\lambda}{2}\{t_r^2 + \theta_r(t_r - t_d)^2 t_d\}\right] + h_o\left[\frac{\lambda}{2}\{\theta_o(2t_r t_d^2 - t_r^2 t_d) - t_r^2\}\right]$$
$$+ C\left[\frac{\lambda}{2}\{\theta_r(t_r - t_d)^2\} + \frac{\lambda}{2}\{\theta_o(2t_r t_d - t_r^2)\}\right]$$
$$- sI_e\left[\left(\alpha\frac{t_d^2}{2} + \beta\frac{t_d^3}{3} + \gamma\frac{t_d^4}{4}\right) + M\left(\alpha t_d + \beta\frac{t_d^2}{2} + \gamma\frac{t_d^3}{3}\right) - \frac{\lambda}{2}(2M + t_d)t_d\right]\Bigg].$$

3.1.1 Optimal Decision

The necessary and sufficient conditions are developed to determine the best ordering policies that optimizes the total variable cost per unit time. The necessary condition for the total variable cost per unit time $Z_{ij}(t_o, T)$ to be minimum are $\frac{\partial Z_{ij}(t_o,T)}{\partial t_o} = 0$ and $\frac{\partial Z_{ij}(t_o,T)}{\partial T} = 0$ for $i = 1$ when $t_r > t_d$ and $j = 1, 2, 3, 4$. The value of (t_o, T) obtained from $\frac{\partial Z_{ij}(t_o,T)}{\partial t_o} = 0$ and $\frac{\partial Z_i\, j(t_o,T)}{\partial T} = 0$ and for which the sufficient condition $\left\{\left(\frac{\partial^2 Z_{ij}(t_o,T)}{\partial t_o^2}\right)\left(\frac{\partial^2 Z_{ij}(t_o,T)}{\partial T^2}\right) - \left(\frac{\partial^2 Z_{ij}(t_o,T)}{\partial t_o\, \partial T}\right)^2\right\} > 0$ is satisfied which gives a minimum for the total variable cost per unit time $Z_{ij}(t_o, T)$.

Optimality condition for sub-case 1.1: $0 < M \leq t_d$

The necessary conditions for the total variable cost in equation (16) to be the minimum are $\frac{\partial Z_{11}(t_o,T)}{\partial t_o} = 0$ and $\frac{\partial Z_{11}(t_o,T)}{\partial T} = 0$, which give

$$\frac{\partial Z_{11}(t_o, T)}{\partial t_o} = \frac{\lambda}{T}\{W_{11}t_o - \mathrm{X}_{11} - (C_b + C_\pi\delta)T\} = 0 \tag{20}$$

and

$$T = \frac{1}{(C_b + C_\pi\delta)}(W_{11}t_o - \mathrm{X}_{11}) \tag{21}$$

Note that

$$W_{11}t_o - \mathrm{X}_{11} = \big[h_o(t_d\theta_o(t_o - t_d) + t_o) + C\theta_o(t_o - t_d) + (C_b + C_\pi\delta)t_o + cI_c\big((t_o - M) + \theta_o(t_d - M)(t_o - t_d)\big)\big] > 0$$

since $(t_d - M) \geq 0, (t_o - M), (t_o - t_d) > 0$

Similarly

$$\frac{\partial Z_{11}(t_o, T)}{\partial T} = -\frac{\lambda}{T^2}\left\{\frac{1}{2}W_{11}t_o^2 - \mathrm{X}_{11}t_o + Y_{11} - \frac{T^2}{2}(C_b + C_\pi\delta)\right\} = 0 \tag{22}$$

T from equation (21) is substituted into equation (22) which yields

$$W_{11}\big(W_{11} - (C_b + C_\pi\delta)\big)t_o^2 - 2\mathrm{X}_{11}\big(W_{11} - (C_b + C_\pi\delta)\big)t_o - \big(2(C_b + C_\pi\delta)Y_{11} - \mathrm{X}_{11}^2\big) = 0 \tag{23}$$

Let $\Delta_{11} = W_{11}\big(W_{11} - (C_b + C_\pi\delta)\big)t_d^2 - 2\mathrm{X}_{11}\big(W_{11} - (C_b + C_\pi\delta)\big)t_d - \big(2(C_b + C_\pi\delta)Y_{11} - \mathrm{X}_{11}^2\big)$, then the outcome shown below is attained.

Lemma 1.1

(i) If $\Delta_{11} \leq 0$, then the solution of $t_o \in [t_d, \infty)$ (say t_{o11}^*) which satisfies equation (23) not only exists but also is unique.

(ii) If $\Delta_{11} > 0$, then the solution of $t_o \in [t_d, \infty)$ which satisfies equation (23) does not exist.

Proof of (i): From equation (23), a new function $F_{11}(t_o)$ is defined as follows

$$F_{11}(t_o) = W_{11}\big(W_{11} - (C_b + C_\pi\delta)\big)t_o^2 - 2X_{11}\big(W_{11} - (C_b + C_\pi\delta)\big)t_o - (2(C_b + C_\pi\delta)Y_{11} - X_{11}^2), \qquad t_o \in [t_d, \infty) \qquad (24)$$

Taking the first-order derivative of $F_{11}(t_o)$ with respect to $t_o \in [t_d, \infty)$ yields

$$\frac{F_{11}(t_o)}{dt_o} = 2(W_{11}t_o - X_{11})\big(W_{11} - (C_b + C_\pi\delta)\big) > 0$$

Because $(W_{11}t_o - X_{11}) > 0$ and $\big(W_{11} - (C_b + C_\pi\delta)\big) = \big[h_o[\theta_o t_d + 1] + C\theta_o + cI_c[(t_d - M)\theta_o + 1]\big] > 0$

Hence $F_{11}(t_o)$ is a strictly increasing of t_o in the interval $[t_d, \infty)$. Moreover, $\lim_{t_o \to \infty} F_{11}(t_o) = \infty$ and $F_{11}(t_d) = \Delta_{11} \leq 0$. Therefore, by applying intermediate value theorem, there exists a unique t_o say $t_{11}^* \in [t_d, \infty)$ such that $F_{11}(t_{o11}^*) = 0$. Hence t_{o11}^* is the unique solution of equation (23). Thus, the value of t_o (denoted by t_{o11}^*) can be found from equation (23) and is given by

$$t_{o11}^* = \frac{X_{11}}{W_{11}} + \frac{1}{W_{11}}\sqrt{\frac{(2W_{11}Y_{11} - X_{11}^2)(C_b + C_\pi\delta)}{\big(W_{11} - (C_b + C_\pi\delta)\big)}} \qquad (25)$$

Once t_{o11}^* is obtained, then the value of T (denoted by T_{11}^*) can be found from equation (21) and is given by

$$T_{11}^* = \frac{1}{(C_b + C_\pi\delta)}(W_{11}t_{o11}^* - X_{11}) \qquad (26)$$

Equations (25) and (26) give the optimal values of t_{o11}^* and T_{11}^* for the cost function in equation (16) only if X_{11} satisfies the inequality given in equation (27)

$$X_{11}^2 < 2W_{11}Y_{11} \qquad (27)$$

Proof of (ii): If $\Delta_{11} > 0$, then from equation (24), $F_{11}(t_o) > 0$. Since $F_{11}(t_o)$ is a strictly increasing function of $t_o \in [t_d, \infty)$, $F_{11}(t_o) > 0$ for all $t_o \in [t_d, \infty)$. Thus, a value of $t_o \in [t_d, \infty)$ cannot be found such that $F_{11}(t_o) = 0$. This completes the proof.

Theorem 1.1

(i) If $\Delta_{11} \leq 0$, then the total variable cost $Z_{11}(t_o, T)$ is convex and reaches its global minimum at the point $(t_{o11}^*,\ T_{11}^*)$, where $(t_{o11}^*,\ T_{11}^*)$ is the point which satisfies equations (23) and (20).

(ii) If $\Delta_1 > 0$, then the total variable cost $Z_{11}(t_o, T)$ has a minimum value at the point $(t_{o11}^*,\ T_{11}^*)$ where $t_{o11}^* = t_d$ and $T_{11}^* = \frac{1}{(C_b + C_\pi\delta)}(W_{11}t_d - X_{11})$

Proof of (i): When $\Delta_{11} \leq 0$, it is observed that t_{o11}^* and T_{11}^* are the unique solutions of equations (23) and (20) from Lemma l.1(i). Taking the second derivative of $Z_{11}(t_o, T)$ with respect to t_o and T, and then finding the values of these functions at the point $(t_{o11}^*,\ T_{11}^*)$ yields

$$\left.\frac{\partial^2 Z_{11}(t_o,\ T)}{\partial t_o^2}\right|_{(t_{o11}^*,\ T_{11}^*)} = \frac{\lambda}{T_{11}^*}W_{11} > 0$$

$$\left.\frac{\partial^2 Z_{11}(t_o,\ T)}{\partial t_o \partial T}\right|_{(t_{o11}^*,\ T_{11}^*)} = -\frac{\lambda}{T_{11}^*}(C_b + C_\pi\delta)$$

$$\left.\frac{\partial^2 Z_{11}(t_o,\ T)}{\partial T^2}\right|_{(t^*_{o11},\ T^*_{11})} = \frac{\lambda}{T^*_{11}}(C_b + C_\pi \delta) > 0$$

and

$$\left(\left.\frac{\partial^2 Z_{11}(t_o,\ T)}{\partial t_o^2}\right|_{(t^*_{o11},\ T^*_{11})}\right)\left(\left.\frac{\partial^2 Z_{11}(t_o,\ T)}{\partial T^2}\right|_{(t^*_{o11},\ T^*_{11})}\right) - \left(\left.\frac{\partial^2 Z_{11}(t_o,\ T)}{\partial t_o \partial T}\right|_{(t^*_{o11},\ T^*_{11})}\right)^2$$
$$= \frac{\lambda^2(C_b + C_\pi \delta)}{T^{*2}_{11}}\left[h_o[\theta_o t_d + 1] + C\theta_o + cI_c[(t_d - M)\theta_o + 1]\right] > 0 \quad (28)$$

It is therefore concluded from equation (28) and Lemma 1.1 that $Z_{11}(t^*_{o11},\ T^*_{11})$ is convex and $(t^*_{o11},\ T^*_{11})$ is the global minimum point of $Z_{11}(t_o,\ T)$. Hence the values of t_o and T in equations (25) and (26) are optimal.

Proof of (ii): When $\Delta_{11} > 0$, then $F_{11}(t_o) > 0$ for all $t_o \in [t_d,\ \infty)$. Thus, $\frac{\partial Z_{11}(t_o,\ T)}{\partial T} = \frac{F_{11}(t_o)}{T^2} > 0$ for all $t_o \in [t_d, \infty)$ which implies $Z_{11}(t_o,\ T)$ is an increasing function of T. Thus $Z_{11}(t_o,\ T)$ has a minimum value when T is minimum. Therefore, $Z_{11}(t_o,\ T)$ has a minimum value at the point $(t^*_{o11},\ T^*_{11})$ where $t^*_{o11} = t_d$ and $T^*_{11} = \frac{1}{(C_b + C_\pi \delta)}(W_{11} t_d - \mathrm{X}_{11})$. This completes the proof.

Optimality condition for sub-case 1.2: $t_d < M \le t_r$

The necessary conditions for the total variable cost in equation (17) to be the minimum are $\frac{\partial Z_{12}(t_o,T)}{\partial t_o} = 0$ and $\frac{\partial Z_{12}(t_o,T)}{\partial T} = 0$, which give

$$\frac{\partial Z_{12}(t_o,T)}{\partial t_o} = \frac{\lambda}{T}\{W_{12} t_o - \mathrm{X}_{12} - (C_b + C_\pi \delta)T\} = 0 \quad (29)$$

and

$$T = \frac{1}{(C_b + C_\pi \delta)}(W_{12} t_o - \mathrm{X}_{12}) \quad (30)$$

Note that

$$W_{12} t_o - \mathrm{X}_{12} = \left[h_o(t_d \theta_o (t_o - t_d) + t_o) + C\theta_o(t_o - t_d) + (C_b + C_\pi \delta)t_o + cI_c\big((t_o - M)\big)\right] > 0$$

since $(t_o - M), (t_o - t_d) > 0$

Similarly

$$\frac{\partial Z_{12}(t_o,T)}{\partial T} = -\frac{\lambda}{T^2}\left\{\frac{1}{2} W_{12} t_o^2 - \mathrm{X}_{12} t_o + Y_{12} - \frac{T^2}{2}(C_b + C_\pi \delta)\right\} = 0 \quad (31)$$

Replacing T from equation (30) into equation (31) yields

$$W_{12}\big(W_{12} - (C_b + C_\pi \delta)\big)t_o^2 - 2\mathrm{X}_{12}\big(W_{12} - (C_b + C_\pi \delta)\big)t_o - (2(C_b + C_\pi \delta)Y_{12} - \mathrm{X}_{12}^2) = 0 \quad (32)$$

Let $\Delta_{12} = W_{12}\big(W_{12} - (C_b + C_\pi \delta)\big)t_r^2 - 2\mathrm{X}_{12}\big(W_{12} - (C_b + C_\pi \delta)\big)t_r - (2(C_b + C_\pi \delta)Y_{12} - \mathrm{X}_{12}^2)$, then the following result is obtained.

Lemma 1.2

(i) If $\Delta_{12} \le 0$, then the solution of $t_o \in [t_r,\ \infty)$ (say t^*_{o12}) which satisfies equation (32) not only exists but also is unique.

(ii) If $\Delta_{12} > 0$, then the solution of $t_o \in [t_r, \infty)$ which satisfies equation (32) does not exist.

Proof: The process of proof is similar to Lemma 1.1.

Thus, the value of t_o (denoted by t_{o12}^*) can be found from equation (32) and is given by

$$t_{o12}^* = \frac{X_{12}}{W_{12}} + \frac{1}{W_{12}} \sqrt{\frac{(2W_{12}Y_{12} - X_{12}^2)(C_b + C_\pi \delta)}{\left(W_{12} - (C_b + C_\pi \delta)\right)}} \tag{33}$$

Once t_{o12}^* is obtained, then the value of T (denoted by T_{12}^*) can be found from equation (30) and is given by

$$T_{12}^* = \frac{1}{(C_b + C_\pi \delta)} (W_{12} t_{o12}^* - X_{12}) \tag{34}$$

Equations (33) and (34) give the optimal values of t_{o12}^* and T_{12}^* for the cost function in equation (17) only if X_{12} satisfies the inequality given in equation (35)

$$X_{12}^2 < 2W_{12}Y_{12} \tag{35}$$

Theorem 1.2

(i) If $\Delta_{12} \leq 0$, then the total variable cost $Z_{12}(t_o, T)$ is convex and reaches its global minimum at the point (t_{o12}^*, T_{12}^*), where (t_{o12}^*, T_{12}^*) is the point which satisfies equations (32) and (29).

(ii) If $\Delta_{12} > 0$, then the total variable cost $Z_{12}(t_o, T)$ has a minimum value at the point (t_{o12}^*, T_{12}^*) where $t_{o12}^* = t_r$ and $T_{12}^* = \frac{1}{(C_b + C_\pi \delta)} (W_{12} t_r - X_{12})$

Proof. The process of proof is similar to Theorem 1.1

Optimality condition for sub-case 1.3: $t_r < M \leq t_o$

The necessary conditions for the total variable cost $Z_{13}(t_o, T)$ in equation (18) to be the minimum are $\frac{\partial Z_{13}(t_o,T)}{\partial t_o} = 0$ and $\frac{\partial Z_{13}(t_o,T)}{\partial T} = 0$, which give

$$\frac{\partial Z_{13}(t_o, T)}{\partial t_o} = \frac{\lambda}{T} \{W_{13} t_o - X_{13} - (C_b + C_\pi \delta) T\} = 0 \tag{36}$$

and

$$T = \frac{1}{(C_b + C_\pi \delta)} (W_{13} t_o - X_{13}) \tag{37}$$

Note that

$$W_{13} t_o - X_{13} = \left[h_o(t_d \theta_o (t_o - t_d) + t_o) + C\theta_o (t_o - t_d) + (C_b + C_\pi \delta) t_o + cI_c\left((t_o - M)\right)\right] > 0$$

since $(t_o - M), (t_o - t_d) > 0$

Similarly

$$\frac{\partial Z_{13}(t_o, T)}{\partial T} = -\frac{\lambda}{T^2} \left\{\frac{1}{2} W_{13} t_o^2 - X_{13} t_o + Y_{13} - \frac{T^2}{2} (C_b + C_\pi \delta)\right\} = 0 \tag{38}$$

Replacing T from equation (37) into equation (38) yields

$W_{13}(W_{13}-(C_b+C_\pi\delta))t_o^2-2X_{13}(W_{13}-(C_b+C_\pi\delta))t_o-(2(C_b+C_\pi\delta)Y_{13}-X_{13}^2)=0(39)$

Let $\Delta_{13}=W_{13}(W_{13}-(C_b+C_\pi\delta))M^2-2X_{13}(W_{13}-(C_b+C_\pi\delta))M-(2(C_b+C_\pi\delta)Y_{13}-X_{13}^2)$, then the following result is obtained.

Lemma 1.3

(i) If $\Delta_{13}\leq 0$, then the solution of $t_o\in[M,\infty)$ (say t_{o13}^*) which satisfies equation (39) does not only exists but also unique.

(ii) If $\Delta_{13}>0$, then the solution of $t_o\in[M,\infty)$ which satisfies equation (39) does not exist.

Proof: The process of proof is similar to Lemma 1.1.

Thus, the value of t_o (denoted by t_{o13}^*) can be found from equation (39) and is given by

$$t_{o13}^*=\frac{X_{13}}{W_{13}}+\frac{1}{W_{13}}\sqrt{\frac{(2W_{13}Y_{13}-X_{13}^2)(C_b+C_\pi\delta)}{(W_{13}-(C_b+C_\pi\delta))}} \tag{40}$$

Once t_{o13}^* is obtained, then the value of T (denoted by T_{13}^*) can be found from equation (37) and is given by

$$T_{13}^*=\frac{1}{(C_b+C_\pi\delta)}(W_{13}t_{o13}^*-X_{13}) \tag{41}$$

Equations (40) and (41) give the optimal values of t_{o13}^* and T_{13}^* for the cost function in equation (18) only if X_{13} satisfies the inequality given in equation (42)

$$X_{13}^2<2W_{13}Y_{13} \tag{42}$$

Theorem 1.3

(i) If $\Delta_{13}\leq 0$, then the total variable cost $Z_{13}(t_o,T)$ is convex and reaches its global minimum at the point (t_{o13}^*,T_{13}^*), where (t_{o13}^*,T_{13}^*) is the point which satisfies equations (39) and (36).

(ii) If $\Delta_{13}>0$, then the total variable cost $Z_{13}(t_o,T)$ has a minimum value at the point (t_{o13}^*,T_{13}^*) where $t_{o13}^*=M$ and $T_{13}^*=\frac{1}{(C_b+C_\pi\delta)}(W_{13}M-X_{13})$

Proof: The process of proof is similar to Theorem 1.1.

Optimality condition for sub-case 1.4: $M>t_o$

The necessary conditions for the total variable cost $Z_{14}(t_o,T)$ in equation (19) to be the minimum are $\frac{\partial Z_{14}(t_o,T)}{\partial t_o}=0$ and $\frac{\partial Z_{14}(t_o,T)}{\partial T}=0$, which give

$$\frac{\partial Z_{14}(t_o,T)}{\partial t_o}=\frac{\lambda}{T}\{W_{14}t_o-X_{14}-(C_b+C_\pi\delta)T\}=0 \tag{43}$$

and

$$T=\frac{1}{(C_b+C_\pi\delta)}(W_{14}t_o-X_{14}) \tag{44}$$

Note that

$$W_{14}t_o - X_{14} = \left[h_o(t_d\theta_o(t_o - t_d) + t_o) + C\theta_o(t_o - t_d) + (C_b + C_\pi\delta)t_o + sI_e\left[(t_o - t_d) + \left(\alpha t_d + \beta\frac{t_d^2}{2} + \gamma\frac{t_d^3}{3}\right)\frac{1}{\lambda} - M\right]\right] > 0$$

Similarly

$$\frac{\partial Z_{14}(t_o, T)}{\partial T} = -\frac{\lambda}{T^2}\left\{\frac{1}{2}W_{14}t_o^2 - X_{14}t_o + Y_{14} - \frac{T^2}{2}(C_b + C_\pi\delta)\right\} = 0 \quad (45)$$

Replacing T from equation (44) into equation (45) yields

$$W_{14}\big(W_{14} - (C_b + C_\pi\delta)\big)t_o^2 - 2X_{14}\big(W_{14} - (C_b + C_\pi\delta)\big)t_o - \big(2(C_b + C_\pi\delta)Y_{14} - X_{14}^2\big) = 0 \quad (46)$$

Let $\Delta_{14a} = W_{14}\big(W_{14} - (C_b + C_\pi\delta)\big)t_r^2 - 2X_{14}\big(W_{14} - (C_b + C_\pi\delta)\big)t_r - \big(2(C_b + C_\pi\delta)Y_{14} - X_{14}^2\big)$ and $\Delta_{14b} = W_{14}\big(W_{14} - (C_b + C_\pi\delta)\big)M^2 - 2X_{14}\big(W_{14} - (C_b + C_\pi\delta)\big)M - \big(2(C_b + C_\pi\delta)Y_{14} - X_{14}^2\big)$, then the following result is obtained.

Lemma 1.4

(i) If $\Delta_{14a} \le 0 \le \Delta_{14b}$, then the solution of $t_o \in [t_r, M]$ (say t_{o14}^*) which satisfies equation (46) does not only exists but also unique.

(ii) If $\Delta_{14b} < 0$, then the solution of $t_o \in [t_r, M]$ which satisfies equation (46) does not exist.

Proof : The process of proof is similar to Lemma 1.1.

Thus, the value of t_o (denoted by t_{o14}^*) can be found from equation (46) is given by

$$t_{o14}^* = \frac{X_{14}}{W_{14}} + \frac{1}{W_{14}}\sqrt{\frac{(2W_{14}Y_{14} - X_{14}^2)(C_b + C_\pi\delta)}{\big(W_{14} - (C_b + C_\pi\delta)\big)}} \quad (47)$$

Once t_{o14}^* is obtained, then the value of T (denoted by T_{14}^*) can be found from equation (44) and is given by

$$T_{14}^* = \frac{1}{(C_b + C_\pi\delta)}(W_{14}t_{o14}^* - X_{14}) \quad (48)$$

Equations (47) and (48) give the optimal values of t_{o14}^* and T_{14}^* for the cost function in equation (19) only if X_{14} satisfies the inequality given in equation (49)

$$X_{14}^2 < 2W_{14}Y_{14} \quad (49)$$

Theorem 1.4

(i) If $\Delta_{14a} \le 0 \le \Delta_{14b}$, then the total variable cost $Z_{14}(t_o, T)$ is convex and reaches its global minimum at the point (t_{o14}^*, T_{14}^*), where (t_{o14}^*, T_{14}^*) is the point which satisfies equations (46) and (43).

(ii) If $\Delta_{14b} < 0$, then the total variable cost $Z_{14}(t_o, T)$ has a minimum value at the point (t_{o14}^*, T_{14}^*) where $t_{o14}^* = M$ and $T_{14}^* = \frac{1}{(C_b + C_\pi\delta)}(W_{14}M - X_{14})$

(iii) If $\Delta_{14a} > 0$, then the total variable cost $Z_{14}(t_o, T)$ has a minimum value at the point (t_{o14}^*, T_{14}^*) where $t_{o14}^* = t_r$ and $T_{14}^* = \frac{1}{(C_b + C_\pi\delta)}(W_{14}t_r - X_{14})$

Proof. The process of proof is similar to Theorem 1.1

Thus, the economic order quantity (EOQ) corresponding to the optimal cycle length T^* will be computed as follows:

$EOQ^* =$The maximum inventory $+$the backordered units during the shortage period.

$$= \frac{\lambda}{\theta_o}\left(e^{\theta_o(t_o^*-t_d)} - e^{\theta_o(t_r-t_d)}\right) + \left(\alpha t_d + \beta\frac{t_d^2}{2} + \gamma\frac{t_d^3}{3}\right) + \frac{\lambda}{\theta_r}\left(e^{\theta_r(t_r-t_d)} - 1\right) + \frac{\lambda}{\delta}\left[ln[1+\delta(T^*-t_o^*)]\right] \quad (50)$$

3.1.2 Numerical Examples

This section provides some numerical examples to illustrate the model established.

Example 3.1.1 (Sub-case 1.1)

Consider an inventory system with the following input parameters: $A = \$350$/order, $C = \$45$/unit/year, $S = \$65$/unit/year, $h_o = \$5$/unit/year, $h_r = \$12$/unit/year, $\theta_o = 0.05$ units/year, $\theta_r = 0.03$ units/year, $\alpha = 980$ units, $\beta = 180$ units, $\gamma = 15$ units, $\lambda = 450$ units, $t_d = 0.1971$ year (72 days), $t_r = 0.2136$ year (78 days), $C_b = \$20$/unit/year, $C_\pi = \$5$/unit/year, $\delta = 0.8, M = 0.0684$ year (25 days), $I_c = 0.1$, $I_e = 0.08$. It is observed that $M \le t_d$, $\Delta_{11}= -58.0529 < 0$, $X_{11}^2 = 0.5878$, $2W_{11}Y_{11} = 106.7819$ and hence $X_{11}^2 < 2W_{11}Y_{11}$. Substituting the above values in equations $(25), (26), (16)$ and (50), the value of optimal time at which the inventory level reaches zero in the owned ware-house, cycle length, total variable cost and economic order quantity are respectively obtained as follows:$t_{o11}^* = 0.4311$ year (157 days), $T_{11}^* = 0.6116$ year (223 days), $Z_{11}(t_{o11}^*, T_{11}^*) = \$2175.1477$ per year and $EOQ_{11}^* = 378.4898$ units per year.

Example 3.1.2 (Sub-case 1.2)

The data are same as in Example 3.1.1 except that $M = 0.2382$ year (87 days). It is observed that $M > t_d$, $\Delta_{12}= -49.2285 < 0$, $X_{12}^2 = 2.3259$, $2W_{12}Y_{12} = 94.1807$ and hence $X_{12}^2 < 2W_{12}Y_{12}$. Substituting the above values in equations $(33), (34), (17)$ and (50), the value of optimal time at which the inventory level reaches zero in the owned ware-house, cycle length, total variable cost and economic order quantity are respectively obtained as follows: $t_{o12}^* = 0.4244$ year (155 days), $T_{12}^* = 0.5695$ year (208 days), $Z_{12}(t_{o12}^*, T_{12}^*) = \$1567.2293$ per year and $EOQ_{12}^* = 793.8139$ units per year.

Example 3.1.3 (Sub-case 1.3)

The data are same as in Example 3.1.1 except that $M = 0.2464$ year (90 days). It is observed that $M > t_r$, $\Delta_{13}= -44.0696 < 0$, $X_{13}^2 = 2.4398$, $2W_{13}Y_{13} = 94.0811$ and hence $X_{13}^2 < 2W_{13}Y_{13}$. Substituting the above values in equations $(40), (41), (18)$ and (50), the value of optimal time at which the inventory level reaches zero in the owned ware-house, cycle length, total variable cost and economic order quantity are respectively obtained as follows: $t_{o13}^* = 0.4250$ year (155 days), $T_{13}^* = 0.5689$ year (208 days), $Z_{13}(t_{o13}^*, T_{13}^*) = \$1553.7399$ per year and $EOQ_{13}^* = 661.6315$ units per year.

Example 3.1.4 (Sub-case 1.4)

The data are same as in Example 3.1.1 except that $M = 0.3559$ year (130 days). It is observed that $\Delta_{14a}= -26.4815 < 0$, $\Delta_{14b}= 6.7339 > 0$, $X_{14}^2 = 1.1149$, $2W_{14}Y_{14} = 65.0493$. Here $\Delta_{14a}\le 0 \le \Delta_{14b}$ and $X_{14}^2 < 2W_{14}Y_{14}$. Substituting the above values in equations $(47), (48), (19)$ and (50), the value of optimal time at which the inventory level reaches zero in the owned ware-house, cycle length, total variable cost and economic order quantity are respectively obtained as follows: $t_{o14}^* = 0.3325$ year (121 days), $T_{14}^* = 0.4617$ year (169 days), $Z_{14}(t_{o14}^*, T_{14}^*) = \$1394.9979$ per year and $EOQ_{14}^* = 565.3016$ units per year. It is also seen that $M > t_o$.

3.1.3 Summary of numerical examples

Table 3.1.3 Comparison of Results

Sub-cases	t_o^*(years)	T^*(years)	$Z(t_o^*, T^*)$(\$)	EOQ^*(units per year)
1.1	0.4311	0.6116	2175.1477	378.4898
1.2	0.4244	0.5695	1567.2293	793.8139
1.3	0.4250	0.5689	1553.7399	661.6315
1.4	0.3325	0.4617	1394.9979	565.3016

Sub-case 1.4 provides the optimal deterioration period and total variable cost while sub-case 1.2 provides the optimal EOQ, as such, cases 1.2 and 1.4 are the feasible cases in which an optimal case can be determined. Averagely, 3.8164 units per day can be ordered at a cost of $7.5348 when the time to settle the account is from the beginning of the deterioration period to the time at which the inventory level reaches zero in a rented ware-house (Sub-case 1.2) while 3.3450 units per day can be ordered at a cost of $8.2544 when the time to settle the account exceeds the time at which the inventory level reaches zero in the owned ware-house (Sub-case 1.4). This shows that considering the credit period to be from when deterioration starts to when the inventory level reaches zero in a rented ware-house provides both optimal total variable cost and profit which proved **sub-case 1.2** optimal among others.

3.2 Case II: when $t_d < t_r$ (Deterioration starts after the inventory level in the rented ware-house becomes zero)

Figure 3.2 designates behaviours of the inventory system. During the time interval $[0,\ t_r]$, the inventory level $I_r(t)$ in the rented ware-house is depleting gradually due to market demand only and it is assumed to be a quadratic function of time t whereas in the owned ware-house inventory level remains unchanged. At time interval $[t_r,\ t_d]$ the inventory level $I_o(t)$ in the owned ware-house is depleting due to demand from the consumers and is also assumed to be a quadratic function of time t. At time interval $[t_d,\ t_o]$, the inventory level $I_o(t)$ in the owned ware-house depletes to zero due to the combined effects of demand from the consumers and deterioration. Shortages occur at the time $t = t_o$ and are partially backlogged in the interval $[t_o, T]$. The whole process of the inventory is repeated.

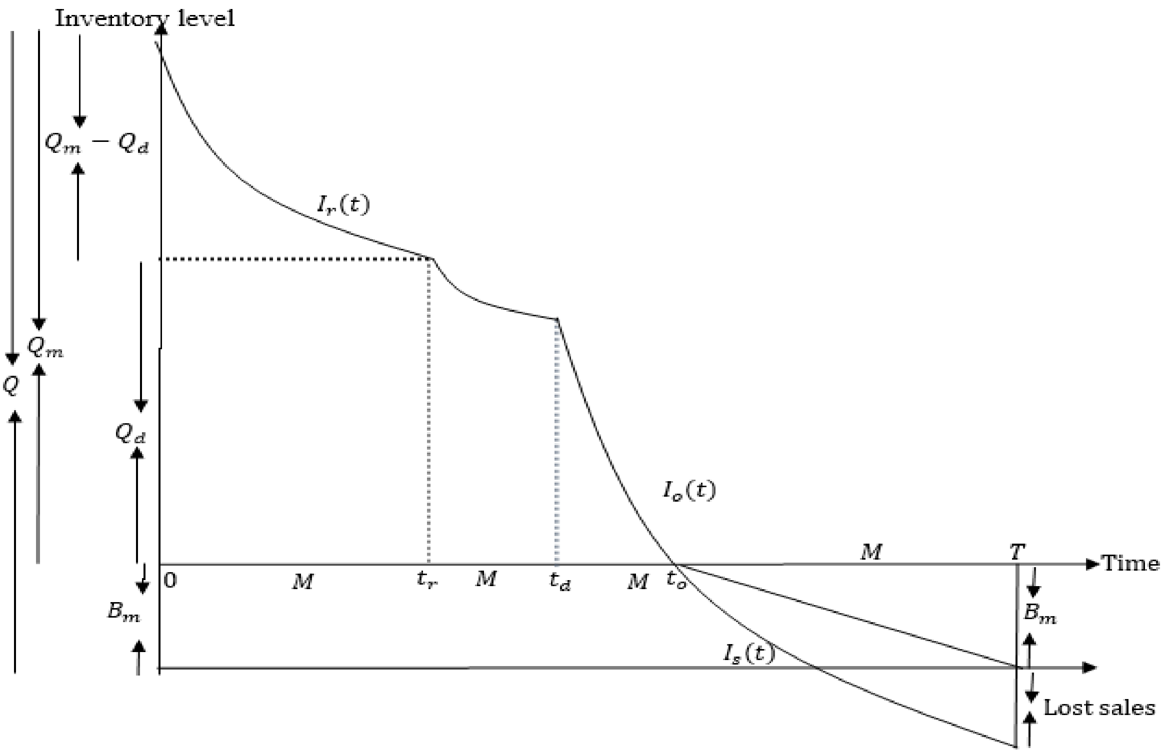

Figure 3.2: Two-ware-house inventory system when $t_d > t_r$

The differential equations that describe the inventory level in both rented ware-house and owned ware-house at any time t over the period $[0,\ T]$ are given by

$$\frac{dI_r(t)}{dt} = -(\alpha + \beta t + \gamma t^2), \qquad 0 \le t \le t_r \tag{51}$$

$$\frac{dI_o(t)}{dt} = -(\alpha + \beta t + \gamma t^2), \qquad t_r \le t \le t_d \tag{52}$$

$$\frac{dI_o(t)}{dt} + \theta_o I_o(t) = -\lambda, \qquad t_d \le t \le t_o \qquad (53)$$

$$\frac{dI_s(t)}{dt} = -\frac{\lambda}{1+\delta(T-t)}, \qquad t_o \le t \le T \qquad (54)$$

with boundary conditions $I_r(t_r) = 0, I_o(t_r) = Q_d, I_o(t_o) = 0$ and $I_s(t_o) = 0$.

The solutions of equations (51), (52), (53) and (54) are as follows

$$I_r(t) = \alpha(t_r - t) + \frac{\beta}{2}(t_r^2 - t^2) + \frac{\gamma}{3}(t_r^3 - t^3), \qquad 0 \le t \le t_r \qquad (55)$$

$$I_o(t) = Q_d + \alpha(t_r - t) + \frac{\beta}{2}(t_r^2 - t^2) + \frac{\gamma}{3}(t_r^3 - t^3), \qquad t_r \le t \le t_d \qquad (56)$$

$$I_o(t) = \frac{\lambda}{\theta_o}\left(e^{\theta_o(t_o - t)} - 1\right), \qquad t_d \le t \le t_o \qquad (57)$$

$$I_s(t) = -\frac{\lambda}{\delta}\left[ln[1+\delta(T-t_o)] - ln[1+\delta(T-t)]\right], \qquad t_o \le t \le T \qquad (58)$$

Considering continuity of $I_o(t)$ at $t = t_d$, it follows from equations (56) and (57) that

$$Q_d = \alpha(t_d - t_r) + \frac{\beta}{2}(t_d^2 - t_r^2) + \frac{\gamma}{3}(t_d^3 - t_r^3) + \frac{\lambda}{\theta_o}\left(e^{\theta_o(t_o - t_d)} - 1\right) \qquad (59)$$

Now, at $t = 0$ when $I_r(t) = Q_m - Q_d$ and solving equation (55) to get the maximum inventory level per cycle as

$$Q_m = \alpha t_d + \frac{\beta}{2}t_d^2 + \frac{\gamma}{3}t_d^3 + \frac{\lambda}{\theta_o}\left(e^{\theta_o(t_o - t_d)} - 1\right) \qquad (60)$$

The maximum backordered units B_m is obtained at $t = T$, and then from equation (58), it follows that

$$B_m = -I_s(T) = \frac{\lambda}{\delta}\left[ln[1+\delta(T-t_o)]\right] \qquad (61)$$

Thus the order size during total time interval $[0,\ T]$ is

$$Q = Z + B_m = \alpha t_d + \frac{\beta}{2}t_d^2 + \frac{\gamma}{3}t_d^3 + \frac{\lambda}{\theta_o}\left(e^{\theta_o(t_o - t_d)} - 1\right) + \frac{\lambda}{\delta}\left[ln[1+\delta(T-t_o)]\right], \qquad t_o \le t \le T \qquad (62)$$

The total variable cost per unit time $Z(t_o, T)$ is given by

$$Z(t_o, T) = \begin{cases} Z_{21}(t_o, T), & \text{Sub} - \text{case } 2.1 \quad 0 < M \le t_r \\ Z_{22}(t_o, T), & \text{Sub} - \text{case } 2.2 \quad t_r < M \le t_d \\ Z_{23}(t_o, T), & \text{Sub} - \text{case } 2.3 \quad t_d < M \le t_o \\ Z_{24}(t_o, T), & \text{Sub} - \text{case } 2.4 \quad M > t_o \end{cases} \qquad (63)$$

where

$Z_{21}(t_o, T) =$ (Ordering cost + inventory holding cost for rented ware-house+ inventory holding cost for owned ware-house + deterioration cost + backordered cost+ interest charge – interest earned)

$$
\begin{aligned}
&= \frac{1}{T}\Bigg\{A + h_r\left[\int_0^{t_r} I_r(t)dt\right] + h_o\left[\int_0^{t_r} I_o(t)dt + \int_{t_r}^{t_d} I_o(t)dt + \int_{t_d}^{t_o} I_o(t)dt\right] + C\left[\theta_o \int_{t_d}^{t_o} I_o(t)dt\right] \\
&\qquad + C_b\left[\int_{t_o}^{T} -I_s(t)dt\right] + C_\pi\lambda \int_{t_o}^{T}\left(1 - \frac{\lambda}{1+\delta(T-t)}\right)dt \\
&\qquad + cI_c\left[\int_M^{t_r} I_r(t)dt + \int_M^{t_r} I_o(t)dt + \int_{t_r}^{t_d} I_o(t)dt + \int_{t_d}^{t_o} I_o(t)dt\right] \\
&\qquad - sI_e\left[\int_0^{M}(\alpha+\beta t+\gamma t^2)tdt\right]\Bigg\}
\end{aligned}
$$

$$
= \frac{\lambda}{T}\left\{\frac{1}{2}W_{21}t_o^2 - \mathrm{X}_{21}t_1 + Y_{21} + \frac{(C_b + C_\pi\delta)}{2}T^2 - (C_b + C_\pi\delta)t_oT\right\} \tag{64}
$$

where

$W_{21} = [h_o[\theta_o t_d + 1] + C\theta_o + (C_b + C_\pi\delta) + cI_c[\theta_o(t_d - M) + 1]]$, $\mathrm{X}_{21} = \left[h_o\theta_o t_d^2 + C\theta_o t_d + cI_c[M + \theta_o t_d(t_d - M)]\right]$ and

$$
\begin{aligned}
Y_{21} = \frac{1}{\lambda}\Bigg[&A + h_r\left\{\frac{\alpha}{2}t_r^2 + \frac{\beta}{3}t_r^3 + \frac{\gamma}{4}t_r^4\right\} + h_o\left[\frac{\alpha}{2}(t_d^2 - t_r^2) + \frac{\beta}{3}(t_d^3 - t_r^3) + \frac{\gamma}{4}(t_d^4 - t_r^4) + \frac{\lambda\theta_o t_d^3}{2} - \frac{\lambda}{2}t_d^2\right] \\
&+ C\frac{\lambda}{2}\theta_o t_d^2 \\
&+ cI_c\left[\frac{\alpha}{2}(t_d - M)^2 + \frac{\beta}{6}(2t_d + M)(t_d - M)^2 + \frac{\gamma}{12}(3t_d^2 + 2t_dM + M^2)(t_d - M)^2 - \frac{\lambda}{2}t_d^2\right. \\
&\left. + \lambda M t_d + \frac{\lambda\theta_o t_d^2(t_d - M)}{2}\right] - sI_e\left(\frac{\alpha}{2}M^2 + \frac{\beta}{3}M^3 + \frac{\gamma}{4}M^4\right)\Bigg].
\end{aligned}
$$

$Z_{22}(t_o, T) =$ (Ordering cost + inventory holding cost for rented ware-house+ inventory holding cost for owned ware-house + deterioration cost + backordered cost+ interest charge – interest earned)

$$
\begin{aligned}
&= \frac{1}{T}\Bigg\{A + h_r\left[\int_0^{t_r} I_r(t)dt\right] + h_o\left[\int_0^{t_r} I_o(t)dt + \int_{t_r}^{t_d} I_o(t)dt + \int_{t_d}^{t_o} I_o(t)dt\right] + C\left[\theta_o \int_{t_d}^{t_o} I_o(t)dt\right] \\
&\qquad + C_b\left[\int_{t_o}^{T} -I_s(t)dt\right] + C_\pi\lambda \int_{t_o}^{T}\left(1 - \frac{\lambda}{1+\delta(T-t)}\right)dt + cI_c\left[\int_M^{t_d} I_o(t)dt + \int_{t_d}^{t_o} I_o(t)dt\right] \\
&\qquad - sI_e\left[\int_0^{t_r}(\alpha+\beta t+\gamma t^2)tdt + \int_{t_r}^{M}(\alpha+\beta t+\gamma t^2)tdt\right]\Bigg\}
\end{aligned}
$$

$$
= \frac{\lambda}{T}\left\{\frac{1}{2}W_{22}t_o^2 - \mathrm{X}_{22}t_o + Y_{22} + \frac{(C_b + C_\pi\delta)}{2}T^2 - (C_b + C_\pi\delta)t_oT\right\} \tag{65}
$$

where

$W_{22} = [h_o[\theta_o t_d + 1] + C\theta_o + (C_b + C_\pi\delta) + cI_c[\theta_o(t_d - M) + 1]]$, $\mathrm{X}_{22} = \left[h_o\theta_o t_d^2 + C\theta_o t_d + cI_c[M + \theta_o t_d(t_d - M)]\right]$ and

$$Y_{22} = \frac{1}{\lambda}\Bigg[A + h_r\left\{\frac{\alpha}{2}t_r^2 + \frac{\beta}{3}t_r^3 + \frac{\gamma}{4}t_r^4\right\} + h_o\left[\frac{\alpha}{2}(t_d^2 - t_r^2) + \frac{\beta}{3}(t_d^3 - t_r^3) + \frac{\gamma}{4}(t_d^4 - t_r^4) + \frac{\lambda\theta_o t_d^3}{2} - \frac{\lambda}{2}t_d^2\right] + C\frac{\lambda}{2}\theta_o t_d^2 + cI_c\left[\frac{\alpha}{2}(t_d - M)^2 + \frac{\beta}{6}(2t_d + M)(t_d - M)^2 + \frac{\gamma}{12}(3t_d^2 + 2t_d M + M^2)(t_d - M)^2 - \frac{\lambda}{2}t_d^2 + \lambda M t_d + \frac{\lambda\theta_o t_d^2(t_d - M)}{2}\right] - sI_e\left(\frac{\alpha}{2}M^2 + \frac{\beta}{3}M^3 + \frac{\gamma}{4}M^4\right)\Bigg].$$

$Z_{23}(t_o, T) =$ (Ordering cost + inventory holding cost for rented ware-house+ inventory holding cost for owned ware-house + deterioration cost + backordered cost+ interest charge – interest earned)

$$= \frac{1}{T}\Bigg\{A + h_r\left[\int_0^{t_r} I_r(t)dt\right] + h_o\left[\int_0^{t_r} I_o(t)dt + \int_{t_r}^{t_d} I_o(t)dt + \int_{t_d}^{t_o} I_o(t)dt\right] + C\left[\theta_o\int_{t_d}^{t_o} I_o(t)dt\right] + C_b\left[\int_{t_o}^{T} -I_s(t)dt\right] + C_\pi\lambda\int_{t_o}^{T}\left(1 - \frac{\lambda}{1 + \delta(T - t)}\right)dt + cI_c\left[\int_M^{t_o} I_o(t)dt\right] - sI_e\left[\int_0^{t_r}(\alpha + \beta t + \gamma t^2)tdt + \int_{t_r}^{t_d}(\alpha + \beta t + \gamma t^2)tdt + \int_{t_d}^{M}\lambda tdt\right]\Bigg\}$$

$$= \frac{\lambda}{T}\left\{\frac{1}{2}W_{23}t_o^2 - X_{23}t_o + Y_{23} + \frac{(C_b + C_\pi\delta)}{2}T^2 - (C_b + C_\pi\delta)t_o T\right\} \qquad (66)$$

where

$W_{23} = [h_o[\theta_o t_d + 1] + C\theta_o + (C_b + C_\pi\delta) + cI_c]$, $X_{23} = [h_o\theta_o t_d^2 + C\theta_o t_d + cI_c M]$ and

$$Y_{23} = \frac{1}{\lambda}\Bigg[A + h_r\left\{\frac{\alpha}{2}t_r^2 + \frac{\beta}{3}t_r^3 + \frac{\gamma}{4}t_r^4\right\} + h_o\left[\frac{\alpha}{2}(t_d^2 - t_r^2) + \frac{\beta}{3}(t_d^3 - t_r^3) + \frac{\gamma}{4}(t_d^4 - t_r^4) + \frac{\lambda\theta_o t_d^3}{2} - \frac{\lambda}{2}t_d^2\right] + C\frac{\lambda}{2}\theta_o t_d^2 + cI_c\frac{\lambda}{2}M^2 - sI_e\left[\left(\alpha\frac{t_d^2}{2} + \beta\frac{t_d^3}{3} + \gamma\frac{t_d^4}{4}\right) + \frac{\lambda M^2}{2} - \frac{\lambda t_d^2}{2}\right]\Bigg].$$

and

$Z_{24}(t_o, T) =$ (Ordering cost + inventory holding cost for rented ware-house+ inventory holding cost for owned ware-house + deterioration cost + backordered cost – interest earned)

$$= \frac{1}{T}\Bigg\{A + h_r\left[\int_0^{t_r} I_r(t)dt\right] + h_o\left[\int_0^{t_r} I_o(t)dt + \int_{t_r}^{t_d} I_o(t)dt + \int_{t_d}^{t_o} I_o(t)dt\right] + C\left[\theta_o\int_{t_d}^{t_o} I_o(t)dt\right] + C_b\left[\int_{t_o}^{T} -I_s(t)dt\right] + C_\pi\lambda\int_{t_o}^{T}\left(1 - \frac{\lambda}{1 + \delta(T - t)}\right)dt - sI_e\Bigg[\int_0^{t_r}(\alpha + \beta t + \gamma t^2)tdt + (M - t_o)\int_0^{t_r}(\alpha + \beta t + \gamma t^2)dt + \int_{t_r}^{t_d}(\alpha + \beta t + \gamma t^2)tdt + (M - t_o)\int_{t_r}^{t_d}(\alpha + \beta t + \gamma t^2)dt + \int_{t_d}^{t_o}\lambda tdt + (M - t_o)\int_{t_d}^{t_o}(\alpha + \beta t + \gamma t^2)dt\Bigg]\Bigg\}$$

$$= \frac{\lambda}{T}\left\{\frac{1}{2}W_{24}t_o^2 - X_{24}t_o + Y_{24} + \frac{(C_b + C_\pi\delta)}{2}T^2 - (C_b + C_\pi\delta)t_o T\right\} \qquad (67)$$

where

$W_{24} = [h_o[\theta_o t_d + 1] + C\theta_o + (C_b + C_\pi\delta) + sI_e], \mathrm{X}_{24} = \left[h_o\theta_o t_d^2 + C\theta_o t_d + sI_e\left[t_d + M - \frac{1}{\lambda}\left(\alpha t_d + \beta\frac{t_d^2}{2} + \gamma\frac{t_d^3}{3}\right)\right]\right]$ and

$$Y_{24} = \frac{1}{\lambda}\left[A + h_r\left\{\frac{\alpha}{2}t_r^2 + \frac{\beta}{3}t_r^3 + \frac{\gamma}{4}t_r^4\right\} + h_o\left[\frac{\alpha}{2}(t_d^2 - t_r^2) + \frac{\beta}{3}(t_d^3 - t_r^3) + \frac{\gamma}{4}(t_d^4 - t_r^4) + \frac{\lambda\theta_o t_d^3}{2} - \frac{\lambda}{2}t_d^2\right]\right.$$
$$\left. + C\frac{\lambda}{2}\theta_o t_d^2 - sI_e\left[\left(\alpha\frac{t_d^2}{2} + \beta\frac{t_d^3}{3} + \gamma\frac{t_d^4}{4}\right) + M\left(\alpha t_d + \beta\frac{t_d^2}{2} + \gamma\frac{t_d^3}{3}\right) - \frac{\lambda t_d^2}{2} - Mt_d\lambda\right]\right].$$

3.2.1 Optimal Decision

In order to find the optimal ordering policies that minimize the total variable cost per unit time, the necessary and sufficient conditions are established. The necessary condition for the total variable cost per unit time $Z_{ij}(t_o, T)$ to be minimum are $\frac{\partial Z_{ij}(t_o,T)}{\partial t_o} = 0$ and $\frac{\partial Z_{ij}(t_o,T)}{\partial T} = 0$ for $i = 2$ when $t_d > t_r$ and $j = 1, 2, 3, 4$. The value of (t_o, T) obtained from $\frac{\partial Z_{ij}(t_o,T)}{\partial t_o} = 0$ and $\frac{\partial Z_{i\,j}(t_o,T)}{\partial T} = 0$ and for which the sufficient condition $\left\{\left(\frac{\partial^2 Z_{ij}(t_o,T)}{\partial t_o^2}\right)\left(\frac{\partial^2 Z_{ij}(t_o,T)}{\partial T^2}\right) - \left(\frac{\partial^2 Z_{ij}(t_o,T)}{\partial t_o\,\partial T}\right)^2\right\} > 0$ is satisfied gives a minimum for the total variable cost per unit time $Z_{ij}(t_o, T)$.

Optimality condition for sub-case 2.1: $0 < M \leq t_r$

The necessary conditions for the total variable cost $Z_{21}(t_o, T)$ in equation (64) to be the minimum are $\frac{\partial Z_{21}(t_o,T)}{\partial t_o} = 0$ and $\frac{\partial Z_{21}(t_o,T)}{\partial T} = 0$, which give

$$\frac{\partial Z_{21}(t_o, T)}{\partial t_o} = \frac{\lambda}{T}\{W_{21}t_o - \mathrm{X}_{21} - (C_b + C_\pi\delta)T\} = 0 \qquad (68)$$

and

$$T = \frac{1}{(C_b + C_\pi\delta)}(W_{21}t_o - \mathrm{X}_{21}) \qquad (69)$$

Note that

$$W_{21}t_o - \mathrm{X}_{21} = [h_o(t_d\theta_o(t_o - t_d) + t_o) + C\theta_o(t_o - t_d) + (C_b + C_\pi\delta)t_o + cI_c((t_o - M) + \theta_o(t_d - M)(t_o - t_d))] > 0$$

since $(t_d - M), (t_o - M), (t_o - t_d), > 0$

Similarly

$$\frac{\partial Z_{21}(t_o, T)}{\partial T} = -\frac{\lambda}{T^2}\left\{\frac{1}{2}W_{21}t_o^2 - \mathrm{X}_{21}t_o + Y_{21} - \frac{T^2}{2}(C_b + C_\pi\delta)\right\} = 0 \qquad (70)$$

Replacing T from equation (69) into equation (70) yields

$$W_{21}(W_{21} - (C_b + C_\pi\delta))t_o^2 - 2\mathrm{X}_{21}(W_{21} - (C_b + C_\pi\delta))t_o - (2(C_b + C_\pi\delta)Y_{21} - \mathrm{X}_{21}^2) = 0 \quad (71)$$

Let $\Delta_{21} = W_{21}(W_{21} - (C_b + C_\pi\delta))t_r^2 - 2\mathrm{X}_{21}(W_{21} - (C_b + C_\pi\delta))t_r - (2(C_b + C_\pi\delta)Y_{21} - \mathrm{X}_{21}^2)$, then the following result is obtained.

Lemma 3.2.1

(i) If $\Delta_{21} \leq 0$, then the solution of $t_o \in [t_r, \infty)$ (say t^*_{o21}) which satisfies equation (71) does not only exist but also unique.

(ii) If $\Delta_{21} > 0$, then the solution of $t_1 \in [t_r, \infty)$ which satisfies equation (71) does not exist.

Proof: The process of proof is similar to Lemma 3.1.1.

Thus, the value of t_o (denoted by t^*_{o21}) can be found from equation (71) and is given by

$$t^*_{o21} = \frac{X_{21}}{W_{21}} + \frac{1}{W_{21}}\sqrt{\frac{(2W_{21}Y_{21} - X^2_{21})(C_b + C_\pi\delta)}{\left(W_{21} - (C_b + C_\pi\delta)\right)}} \tag{72}$$

Once t^*_{o21} is obtained, then the value of T (denoted by T^*_{21}) can be found from equation (69) and is given by

$$T^*_{21} = \frac{1}{(C_b + C_\pi\delta)}(W_{21}t^*_{o21} - X_{21}) \tag{73}$$

Equations (72) and (73) give the optimal values of t^*_{o21} and T^*_{21} for the cost function in equation (33) only if X_{21} satisfies the inequality given in equation (74)

$$X^2_{21} < 2W_{21}Y_{21} \tag{74}$$

Theorem 3.2.1

(i) If $\Delta_{21} \leq 0$, then the total variable cost $Z_{21}(t_o, T)$ is convex and reaches its global minimum at the point (t^*_{o21}, T^*_{21}), where (t^*_{o21}, T^*_{21}) is the point which satisfies equations (71) and (68).

(ii) If $\Delta_{21} > 0$, then the total variable cost $Z_{21}(t_o, T)$ has a minimum value at the point (t^*_{o21}, T^*_{21}) where $t^*_{o21} = t_r$ and $T^*_{21} = \frac{1}{(C_b + C_\pi\delta)}(W_{21}t_r - X_{21})$

Proof: The process of proof is similar to Theorem 3.1.1.

Optimality condition for sub-case 2.2: $t_r < M \leq t_d$

The necessary conditions for the total variable cost $Z_{22}(t_o, T)$ in equation (34) to be the minimum are $\frac{\partial Z_{22}(t_o,T)}{\partial t_o} = 0$ and $\frac{\partial Z_{22}(t_o,T)}{\partial T} = 0$, which give

$$\frac{\partial Z_{22}(t_o, T)}{\partial t_o} = \frac{\lambda}{T}\{W_{22}t_o - X_{22} - (C_b + C_\pi\delta)T\} = 0 \tag{75}$$

and

$$T = \frac{1}{(C_b + C_\pi\delta)}(W_{22}t_o - X_{22}) \tag{76}$$

Note that

$$W_{22}t_o - X_{22} = \left[h_o(t_d\theta_o(t_o - t_d) + t_o) + C\theta_o(t_o - t_d) + (C_b + C_\pi\delta)t_o \right. \\ \left. + cI_c\left((t_o - M) + \theta_o(t_d - M)(t_o - t_d)\right)\right] > 0$$

since $(t_d - M), (t_o - M), (t_o - t_d) > 0$

Similarly

$$\frac{\partial Z_{22}(t_o,T)}{\partial T} = -\frac{\lambda}{T^2}\left\{\frac{1}{2}W_{22}t_o^2 - \mathrm{X}_{22}t_o + Y_{22} - \frac{T^2}{2}(C_b + C_\pi\delta)\right\} = 0 \tag{77}$$

Replacing T from equation (76) into equation (77) yields

$$W_{22}\big(W_{22} - (C_b + C_\pi\delta)\big)t_o^2 - 2\mathrm{X}_{22}\big(W_{22} - (C_b + C_\pi\delta)\big)t_o - \big(2(C_b + C_\pi\delta)Y_{22} - \mathrm{X}_{22}^2\big) = 0 \tag{78}$$

Let $\Delta_{22} = W_{22}\big(W_{22} - (C_b + C_\pi\delta)\big)t_d^2 - 2\mathrm{X}_{22}\big(W_{22} - (C_b + C_\pi\delta)\big)t_d - \big(2(C_b + C_\pi\delta)Y_{22} - \mathrm{X}_{22}^2\big)$, then the following result is obtained.

Lemma 3.2.2

(i) If $\Delta_{22} \le 0$, then the solution of $t_o \in [t_d, \infty)$ (say t_{o22}^*) which satisfies equation (78) does not only exist but also unique.

(ii) If $\Delta_{22} > 0$, then the solution of $t_o \in [t_d, \infty)$which satisfies equation (78) does not exist.

Proof: The process of proof is similar to Lemma 3.1.1.

Thus, the value of t_o (denoted by t_{o22}^*) can be found from equation (78) and is given by

$$t_{o22}^* = \frac{\mathrm{X}_{22}}{W_{22}} + \frac{1}{W_{22}}\sqrt{\frac{(2W_{22}Y_{22} - \mathrm{X}_{22}^2)(C_b + C_\pi\delta)}{\big(W_{22} - (C_b + C_\pi\delta)\big)}} \tag{79}$$

Once t_{o22}^* is obtained, then the value of T (denoted by T_{22}^*) can be found from equation (76) and is given by

$$T_{22}^* = \frac{1}{(C_b + C_\pi\delta)}(W_{22}t_{o22}^* - \mathrm{X}_{22}) \tag{80}$$

Equations (79) and (80) give the optimal values of t_{o22}^* and T_{22}^* for the cost function in equation (34) only if X_{22} satisfies the inequality given in equation (81)

$$\mathrm{X}_{22}^2 < 2W_{22}Y_{22} \tag{81}$$

Theorem 3.2.2

(i) If $\Delta_{22} \le 0$, then the total variable cost $Z_{22}(t_o, T)$ is convex and reaches its global minimum at the point $(t_{o22}^*,\ T_{22}^*)$, where $(t_{o22}^*,\ T_{22}^*)$ is the point which satisfies equations (78) and (75).

(ii) If $\Delta_{22} > 0$, then the total variable cost $Z_{22}(t_o, T)$ has a minimum value at the point $(t_{o22}^*,\ T_{22}^*)$ where $t_{o22}^* = t_d$ and $T_{22}^* = \frac{1}{(C_b + C_\pi\delta)}(W_{22}t_r - \mathrm{X}_{22})$

Proof: The process of proof is similar to Theorem 3.1.1.

Optimality condition for sub-case 2.3: $t_d < M \le t_o$

The necessary conditions for the total variable cost $Z_{23}(t_o, T)$ in equation (35) to be the minimum are $\frac{\partial Z_{23}(t_o,T)}{\partial t_o} = 0$ and $\frac{\partial Z_{23}(t_o,T)}{\partial T} = 0$, which give

$$\frac{\partial Z_{23}(t_o,T)}{\partial t_o} = \frac{\lambda}{T}\{W_{23}t_o - \mathrm{X}_{23} - (C_b + C_\pi\delta)T\} = 0 \tag{82}$$

and

$$T = \frac{1}{(C_b + C_\pi \delta)}(W_{23} t_o - X_{23}) \quad (83)$$

Note that

$$W_{23} t_o - X_{23} = [h_o(t_d \theta_o (t_o - t_d) + t_o) + C\theta_o(t_o - t_d) + (C_b + C_\pi \delta) t_o + cI_c(t_o - M)] > 0$$

since $(t_o - M), (t_o - t_d) > 0$

Similarly

$$\frac{\partial Z_{23}(t_o, T)}{\partial T} = -\frac{\lambda}{T^2}\left\{\frac{1}{2} W_{23} t_o^2 - X_{23} t_o + Y_{23} - \frac{T^2}{2}(C_b + C_\pi \delta)\right\} = 0 \quad (84)$$

$$W_{23}\big(W_{23} - (C_b + C_\pi \delta)\big) t_o^2 - 2X_{23}\big(W_{23} - (C_b + C_\pi \delta)\big) t_o - (2(C_b + C_\pi \delta) Y_{23} - X_{23}^2) = 0 (85)$$

Let $\Delta_{23} = W_{23}\big(W_{23} - (C_b + C_\pi \delta)\big) M^2 - 2X_{23}\big(W_{23} - (C_b + C_\pi \delta)\big) M - (2(C_b + C_\pi \delta) Y_{23} - X_{23}^2)$, then the following result is obtained.

Lemma 3.2.3

(i) If $\Delta_{23} \le 0$, then the solution of $t_o \in [M, \infty)$ (say t_{o23}^*) which satisfies equation (85) does not only exists but also unique.

(ii) If $\Delta_{23} > 0$, then the solution of $t_o \in [M, \infty)$which satisfies equation (85) does not exist.

Proof: The process of proof is similar to Lemma 3.1.1.

Thus, the value of t_o (denoted by t_{o23}^*) can be found from equation (85) and is given by

$$t_{o23}^* = \frac{X_{23}}{W_{23}} + \frac{1}{W_{23}}\sqrt{\frac{(2W_{23} Y_{23} - X_{23}^2)(C_b + C_\pi \delta)}{\big(W_{23} - (C_b + C_\pi \delta)\big)}} \quad (86)$$

Once t_{o23}^* is obtained, then the value of T (denoted by T_{23}^*) can be found from equation (82) and is given by

$$T_{23}^* = \frac{1}{(C_b + C_\pi \delta)}(W_{23} t_{o23}^* - X_{23}) \quad (87)$$

Equations (86) and (87) give the optimal values of t_{o23}^* and $T_{23}^*$for the cost function in equation (35) only if X_{23} satisfies the inequality given in equation (88)

$$X_{23}^2 < 2W_{23} Y_{23} \quad (88)$$

Theorem 3.2.3

(i) If $\Delta_{23} \le 0$,then the total variable cost $Z_{23}(t_o, T)$ is convex and reaches its global minimum at the point (t_{o23}^*, T_{23}^*), where (t_{o23}^*, T_{23}^*) is the point which satisfies equations (85) and (82).

(ii) If $\Delta_{23} > 0$, then the total variable cost $Z_{23}(t_o, T)$ has a minimum value at the point (t_{o23}^*, T_{23}^*) where $t_{o23}^* = M$ and $T_{23}^* = \frac{1}{(C_b + C_\pi \delta)}(W_{23} M - X_{23})$

Proof: The process of proof is similar to Theorem 3.1.1.

Optimality condition for sub-case 2.4: $M > t_o$.

The necessary conditions for the total variable cost $Z_{24}(t_o, T)$ in equation (67) to be the minimum are $\frac{\partial Z_{24}(t_o,T)}{\partial t_o} = 0$ and $\frac{\partial Z_{24}(t_o,T)}{\partial T} = 0$, which give

$$\frac{\partial Z_{24}(t_o, T)}{\partial t_o} = \frac{\lambda}{T}\{W_{24}t_o - \mathrm{X}_{24} - (C_b + C_\pi \delta)T\} = 0 \quad (89)$$

and

$$T = \frac{1}{(C_b + C_\pi \delta)}(W_{24}t_o - \mathrm{X}_{24}) \quad (90)$$

Note that

$$W_{24}t_o - \mathrm{X}_{24} = \Bigg[h_o(t_d\theta_o(t_o - t_d) + t_o) + C\theta_o(t_o - t_d) + (C_b + C_\pi\delta)t_o + sI_e\left[(t_o - t_d) + \left(\alpha t_d + \beta\frac{t_d^2}{2} + \gamma\frac{t_d^3}{3}\right)\frac{1}{\lambda} - M\right]\Bigg] > 0$$

since , $(t_o - t_d) > 0$

Similarly

$$\frac{\partial Z_{24}(t_o, T)}{\partial T} = -\frac{\lambda}{T^2}\left\{\frac{1}{2}W_{24}t_o^2 - \mathrm{X}_{24}t_o + Y_{24} - \frac{T^2}{2}(C_b + C_\pi\delta)\right\} = 0 \quad (91)$$

Replacing T from equation (90) into equation (91) yields

$$W_{24}\big(W_{24} - (C_b + C_\pi\delta)\big)t_o^2 - 2\mathrm{X}_{24}\big(W_{24} - (C_b + C_\pi\delta)\big)t_o - (2(C_b + C_\pi\delta)Y_{24} - \mathrm{X}_{24}^2) = 0 \quad (92)$$

Let $\Delta_{24a} = W_{24}\big(W_{24} - (C_b + C_\pi\delta)\big)t_d^2 - 2\mathrm{X}_{24}\big(W_{24} - (C_b + C_\pi\delta)\big)t_d - (2(C_b + C_\pi\delta)Y_{24} - \mathrm{X}_{24}^2)$ and $\Delta_{24b} = W_{24}\big(W_{24} - (C_b + C_\pi\delta)\big)M^2 - 2\mathrm{X}_{24}\big(W_{24} - (C_b + C_\pi\delta)\big)M - (2(C_b + C_\pi\delta)Y_{24} - \mathrm{X}_{24}^2)$, then the following result is obtained.

Lemma 3.2.4

(i) If $\Delta_{24a} \leq 0 \leq \Delta_{24b}$, then the solution of $t_o \in [t_d,\ M]$ (say t_{o24}^*) which satisfies equation (92) does not only exists but also unique.

(ii) If $\Delta_{24b} < 0$, then the solution of $t_o \in [t_d, M]$ which satisfies equation (92) does not exist.

Proof: The process of proof is similar to Lemma 3.1.1.

Thus, the value of t_o (denoted by t_{o24}^*) can be found from equation (92) is given by

$$t_{o24}^* = \frac{\mathrm{X}_{24}}{W_{24}} + \frac{1}{W_{24}}\sqrt{\frac{(2W_{24}Y_{24} - \mathrm{X}_{24}^2)(C_b + C_\pi\delta)}{\big(W_{24} - (C_b + C_\pi\delta)\big)}} \quad (93)$$

Once t_{o24}^* is obtained, then the value of T (denoted by T_{24}^*) can be found from equation (90) and is given by

$$T_{24}^* = \frac{1}{(C_b + C_\pi\delta)}(W_{24}t_{o24}^* - \mathrm{X}_{24}) \quad (94)$$

Equations (93) and (94) give the optimal values of t_{o24}^{*} and T_{24}^{*} for the cost function in equation (67) only if X_{24} satisfies the inequality given in equation (95).

$$X_{24}^{2} < 2W_{24}Y_{24} \tag{95}$$

Theorem 3.2.4

(i) If $\Delta_{24a} \leq 0 \leq \Delta_{24b}$, then the total variable cost $Z_{24}(t_o, T)$ is convex and reaches its global minimum at the point $(t_{o24}^{*}, T_{24}^{*})$, where $(t_{o24}^{*}, T_{24}^{*})$is the point which satisfies equations (92) and (90).

(ii) If $\Delta_{42} < 0$, then the total variable cost $Z_{24}(t_o, T)$ has a minimum value at the point $(t_{o24}^{*}, T_{24}^{*})$ where $t_{o24}^{*} = M$ and $T_{24}^{*} = \frac{1}{(C_b + C_\pi \delta)}(W_{24}M - X_{24})$.

(iii) If $\Delta_{24a} > 0$, then the total variable cost $Z_{24}(t_o, T)$ has a minimum value at the point $(t_{o24}^{*}, T_{24}^{*})$ where $t_{o24}^{*} = t_d$ and $T_{24}^{*} = \frac{1}{(C_b + C_\pi \delta)}(W_{24}t_d - X_{24})$

Proof: The process of proof is similar to Theorem 3.1.1.

Thus, the economic order quantity (EOQ) corresponding to the optimal cycle length T^* will be computed as follows:

$EOQ^* =$The maximum inventory $+$the backordered units during the shortage period.

$$= \alpha t_d + \frac{\beta}{2}t_d^2 + \frac{\gamma}{3}t_d^3 + \frac{\lambda}{\theta_o}\left(e^{\theta_o(t_o^* - t_d)} - 1\right) + \frac{\lambda}{\delta}\left[ln[1 + \delta(T^* - t_o^*)]\right] \tag{96}$$

3.2.2 Numerical Examples

This section provides some numerical examples to illustrate the model established.

Example 3.2.1 (Sub-case 2.1)

The data are same as in Example 3.1.1 except that $t_d = 0.2218$ year (81 days). It is observed that $\Delta_{21} = -53.3986 < 0$, $X_{21}^2 = 0.6836$, $2W_{21}Y_{21} = 103.4295$ and hence $X_{21}^2 < 2W_{21}Y_{21}$. Substituting the above values in equations $(72), (73), (64)$ and (96), the value of optimal time at which the inventory level reaches zero in the owned ware-house, cycle length, total variable cost and economic order quantity are respectively obtained as follows: $t_{o21}^{*} = 0.4257$ year (155 days), $T_{21}^{*} = 0.6013$ year (220 days), $Z_{21}(t_{o21}^{*}, T_{21}^{*}) = \$1896.2495$ per year and $EOQ_{21}^{*} = 388.0180$ units per year.

Example 3.2.2 (Sub-case 2.2)

The data are same as in Example 3.2.1 except that $M = 0.2382$ year (87 days). It is observed that $M > t_r$, $\Delta_{22} = -42.1049 < 0$, $X_{22}^2 = 2.5041$, $2W_{22}Y_{22} = 85.1944$ and hence $X_{22}^2 < 2W_{22}Y_{22}$. Substituting the above values in equations $(79), (80), (65)$ and $((96)$, the value of optimal time at which the inventory level reaches zero in the owned ware-house, cycle length, total variable cost and economic order quantity are respectively obtained as follows: $t_{o22}^{*} = 0.4064$ year (148 days), $T_{22}^{*} = 0.5403$ year (197 days), $Z_{22}(t_{o22}^{*}, T_{22}^{*}) = \$1446.2407$ per year and $EOQ_{22}^{*} = 362.5485$ units per year.

Example 3.2.3 (Sub-case 2.3)

The data are same as in Example 3.2.1 except that $M = 0.2464$ year (90 days). It is observed that $M > t_d$, $\Delta_{23} = -39.3460 < 0$, $X_{23}^2 = 2.6249$, $2W_{23}Y_{23} = 86.8411$ and hence $X_{23}^2 < 2W_{23}Y_{23}$. Substituting the above values in equations $(86), (87), (66)$ and $((96)$, the value of optimal time at which the inventory level reaches zero in the owned ware-house, cycle length, total variable cost and economic order quantity are respectively obtained as follows: $t_{o23}^{*} = 0.4107$ year (150 days), $T_{23}^{*} = 0.5452$ year (199 days), $Z_{23}(t_{o23}^{*}, T_{23}^{*}) = \$1452.6834$ per year and $EOQ_{23}^{*} = 364.7353$ units per year.

Example 3.2.4 (Sub-case 2.4)

The data are same as in Example 3.2.1 except that $M = 0.3559$ year (130 days). It is observed that $\Delta_{24a} = -18.4884 < 0$, $\Delta_{24b} = 15.1201 > 0$, $X_{24}^2 = 0.9060$, $2W_{24}Y_{24} = 53.4469$. Here hence $\Delta_{24a} \leq 0 \leq \Delta_{24b}$ and $X_{24}^2 < 2W_{24}Y_{24}$. Substituting the above values in equations $(93), (94), (67)$ and $((96)$, the value of optimal time at which the inventory level reaches zero in the owned ware-house, cycle length, total variable cost and economic order quantity are respectively obtained as follows: $t_{o24}^{*} = 0.3011$ year (110 days), $T_{24}^{*} = 0.4184$ year (153 days), $Z_{24}(t_{o24}^{*}, T_{24}^{*}) = \$1266.3614$ per year and $EOQ_{24}^{*} = 308.0575$ units per year. It is also seen that $M > t_o$.

3.2.3 Summary of numerical examples

Table 3.2.3 Comparison of Results

Sub-cases	t_o^*(years)	T^*(years)	$Z(t_o^*, T^*)$($)	EOQ^*(Units per year)
2.1	0.4257	0.6013	1896.2495	388.0180
2.2	0.4064	0.5452	1446.2407	362.5485
2.3	0.4107	0.5453	1452.6834	364.7353
2.4	0.3011	0.4184	1266.3614	308.0575

Sub-case 2.4 provide the optimal deterioration period, cycle length and total variable cost while sub-case 2.1 provides the optimal EOQ, as such, case 2.1 and 2.4 are the feasible cases in which an optimal case can be determined. On average, 1.7637 units can be ordered per day at a cost of \$8.6193 when the time to settle the account starts from the beginning of the cycle length to the time in which the product starts decaying (Sub-case 2.1) while 2.0135 units can be ordered per day at a cost of \$8.2769 when the time to settle the account exceeds the time at which the inventory level reaches zero in the owned ware-house (Sub-case 2.4). This indicates that sub-case 2.4 provided the optimal EOQ at minimal cost per day. This shows that considering the credit period to be from the time at which the inventory level reaches zero in the owned ware-house provides both optimal total variable cost and profit which proved **sub-case 2.4** optimal among others.

Generally, determining the optimal among the optimal cases of these two different models would prove the optimal among the models. Using the same data of both numerical examples the optimal sub-cases of these two cases can be compared, and it is discovered from sub-case 1.2 of Case I that t_o^*= 0.4244y, T^*=0.5695y, $Z(t_o^*, T^*)$=(\$)1567.2293 and EOQ^*=793.8139 while in sub-case 2.4 of Case II t_o^*= 0.3011y, T^*=0.4184y, $Z(t_o^*, T^*)$=(\$)12663614 and EOQ^*= 308.0575 which shows that Case I provided the optimal order quantity while Case II provided the optimal total variable cost but, in Case I 3.8164 units are ordered per day at \$7.5348 while in Case II 2.0135 units are ordered per day at \$8.2769 which clearly shows that Case I is an optimal model.

SENSITIVITY ANALYSIS

The sensitivity analysis associated with different parameters is performed by changing each of the parameters from -40% to 40% taking one parameter at a time and keeping the remaining parameters unchanged. The effects of these parameters on length of time at which the inventory level reaches zero in the owned ware-house, cycle length, total variable cost and the economic order quantity per cycle for all the examples in Case I and Case II have been presented in the Tables below.

Table 4.1 Effect of credit period (δ) on decision Variables

Sub-cases	% change in δ	% change in t_o^*	% change in T^*	% change in EOQ^*	% change in $Z(t_o^*, T^*)$
1.1		-0.8812	1.1591	-1.2672	-5.4415
1.2	-40%	-0.6237	1.1103	-21.9649	-0.8968
1.3		-0.6129	1.1073	-25.5357	-0.8902
1.4		-0.7337	1.1752	-27.1389	-0.9836
2.1		-0.8579	1.1572	1.2006	-1.0262
2.2		-0.5847	1.0988	0.9891	-0.8726
2.3		-0.5784	1.0969	0.9906	-0.8687
2.4		-0.7362	1.1760	0.9521	-0.9852
1.1		-0.4290	0.5612	-0.6028	-2.7622
1.2	-20%	-0.3034	0.5376	-10.5492	-0.4362
1.3		-0.2981	0.5361	-12.2620	-0.4330
1.4		-0.3571	0.5690	-13.0507	-0.4787
2.1		-0.4176	0.5603	0.5714	-0.4995
2.2		-0.2844	0.5319	0.4728	-0.4244
2.3		-0.2813	0.5310	0.4735	-0.4225
2.4		-0.3583	0.5695	0.4558	-0.4795
1.1		0.4075	-0.5279	0.5492	2.8383
1.2	$+20\%$	0.2877	-0.5055	9.7801	0.4137
1.3		0.2827	-0.5041	11.3643	0.4106
1.4		0.3391	-0.5353	12.1262	0.4545
2.1		0.3966	-0.5270	-0.5211	0.4744
2.2		0.2696	-0.5002	-0.4345	0.4024
2.3		0.2667	-0.4993	-0.4351	0.4005
2.4		0.3402	-0.5357	-0.4200	0.4553
1.1		0.7951	-1.0253	1.0516	5.7464
1.2	$+40\%$	0.5610	-0.9818	18.8742	0.8066
1.3		0.5512	-0.9790	21.9280	0.8005
1.4		0.6614	-1.0397	23.4250	0.8867
2.1		0.7738	-1.0237	-0.9982	0.9257
2.2		0.5256	-0.9714	-0.8352	0.7844
2.3		0.5199	-0.9698	-0.8364	0.7808
2.4		0.6637	-1.0405	-0.8082	0.8882

Table 4.2 Effect of interest charges (I_c) on decision Variables

Sub-cases	% change in I_c	% change in t_o^*	% change in T^*	% change in EOQ^*	% change in $Z(t_o^*, T^*)$
1.1		9.6153	5.1399	3.9474	-2.8432
1.2	-40%	6.4488	4.3557	41.3412	-1.7662
1.3		6.2296	4.2385	49.1319	-1.6435
1.4		0.000	0.000	0.000	0.000
2.1		9.2624	4.8210	3.5587	-5.9483
2.2		6.1249	4.1944	2.8701	-1.6645
2.3		5.9755	4.1177	2.8253	-1.5543
2.4		0.000	0.000	0.000	0.000
1.1		4.4982	2.3805	1.8310	-1.3971
1.2	-20%	2.9966	2.0211	15.6815	-0.8320
1.3		2.8935	1.9661	18.6145	-0.7735
1.4		0.000	0.000	0.000	0.000
2.1		4.3323	2.2264	1.6468	-2.8800
2.2		2.8448	1.9455	1.3310	-0.7840
2.3		2.7745	1.9096	1.3100	-0.7310
2.4		0.000	0.000	0.000	0.000
1.1		-3.9913	-2.0729	-1.5992	1.3491
1.2	$+20\%$	-2.6273	-1.7678	-10.6497	0.7462
1.3		-2.5351	-1.7188	-12.6168	0.6927
1.4		0.000	0.000	0.000	0.000
2.1		-3.8429	-1.9276	-1.4319	2.7164
2.2		-2.4923	-1.7004	-1.1630	0.7032
2.3		-2.4292	-1.6687	-1.14435	0.6536
2.4		0.000	0.000	0.000	0.000
1.1		-7.5612	-3.8923	-3.0075	2.6520
1.2	$+40\%$	-4.9509	-3.3277	-18.3959	1.4195
1.3		-4.7756	-3.2349	-21.7760	1.3169
1.4		0.000	0.000	0.000	0.000
2.1		-7.2792	-3.6093	-2.6867	5.2894
2.2		-4.6949	-3.1998	-2.1884	1.3379
2.3		-4.5749	-3.1398	-2.1530	1.2418
2.4		0.000	0.000	0.000	0.000

Table 4.3 Effect of interest earn (I_e) on decision Variables

Sub-cases	% change in I_e	% change in t_o^*	% change in T^*	% change in EOQ^*	% change in $Z(t_o^*, T^*)$
1.1		0.3420	0.3599	0.2528	0.3917
1.2	-40%	3.7352	4.1520	4.0674	5.3708
1.3		3.8732	4.3163	5.6208	5.6254
1.4		22.9868	19.6767	15.5582	11.1561
2.1		0.3519	0.3720	0.2507	0.4209
2.2		4.9672	5.5733	3.6390	7.4129
2.3		4.6624	5.2397	3.4315	7.0023
2.4		28.1517	24.5809	14.8570	15.4099
1.1		0.1712	0.1801	0.1265	0.1960
1.2	-20%	1.8866	2.0971	2.0565	2.7127
1.3		1.9570	2.1809	2.8428	2.8424
1.4		10.9142	9.5807	8.1137	6.1482
2.1		0.1761	0.1862	0.1255	0.2107
2.2		2.5172	2.8244	1.8449	3.7567
2.3		2.3609	2.6533	1.7383	3.5458
2.4		13.4926	12.1091	7.3000	8.5559
1.1		-0.1715	-0.1804	-0.1268	-0.1964
1.2	$+20\%$	-1.9270	-2.1420	-2.1050	-2.7708
1.3		-2.0007	-2.2296	-2.9120	-2.9058
1.4		-10.0527	-9.2914	-8.9154	-7.3320
2.1		-0.1764	-0.1865	-0.1257	-0.2111
2.2		-2.5904	-2.9065	-1.9000	-3.8659
2.3		-2.4253	-2.7256	-1.7870	-3.6425
2.4		-12.7355	-12.0983	-7.2649	-10.4617
1.1		-0.4292	-0.4517	-0.3174	-0.4917
1.2	$+40\%$	-4.8993	-5.4460	-5.3608	-7.0447
1.3		-5.0903	-5.6727	-7.4207	-7.3932
1.4		-24.0621	-23.1501	-24.2001	-20.8027
2.1		-0.4417	-0.4670	-0.3147	-0.5284
2.2		-6.6285	-7.4373	-4.8646	-9.8922
2.3		-6.1964	-6.9636	-4.5681	-9.3062
2.4		-31.3614	-31.1754	-18.6897	-30.6977

Table 4.4 Effect of shortage cost (C_b) on decision Variables

Sub-cases	% change in C_b	% change in t_o^*	% change in T^*	% change in EOQ^*	% change in $Z(t_o^*, T^*)$
1.1		-5.6325	7.8524	4.4093	-9.9386
1.2	-40%	-4.0223	7.5319	22.8794	-5.7837
1.3		-3.9546	7.5116	31.3419	-5.7436
1.4		-4.7016	7.9580	32.5846	-6.3030
2.1		-5.4868	7.8402	4.2626	-6.5634
2.2		-3.7769	7.4556	4.1929	-5.6366
2.3		-3.7372	7.4435	4.1966	-5.6128
2.4		-4.7171	7.9636	4.1566	-6.3125
1.1		-2.3984	3.2127	1.8399	-4.1988
1.2	-20%	-1.7024	3.0788	9.5585	-2.4479
1.3		-1.6732	3.0703	13.0706	-2.4302
1.4		-1.9987	3.2568	13.6096	-2.6793
2.1		-2.3353	3.2076	1.7778	-2.7935
2.2		-1.5968	3.0470	1.7393	-2.3830
2.3		-1.5797	3.0419	1.7409	-2.3725
2.4		-2.0053	3.2592	1.7234	-2.6835
1.1		1.8521	-2.3595	-1.3865	3.2117
1.2	$+20\%$	1.3042	-2.2585	-7.2081	1.8753
1.3		1.2813	-2.2521	-9.8346	1.8610
1.4		1.5399	-2.3928	-10.2628	2.0644
2.1		1.8024	-2.3556	-1.3387	2.1560
2.2		1.2215	-2.2345	-1.3000	1.8229
2.3		1.2081	-2.2307	-1.3013	1.8145
2.4		1.5453	-2.3946	-1.2875	2.0679
1.1		3.3267	-4.1672	-2.4707	5.7508
1.2	$+40\%$	2.3360	-3.9872	-12.8446	3.3589
1.3		2.2947	-3.9759	-17.5119	3.3328
1.4		2.7639	-4.2266	-18.2894	3.7053
2.1		3.2367	-4.1603	-2.3849	3.8717
2.2		2.1867	-3.9446	-2.3097	3.2634
2.3		2.1627	-3.9378	-2.3119	3.2481
2.4		2.7735	-4.2298	-2.2873	3.7116

Table 4.5 Effect of cost of lost sales (C_π) on decision Variables

Sub-cases	% change in C_π	% change in t_o^*	% change in T^*	% change in EOQ^*	% change in $Z(t_o^*, T^*)$
1.1		-0.8812	1.1591	0.6700	-5.4415
1.2	-40%	-0.6237	1.1103	3.4817	-0.8968
1.3		-0.6129	1.1073	4.7572	-0.8902
1.4		-0.7337	1.1752	4.9572	-0.9836
2.1		-0.8579	1.1572	0.6472	-1.0262
2.2		-0.5847	1.0988	0.6315	-0.8726
2.3		-0.5784	1.0969	0.6321	-0.8687
2.4		-0.7362	1.1760	0.6256	-0.9852
1.1		-0.4290	0.5612	0.3253	-2.7622
1.2	-20%	-0.3034	0.5376	1.6906	-0.4362
1.3		-0.2981	0.5361	2.3094	-0.4330
1.4		-0.3571	0.5690	2.4070	-0.4787
2.1		-0.4176	0.5603	0.3142	-0.4995
2.2		-0.2844	0.5319	0.3064	-0.4244
2.3		-0.2813	0.5310	0.3066	-0.4225
2.4		-0.3583	0.5695	0.3035	-0.4795
1.1		0.4075	-0.5279	-0.3075	2.8383
1.2	$+20\%$	0.2877	-0.5055	-1.5985	0.4137
1.3		0.2827	-0.5041	-2.1825	0.4106
1.4		0.3391	-0.5353	-2.2758	0.4545
2.1		0.3966	-0.5270	-0.2970	0.4744
2.2		0.2696	-0.5002	-0.2892	0.4024
2.3		0.2667	-0.4993	-0.2894	0.4005
2.4		0.3402	-0.5357	-0.2864	0.4553
1.1		0.7951	-1.0253	-0.5987	5.7464
1.2	$+40\%$	0.5610	-0.9818	-3.1122	0.8066
1.3		0.5512	-0.9790	-4.2485	0.8005
1.4		0.6614	-1.0397	-4.4310	0.8867
2.1		0.7738	-1.0237	-0.5782	0.9257
2.2		0.5256	-0.9714	-0.5625	0.7844
2.3		0.5199	-0.9698	-0.5631	0.7808
2.4		0.6637	-1.0405	-0.5572	0.8882

RESULTS AND DISCUSSION

Based on the computational results shown in Tables and figures above, the following managerial insights are obtained.

(i) From Table 4.1, when the backlogging parameter (δ) increases, the optimal time at which the inventory level reaches zero in the owned ware-house (t_o^*), economic order quantity (EOQ^*) and total variable cost $\left(Z(t_1^*, T^*)\right)$ increase, while the optimal cycle length (T^*) decrease and vice versa. This is usually the case in real life, as any increase in backorder rate increases the order quantity while any increase in order quantity increases the total variable cost. A balanced backlogging rate is recommended to optimize the total variable cost and profit at the same time.

(ii) From Table 4.2, it is observed that as the interest charge (I_c) increases, the optimal time at which the inventory level reaches zero in the owned ware-house (t_o^*), cycle length (T^*) and order quantity (EOQ^*) decrease, while the total variable cost $(Z(T^*,\ t_o^*))$ increases and vice versa. This means that when interest charge increase, the retailer shall order fewer amounts of goods to take the benefit of trade credit more recurrently. As for $M > t_o$, increase/decrease in interest charge (I_c) does not affect the optimal time at which the inventory level reaches zero in the owned ware-house (t_o^*), cycle length (T^*), economic order quantity (EOQ^*) and total variable cost $\left(Z(t_o^*,\ T^*)\right)$, this is because the interest charge is zero when $M > t_o$. Minimum interest charge is recommended to minimize the total variable cost in this situation.

(iii) From Table 4.3, when the interest earned (I_e) increases, the optimal time at which the inventory level reaches zero in the owned ware-house (t_o^*), cycle length (T^*), economic order quantity (EOQ^*) and total variable cost $\left(Z(t_o^*,\ T^*)\right)$ decrease and vice versa. This implies that the retailer shall order fewer goods to take the benefit of trade credit more recurrently.

(iv) From Table 4.4, it is observed that when shortage cost (C_b) increases, the optimal time at which the inventory level reaches zero in the owned ware-house (t_o^*) and total variable cost $\left(Z(t_o^*,\ T^*)\right)$ increase, while the optimal cycle length (T^*) and economic order quantity (EOQ^*) decrease and vice versa. This implies that when the shortages cost increase, the total variable cost increase and the number of back-ordered goods reduce drastically which in turn lead to a decrease in order quantity.

(v) From Table 4.5, when the cost of lost sales (C_π) increases, the optimal time at which the inventory level reaches zero in the owned ware-house (t_o^*) and total variable cost $\left(Z(t_o^*,\ T^*)\right)$ increase, while the optimal cycle length (T^*) and economic order quantity (EOQ^*) decrease and vice versa. This shows the extreme need of a minimum cost of lost sales to optimize the total variable cost.

CONCLUSION

In this research, an EOQ model for non-instantaneous decaying goods with two-phase demand rates, two-storage facilities and shortages under permissible delay in payments has been established. The demand rate before deterioration sets in is assumed to be a time-dependent quadratic function after which it is considered as a constant function up to when the inventory is completely used up. Shortages considered which are partially backlogged. The length of the Waiting time would determine whether backlogging will be accepted or not, hence, the backlogging rate is variable and depends on the waiting time for the next replenishment. The optimal time at which the inventory level reaches zero in the owned ware-house, cycle length and order quantity that minimizes total variable cost has been determined. Some numerical examples have been given to demonstrate the assumed set of results of the model. Then Sensitivity analysis of some model parameters on optimal solutions have been performed and finally, suggestions toward minimizing the total variable cost of the inventory system have been provided. The model can be extended by taking more realistic assumptions such as variable deterioration rate, inflation rates, reliability of goods, linear holding cost and so on.

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COMPETING INTERESTS

The authors declare that they have no competing interests.

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